On Transcendence of Numbers Related to Sturmian and Arnoux-Rauzy Words

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Abstract

We consider numbers of the form $S_\beta(\boldsymbol{u}) := \sum_{n=0}^{\infty} \frac{u_n}{\beta^n}$, where $\boldsymbol{u} = \langle u_n \rangle_{n=0}^{\infty}$ is an infinite word over a finite alphabet and $\beta \in \mathbb{C}$ satisfies $|\beta| > 1$. Our main contribution is to present a combinatorial criterion on $\boldsymbol{u}$, called echoing, that implies that $S_\beta(\boldsymbol{u})$ is transcendental whenever β is algebraic. We show that every Sturmian word is echoing, as is the Tribonacci word, a leading example of an Arnoux-Rauzy word. We furthermore characterise $\overline{\mathbb{Q}}$-linear independence of sets of the form $\{1, S_\beta(\boldsymbol{u}_1), \ldots, S_\beta(\boldsymbol{u}_k)\}$, where $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ are Sturmian words having the same slope. Finally, we give an application of the above linear independence criterion to the theory of dynamical systems, showing that for a contracted rotation on the unit circle with algebraic slope, its limit set is either finite or consists exclusively of transcendental elements other than its endpoints 0 and 1. This confirms a conjecture of Bugeaud, Kim, Laurent, and Nogueira.

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1 Introduction

A well-known conjecture of Hartmanis and Stearns asserts that for an integer $b \geq 2$ and a sequence $\boldsymbol{u} \in \{0, \ldots, b-1\}^\omega$ that is computable by linear-time Turing machine (in the sense that given input n in unary, the machine outputs the first n elements of $\boldsymbol{u}$ in time $O(n)$), the number $S_b(\boldsymbol{u}) := \sum_{n=0}^{\infty} \frac{u_n}{b^n}$ is either rational or transcendental. This conjecture remains open and is considered to be very difficult [4]. Among many other consequences, the conjecture implies that integer multiplication cannot be done in linear time [7].

A weaker version of the Hartmanis-Stearns conjecture was formulated in 1968 by Cobham, who conjectured that every irrational automatic number is transcendental [9]. In other words, if $b \geq 2$ is an integer and $\boldsymbol{u}$ is an automatic word, then the number $S_b(\boldsymbol{u})$ is either rational or transcendental. Note that every automatic word is morphic, and that morphic words are precisely those that can be generated by so-called tag machines, a restricted class of linear-time Turing machines [4]. The transcendence of irrational automatic numbers over an integer base was proven in 2004 by Adamczewski, Bugeaud, and Luca [3]. It is noted in [1] that extending this result from the class of automatic words to the more general class of morphic words (specifically, for those generated by morphisms with polynomial growth) encompasses recognised open problems in transcendence theory. In another direction, Adamczewski and Faverjon [5] proved a generalisation of Cobham's conjecture to algebraic number bases. Their result entails that for an algebraic number β, with $|\beta| > 1$ and automatic sequence $\boldsymbol{u}$ over a finite alphabet $\{0, 1, \ldots, k-1\}$, the number $S_\beta(\boldsymbol{u})$ either lies in the field $\mathbb{Q}(\beta)$ or is transcendental.

Closely connected with the class of morphic words, one has Sturmian words and, more generally, Arnoux-Rauzy words [6]. A Sturmian word is an infinite word over the alphabet $\{0, 1\}$ that has $n + 1$ factors of length n for all $n \in \mathbb{N}$. In terms of factor complexity, Sturmian words are thereby the simplest non-periodic infinite words. Arnoux-Rauzy words are a generalisation of Sturmian words to the alphabet $\{0, 1, \ldots, k-1\}$ for arbitrary k. Among other properties, an Arnoux-Rauzy word on a k-letter alphabet has factor complexity $(k-1)n+1$. We refer to [6, Definition 3.3] for the precise definition. Perhaps the best-known example of a Sturmian word is the Fibonacci word, while the best-known example of an Arnoux-Rauzy word that is not Sturmian is the Tribonacci word. It so happens that both these words are morphic, although not automatic. The Fibonacci word is the fixed point of the morphism $0 \mapsto 01, 1 \mapsto 0$, while the Tribonacci word is the fixed point of the morphism $0 \mapsto 01, 1 \mapsto 02, 2 \mapsto 0$. More generally, Arnoux-Rauzy words can be generated by iterating a finite set of morphisms via so-called S-adic generation. Sturmian and Arnoux-Rauzy words are also intimately connected with dynamical systems. In their pioneering work [20, 21], Morse and Hedlund showed that every Sturmian word arises as the coding of a translation of the one-dimensional torus and, following the work of Rauzy [23], a subclass of Arnoux-Rauzy words can be realised as natural codings of toral translations in higher dimension [6].

There is an extensive literature on transcendence of Sturmian and Arnoux-Rauzy words over an integer base $b \geq 2$. Danilov [10] proved the transcendence of $S_b(\boldsymbol{u})$ for $\boldsymbol{u}$ the Fibonacci word. This result was significantly strengthened by Ferenczi and Mauduit [11], who proved the transcendence of $S_b(\boldsymbol{u})$, for $\boldsymbol{u}$ either a Sturmian word or an Arnoux-Rauzy word on alphabet $\{0, 1, 2\}$. This result was extended to Arnoux-Rauzy words over alphabet $\{0, 1, \ldots, k-1\}$ for any k in [24]. Meanwhile, Bugeaud *et al.* [8] showed the $\overline{\mathbb{Q}}$-linear independence of sets of the form $\{1, S_b(\boldsymbol{u}_1), S_b(\boldsymbol{u}_2)\}$, where $\boldsymbol{u}_1, \boldsymbol{u}_2$ are Sturmian words having the same slope (where the slope of a Sturmian word is the limiting frequency of 1's, which always exists).

Our interest in this paper is in proving transcendence results for Sturmian and Arnoux-Rauzy words over an algebraic-number base β, with $|\beta| > 1$. Here the picture is less complete compared to the case that β is an integer. Laurent and Nogueira [14] observe that if $\boldsymbol{u}$ is a *characteristic* Sturmian word (cf. Section 3.3), then the transcendence of $S_\beta(\boldsymbol{u})$ follows from a classical result of Loxton and Van der Poorten [17, Theorem 7] concerning transcendence of Hecke-Mahler series. For $\boldsymbol{u}$ having linear subword complexity (which includes all Arnoux-Rauzy words), it follows from [2, Theorem 1] that $S_\beta(\boldsymbol{u})$ is either transcendental or lies in the field $\mathbb{Q}(\beta)$, subject to a non-trivial inequality between the height of β and a combinatorial parameter of $\boldsymbol{u}$ called the *Diophantine exponent*. Most closely related to the present work, recently [18], introduced a criterion that can be used to show transcendence of $S_\beta(\boldsymbol{u})$ for a Sturmian word $\boldsymbol{u}$ and any β.

The main contribution of the present paper is to give a new combinatorial criterion on an infinite word $\boldsymbol{u}$, called *echoing*, that implies that $S_\beta(\boldsymbol{u}) := \sum_{n=0}^{\infty} \frac{u_n}{\beta^n}$ is transcendental for any algebraic number β. The echoing condition is an evolution of the above-mentioned criterion of [18] that allows both handling certain Arnoux-Rauzy words over non-binary alphabets as well as giving a considerably simplified treatment of Sturmian words. To illustrate the utility of the echoing notion, we show that every Sturmian word is echoing, as is the Tribonacci word, a leading example of an Arnoux-Rauzy word. We anticipate that the notion will find further applications among Arnoux-Rauzy words, and note that the thesis [13] contains further examples of such that are echoing. We also employ the echoing condition to give sufficient and necessary conditions for the $\overline{\mathbb{Q}}$-linear independence of a set of Sturmian numbers $\{1, S_\beta(\boldsymbol{u}_1), \ldots, S_\beta(\boldsymbol{u}_k)\}$, where $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$, are Sturmian words that have the same slope.

In Section 7 we give an application of our results to the theory of dynamical systems. We consider the set C of limit points of a contracted rotation f on the unit interval, where f is assumed to have an algebraic contraction factor. The set C is finite if f has a periodic orbit and is otherwise a Cantor set, that is, it is homeomorphic to the Cantor ternary set (equivalently, it is compact, nowhere dense, and has no isolated points). In the latter case we show that all elements of C except its endpoints 0 and 1 are transcendental. Our result confirms a conjecture of Bugeaud, Kim, Laurent, and Nogueira, who proved a special case of this result in [8]. We remark that it is a longstanding open question whether the actual Cantor ternary set contains any irrational algebraic elements.

2 Preliminaries

This section contains some number-theoretic preliminaries that will be used in Section 6.

For functions f and g, we use the Vinogradov notation $f \ll g$ to mean $f = O(g)$.

Let K be a number field of degree d and let $M(K)$ be the set of *places* of K. We divide $M(K)$ into the collection of *infinite places*, which are determined either by an embedding of K in $\mathbb{R}$ or a complex-conjugate pair of embeddings of K in $\mathbb{C}$, and the set of *finite places*, which are determined by prime ideals in the ring $\mathcal{O}_K$ of integers of K.

For $x \in K$ and $v \in M(K)$, define the absolute value $|x|_v$ as follows: $|x|_v := |\sigma(x)|^{1/d}$ in case v corresponds to a real embedding $\sigma : K \to \mathbb{R}$; $|x|_v := |\sigma(x)|^{2/d}$ in case v corresponds to a complex-conjugate pair of embeddings $\sigma, \overline{\sigma} : K \to \mathbb{C}$; finally, $|x|_v := N(\mathfrak{p})^{-\mathrm{ord}_{\mathfrak{p}}(x)/d}$ if v corresponds to a prime ideal $\mathfrak{p}$ in $\mathcal{O}$ and $\mathrm{ord}_{\mathfrak{p}}(x)$ is the order of $\mathfrak{p}$ as a divisor of the ideal $x\mathcal{O}$. With the above definitions we have the *product formula*: $\prod_{v \in M(K)} |x|_v = 1$ for all $x \in K^*$. Given a set of places $S \subseteq M(K)$, the ring $\mathcal{O}_S$ of *S-integers* is the subring comprising all $x \in K$ such $|x|_v \le 1$ for all finite places $v \in S$.

For $m \ge 2$ the *absolute Weil height* of $\boldsymbol{x} = (x_1, \ldots, x_m) \in K^m$ is defined to be

$$H(\boldsymbol{x}) := \prod_{v \in M(K)} \max(|x_1|_v, \ldots, |x_m|_v) .$$

This definition is independent of the choice of field K containing $x_1, \ldots, x_m$. Note the restriction $m \ge 2$ in the above definition. For $x \in K$ we define its height $H(x)$ to be $H(1, x)$. For a non-zero polynomial $f = \sum_{i=0}^{s} a_i X^i \in K[X]$, where $s \ge 1$, we define its height $H(f)$ to be the height of its coefficient vector $(a_0, \ldots, a_s)$.

The following special case of the p-adic Subspace Theorem of Schlickewei is one of the main ingredients of our approach.

▶ **Theorem 1.** *Let $S \subseteq M(K)$ be a finite set of places of K that contains all infinite places. Let $v_0 \in S$ be a distinguished place. Given $m \ge 2$, let $L(x_1, \ldots, x_m)$ be a linear form* *with algebraic coefficients and let* $i_0 \in \{1, \dots, m\}$*. Then for any* $\varepsilon > 0$ *the set of solutions* $\boldsymbol{a} = (a_1, \dots, a_m) \in (\mathcal{O}_S)^m$ *of the inequality*

$$|L(\boldsymbol{a})|_{v_0} \cdot \left(\prod_{\substack{(i,v)\in\{1,\dots,m\}\times S \\ (i,v)\neq(i_0,v_0)}} |a_i|_v \right) \leq H(\boldsymbol{a})^{-\varepsilon}$$

is contained in a finite union of proper linear subspaces of K^m*.*

We will also need the following additional proposition about roots of univariate polynomials.

▶ **Proposition 2.** *[15, Proposition 2.3] Let* $f \in K[X]$ *be a polynomial with at most* $k + 1$ *terms. Assume that* f *can be written as the sum of two polynomials* g *and* h*, where every monomial of* g *has degree at most* d_0 *and every monomial of* h *has degree at least* d_1*. Let* β *be a root of* f *that is not a root of unity. If* $d_1 - d_0 > \frac{\log(k\, H(f))}{\log H(\beta)}$ *then* β *is a common root of* g *and* h*.*

3 Echoing Words

In this section we present the main definition of the paper, the notion of echoing word. Before we present this, by way of motivation we present an informal analysis of the periodicity properties of the Fibonacci and Tribonacci words.

3.1 The Fibonacci Word

Let $\Sigma = \{0, 1\}$ and consider the morphism $\sigma : \Sigma^* \to \Sigma^*$ given by $\sigma(0) = 01$ and $\sigma(1) = 0$. The *Fibonacci word* $F_\infty \in \Sigma^\omega$ is the morphic word

$$F_\infty := \lim_{n\to\infty} \sigma^n(0) = 01001010010010100\dots .$$

In more detail, the Fibonacci word F_∞ is the limit of the sequence of finite words $(F_n)_{n=0}^\infty$ given by $F_n = \sigma^n(0)$ for all n (observe that F_n is a prefix of F_{n+1} for all n, so the limit is well defined). Note that the sequence $(F_n)_{n=0}^\infty$ satisfies the recurrence

$$F_n = F_{n-1}F_{n-2} \quad (n \geq 2)$$

analogous to that satisfied by the sequence of Fibonacci numbers.

The Fibonacci word is not periodic and hence F_∞ is not equal to any its tails $\mathrm{tl}^n(F_\infty)$ for $n > 0$. However, if the shift n is judiciously chosen then, intuitively speaking, the mismatches between F_∞ and $\mathrm{tl}^n(F_\infty)$ are *few and far between.* This intuition will be formalised in the definition of echoing word. It turns out that a particularly good choice of shifts is to take them from the sequence $\langle 1, 2, 3, 5, 8 \dots\rangle$ of Fibonacci numbers: for example, juxtaposing F_∞ and $\mathrm{tl}^5(F_\infty)$ and writing mismatches in bold we see:

$$\begin{aligned} F_\infty &:= 010010\mathbf{10}010010100010\mathbf{10}010010\mathbf{10}0100101001\dots \\ \mathrm{tl}^5(F_\infty) &:= 010010\mathbf{01}010010100010\mathbf{01}010010\mathbf{01}0100101001\dots \end{aligned}$$

Here, we see that each mismatch involves a factor 10 of F_∞ for which the corresponding factor in $\mathrm{tl}^5(F_\infty)$ is the reverse, 01. In fact, we see the same phenomenon for all shifts of F_∞ by an element of the Fibonacci sequence. Furthermore, it turns out that for each successive such shift, the distance between the mismatching factors increases. This is formalised below as the *expanding gaps property.* We will show that the preceding observations about the Fibonacci word generalise to arbitrary Sturmian words.

3.2 The Tribonacci Word

Recall that Sturmian words have factor complexity $p(n) = n + 1$ and thus can be considered as the simplest non-periodic infinite words. A natural candidate for the next simplest such class is the set of Arnoux-Rauzy words. Over a ternary alphabet such words have factor complexity $p(n) = 2n+1$. A prototypical example of an Arnoux-Rauzy word is the Tribonacci word, which we introduce next.

Let $\Sigma = \{0, 1, 2\}$ and consider the morphism $\sigma : \Sigma^* \to \Sigma^*$ given by $\sigma(0) = 01$, $\sigma(1) = 02$, and $\sigma(2) = 0$. The *Tribonacci word* $W_\infty \in \Sigma^\omega$ is the morphic word

$$W_\infty := \sigma^\omega(0) = 0102010010201\ldots .$$

In more detail, the Tribonacci word W_∞ is the limit of the sequence of finite words $(W_n)_{n=0}^\infty$ given by $W_n = \sigma^n(0)$ for all n (again we have that W_n is a prefix of W_{n+1} for all n, so the limit is well defined). Observe that the sequence of words $(W_n)_{n=0}^\infty$ satisfies recurrence

$$W_n = W_{n-1}W_{n-2}W_{n-3} \quad (n \geq 3) .$$

Associated with the Tribonacci word we have the sequence $\langle t_n \rangle_{n=0}^\infty$ of Tribonacci numbers, defined by the recurrence $t_n = t_{n-1} + t_{n-2} + t_{n-3}$ and initial conditions $t_0 = 1, t_1 = 2, t_2 = 4$. Clearly the word W_n has length t_n for all $n \in \mathbb{N}$.

In the spirit of our analysis of the Fibonacci word, we match the Tribonacci word against shifts of itself by elements of the Tribonacci sequence $\langle 1, 2, 4, 7, 13, \ldots \rangle$. By way of example, below we compare W_∞ and $\mathrm{tl}^{13}(W_\infty)$:.

$$\begin{aligned} T_\infty :=& 01020100102010\mathbf{102}0\mathbf{10}0\mathbf{102}01020100102010\mathbf{102010}01020100102010\mathbf{102}\ldots \\ \mathrm{tl}^{13}(T_\infty) :=& 01020100102010\mathbf{201}0\mathbf{01}0\mathbf{201}01020100102010\mathbf{010201}01020100102010\mathbf{201}\ldots \end{aligned}$$

Similar to the example of the Fibonacci word, the mismatches above appear as a fixed set of factors (either 10,20, or 102) in T_∞ that get reversed in $\mathrm{tl}^{13}(T_\infty)$. Unlike with the Fibonacci word, this time the factors may appear close to each other. Nevertheless, by suitably grouping these factors, we recover a form of the expanding gaps property and we are moreover able to show that the mismatches between T_∞ and its shifts are relatively sparse.

3.3 Definition of Echoing Words

Inspired by the respective examples of the Fibonacci and Tribonacci words, we give in this section the formal definition of echoing word.

Given two non-empty intervals $I, J \subseteq \mathbb{N}$, write $I < J$ if $a < b$ for all $a \in I$ and $b \in J$, and define the distance of I and J to be $d(I, J) := \min\{|a - b| : a \in I, b \in J\}$.

▶ **Definition 3.** *Let $\Sigma \subseteq \overline{\mathbb{Q}}$ be a finite alphabet. An infinite word $\boldsymbol{u} = u_0u_1u_2\ldots \in \Sigma^\omega$ is said to be* echoing *if for all $c, \varepsilon_1 > 0$, there exists $d \geq 2$ and for all $n \in \mathbb{N}$ there exist positive integers r_n, s_n and intervals $\{0\} < I_{1,n} < \cdots < I_{d,n} < \{s_n + 1\}$ of total length ℓ_n, such that:*

1. *the sequence $\langle r_n \rangle_{n=0}^\infty$ is unbounded and $s_n \geq cr_n$ for all n;*
2. *for all n it holds that $\{i \in \{0, \ldots, s_n\} : u_i \neq u_{i+r_n}\} \subseteq \bigcup_{j=1}^d I_{j,n}$ and $\ell_n \leq \varepsilon_1 s_n$;*
3. *as $n \to \infty$ we have $d(\{0\}, I_{1,n}) = \omega(\log(r_n + \ell_n))$ and $d(I_{j,n}, I_{j+1,n}) = \omega(\log \ell_n)$ for $1 \leq j \leq d - 1$;*
4. *for all $\beta \in \overline{\mathbb{Q}}$ such that $|\beta| > 1$ and all $n \in \mathbb{N}$ there exist at least two $j \in \{1, \ldots, d\}$ such that $\sum_{i \in I_{j,n}} (u_i - u_{i+r_n})\beta^{-i} \neq 0$.*

Properties 1–4 concern the factors $\langle u_0, \ldots, u_{s_n} \rangle$ and $\langle u_{r_n}, \ldots, u_{r_n+s_n} \rangle$ of $\boldsymbol{u}$. The inequality in Property 1 allows us to take the prefix length s_n to be an arbitrarily large multiple of the shift r_n. We call this the *Long-Overlap Property.* Informally speaking, Property 2 says that mismatches between the above two factors can be grouped into a fixed number d of intervals whose total length ℓ_n is small in proportion to s_n. We call this the *Short-Intervals Property.* Property 3 gives lower bounds on the length of the gaps between the above-mentioned intervals. We call this the *Expanding-Gaps Property.* Property 4 will be used to show, in case β is algebraic, that the words $\langle u_0, \ldots, u_{s_n} \rangle$ and $\langle u_{r_n}, \ldots, u_{r_n+s_n} \rangle$ denote different numbers base β for infinitely many n. We call this the *Non-Vanishing Property.*

The Fibonacci and Tribonacci words are both echoing. The formal proofs will be given respectively in Section 4 and Section 5. As suggested by the examples above, in the case of the Fibonacci word a suitable choice for the sequence $\langle r_n \rangle_{n=0}^{\infty}$ of shifts will be the Fibonacci sequence, while in the case of the Tribonacci word it will be the Tribonacci sequence. In the case of the Fibonacci word (and other Sturmian words) all of the intervals $I_{j,n}$ will be doubletons, whereas in the case of the Tribonacci word their total length ℓ_n grows linearly with s_n.

We conclude this section with some remarks about related work. The notion of a echoing word is reminiscent of the transcendence conditions of [1, 8, 11] in that it concerns periodicity in an infinite word. The ability to choose the parameter c to be arbitrarily large (the Long-Overlap property) is key to our being able to prove transcendence results over an arbitrary algebraic base β. In compensation, we allow for a small number of mismatches, as detailed in the Short-Interval property. This should be contrasted with the notion of stammering word in [1, 3, Section 4], where there is no allowance for such discrepancies and in which the quantity corresponding to c is determined in advance by the word (cf. the notion of the *Diophantine exponent* of a word in [2]).

4 Sturmian Words are Echoing

In this section we show that Sturmian words are echoing and, more generally, that a pointwise linear combination of a collection of Sturmian words having the same slope is echoing.

We will work with a characterisation of Sturmian words in terms of dynamical systems. Write $I := [0, 1)$ for the unit interval and given $x \in \mathbb{R}$ denote the integer part of x by $\lfloor x \rfloor$ and the fractional part of x by $\{x\} := x - \lfloor x \rfloor \in I$. Let $0 < \theta < 1$ be an irrational number and define the *rotation map* $R = R_\theta : I \to I$ by $R(y) = \{y + \theta\}$. Given $x \in I$, the *θ-coding of x* is the infinite word $\boldsymbol{u} = u_1 u_2 u_3 \ldots$ defined by $u_n := 1$ if $R^n(x) \in [0, \theta)$ and $u_n := 0$ otherwise. As shown by Morse and Hedlund, $\boldsymbol{u}$ is a Sturmian word and, up to changing at most two letters, all Sturmian words over a binary alphabet arise as codings of the above type for some choice of θ and x. In particular, for the purposes of establishing our transcendence results we may work exclusively with codings as defined above. The number θ is equal to the slope of the Sturmian word, as defined in Section 1. The θ-coding of 0 is in particular called the *characteristic (or standard)* Sturmian word of slope θ.

The main result of this section is as follows:

▶ **Theorem 4.** *Let $\theta \in (0, 1)$ be irrational. Given a positive integer k, let $c_0, \ldots, c_k \in \mathbb{C}$ and $x_1, \ldots, x_k \in I$ with $c_1, \ldots, c_k$ non-zero. Suppose that $x_i - x_j \notin \mathbb{Z}\theta + \mathbb{Z}$ for all $i \neq j$. Writing $\langle u_n^{(i)} \rangle_{n=0}^{\infty}$ for the θ-coding of x_i, for $i = 1, \ldots, k$, define $u_n := c_0 + \sum_{i=1}^{k} c_i u_n^{(i)}$ for all $n \in \mathbb{N}$. Then $\boldsymbol{u} = \langle u_n \rangle_{n=0}^{\infty}$ is echoing.*

Proof. We start by recalling some basic notions concerning continued fractions (see [12, Chapter 10] for details). Let

$$\theta = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}}$$

be the simple continued-fraction expansion of θ. Given $n \in \mathbb{N}$, we write $\frac{p_n}{q_n}$ for the n-th convergent of the above continued fraction, which is obtained by truncating it at a_n. Then $\langle q_n \rangle_{n=0}^{\infty}$ is a strictly increasing sequence of positive integers such that $\|q_n\theta\| = |q_n\theta - p_n|$, where $\|\alpha\|$ denotes the distance of $\alpha \in \mathbb{R}$ to the nearest integer. We moreover have that $q_n\theta - p_n$ and $q_{n+1}\theta - p_{n+1}$ have opposite signs for all n. Finally we have the *law of best approximation*: $q \in \mathbb{N}$ occurs as one of the q_n just in case $\|q\theta\| < \|q'\theta\|$ for all q' with $0 < q' < q$.

To establish that $\boldsymbol{u}$ is echoing, given $c > 0$ we define $\langle r_n \rangle_{n=0}^{\infty}$ to be the subsequence of $\langle q_n \rangle_{n=0}^{\infty}$ comprising all terms q_n such that $\|q_n\theta\| = q_n\theta - p_n > 0$. We thereby have that either $r_n = q_{2n}$ for all n or $r_n = q_{2n+1}$ for all n, so $\langle r_n \rangle_{n=0}^{\infty}$ is an infinite sequence that diverges to infinity. Next, define $d := (k+1)c$ and for all $n \in \mathbb{N}$ define s_n to be the greatest number such that the words $u_0 \ldots u_{s_n}$ and $u_{r_n} \cdots u_{r_n+s_n}$ have Hamming distance $2d$. Since $\boldsymbol{u}$ is not ultimately periodic, s_n is well-defined.

Short-Intervals Property. Given $n \in \mathbb{N}$, denote the set of positions at which $u_0 \ldots u_{s_n}$ and $u_{r_n} \ldots u_{s_n+r_n}$ differ by

$$\Delta_n := \left\{ m \in \{0, \ldots, s_n\} : u_m \neq u_{m+r_n} \right\}. \tag{1}$$

We claim that for n sufficiently large, $m \in \Delta_n$ if and only if there exists $\ell \in \{1, \ldots, k\}$ such that one of the following two conditions holds:

(i) $R^m(x_\ell) \in [1 - \|r_n\theta\|, 1)$,

(ii) $R^m(x_\ell) \in [\theta - \|r_n\theta\|, \theta)$.

We moreover claim that for all m there is at most one ℓ such that Condition (i) or (ii) holds.

Assuming the claim, since $R^m(x_\ell) \in [1-\|r_n\theta\|, 1)$ if and only if $R^{m+1}(x_\ell) \in [\theta-\|r_n\theta\|, \theta)$, it follows that the elements of Δ_n come in consecutive pairs, i.e., we can write

$$\Delta_n = \bigcup_{j=1}^{d} \{i_{j,n}, i_{j,n} + 1\},$$

where $i_{1,n} < \cdots < i_{d,n}$ are the elements $m \in \Delta_n$ that satisfy Condition (i) above for some ℓ, while $i_{1,n} + 1 < \cdots < i_{d,n} + 1$ are those that satisfy Condition (ii). Defining $I_{j,n} := \{i_{j,n}, i_{j,n} + 1\}$ for $j \in \{1, \ldots, d\}$, we have that Item 2 of Defintion 3 is satisfied. Indeed, since the intervals $I_{1,n}, \ldots, I_{d,n}$ have total length $\ell_n = 2d$, for any choice of $\varepsilon_1 > 0$ we have $\ell_n \leq \varepsilon_1 s_n$ for n sufficiently large.

It remains to prove the claim. To this end note that for a fixed $\ell \in \{1, \ldots, k\}$, for all m we have that $u_m^{(\ell)} \neq u_{m+r_n}^{(\ell)}$ iff exactly one of $R^m(x_\ell)$ and $R^{m+r_n}(x_\ell)$ lies in the interval $[0, \theta)$ iff one of Condition (i) or Condition (ii), above, holds. Moreover, since $x_\ell - x_{\ell'} \neq \theta$ (mod 1) for $\ell \neq \ell'$, we see that for n sufficiently large there is at most one $\ell \in \{1, \ldots, k\}$ such that $u_m^{(\ell)} \neq u_{m+r_n}^{(\ell)}$. We deduce that $u_m \neq u_{m+r_n}$ if and only if $u_m^{(\ell)} \neq u_{m+r_n}^{(\ell)}$ for some $\ell \in \{1, \ldots, k\}$. This concludes the proof of the claim.

Long-Overlap Property Our objective is to show that $s_n \geq cr_n$ for all $n \in \mathbb{N}$. We have already established that there are $d = (k+1)c$ distinct $m \in \Delta_n$ that satisfy Condition (i), above, for some $\ell \in \{1,\dots,k\}$. Thus there exists $\ell_0 \in \{1,\dots,k\}$ and $\Delta'_n \subseteq \Delta_n$ such that $|\Delta'_n| \geq c$ and all $m \in \Delta'_n$ satisfy Condition (i) for $\ell = \ell_0$. In this case we have $\|(m_1 - m_2)\theta\| < \|r_n\theta\|$ for all $m_1, m_2 \in \Delta'_n$. By the law of best approximation it follows that every two distinct elements of Δ'_n have difference strictly greater than r_n. But this contradicts $|\Delta'_n| = c$, given that $\Delta'_n \subseteq \{0,1,\dots,cr_n\}$.

Expanding-Gaps Property. By definition of $i_{1,n},\dots,i_{d,n}$, for all $1 \leq j_1 < j_2 \leq d$ there exists $\ell_1, \ell_2 \in \{1,\dots,k\}$ with $R^{i_{j_1,n}}(x_{\ell_1}), R^{i_{j_2,n}}(x_{\ell_2}) \in [1 - \|r_n\theta\|, 1)$. We deduce that

$$\|(i_{j_2,n} - i_{j_1,n})\theta + x_{\ell_1} - x_{\ell_2}\| \leq \|r_n\theta\| . \tag{2}$$

We claim that the left-hand side of (2) is non-zero. Indeed, the claim holds if $\ell_1 = \ell_2$ because θ is irrational, while the claim also holds in case $\ell_1 \neq \ell_2$ since in this case we have $x_{\ell_1} - x_{\ell_2} \notin \mathbb{Z}\theta + \mathbb{Z}$ by assumption. Since moreover the right-hand side of (2) tends to zero as n tends to infinity, we have that $i_{j_2,n} - i_{j_1,n} = \omega(1)$ as $n \to \infty$. Since $\ell_n = 2d$ is constant, independent of n, we have $i_{j_2,n} - i_{j_1,n} = \omega(\ell_n)$. Finally, we have $i_{1,n} > r_n$ by the requirement that $||R^{i_{n,1}}(x_{\ell_{n,1}})|| < ||r_n\theta||$ and the best-approximation property of r_n. Clearly this entails that $i_{1,n} = \omega(\log(r_n + \ell_n))$. This completes the verification of Item 3 of Definition 3(3).

Non-Vanishing Property. Consider $m \in \Delta_n$ satisfying Condition (i) above, i.e., such that $R^m(x_\ell) \in [1 - \|r_n\theta\|, 1)$ for some $\ell \in \{1,\dots,k\}$. Then we have

$$u^{(\ell)}_m = 0,\ u^{(\ell)}_{m+1} = 1 \quad \text{and} \quad u^{(\ell)}_{m+r_n} = 1,\ u^{(\ell)}_{m+r_n+1} = 0 . \tag{3}$$

Moreover for all $\ell' \neq \ell$ and n sufficiently large we have

$$u^{(\ell')}_m = u^{(\ell')}_{m+r_n} \quad \text{and} \quad u^{(\ell')}_{m+1} = u^{(\ell')}_{m+r_n+1} . \tag{4}$$

From Equations (3) and (4) we deduce that $u_m \neq u_{m+r_n}$, $u_{m+1} \neq u_{m+1+r_n}$, and $u_m + u_{m+1} = u_{m+r_m} + u_{m+r_n+1}$. But this implies that $\beta u_m + u_{m+1} \neq \beta u_{m+r_n} + u_{m+r_n+1}$ for all $\beta \neq 1$. This establishes Item 4 of Definition 3. ◀

5 The Tribonacci Word is Echoing

5.1 The Matching Morphism

In this section we define a morphism in order to understand how the Tribonacci word aligns with shifts of itself. This is an instance of a construction that is used elsewhere to show that for certain morphisms, the associated shift dynamical system has pure discrete spectrum (see the notion of *balanced pairs* in [16] and [22, Definition 6.8]).

Recall that the Tribonacci word W_∞ is defined over the alphabet $\Sigma = \{0,1,2\}$ as a fixed point of the morphism $\sigma(0) = 01, \sigma(1) = 02, \sigma(2) = 0$. We define an alphabet $\Delta = \{a_0,\dots,a_{10}\}$ whose elements are certain ordered pairs of words in Σ^+ having the same Parikh image. For intuition we represent the elements of Δ as tiles as follows:

$$a_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad a_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad a_2 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \quad a_3 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad a_4 = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix} \quad a_5 = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$a_6 = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 \end{bmatrix} \quad a_7 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \quad a_8 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad a_9 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad a_{10} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

We partition Δ into a set $\Delta_0 := \{a_0,\dots,a_7\}$ of *mismatches* and a set $\Delta_1 := \{a_8, a_9, a_{10}\}$ of *matches*.

Define morphisms top, bot : $\Delta^* \to \Sigma^*$ such that top extracts the word on the top of each tile and bottom extracts the word on the bottom, e.g.,

$$\text{top}\left(\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right) = 01 \quad \text{and} \quad \text{bot}\left(\begin{bmatrix}1 & 0 & 2\\ 2 & 0 & 1\end{bmatrix}\right) = 201\,.$$

Below we define the *matching morphism* $\mu : \Delta^* \to \Delta^*$, which is characterised by the following properties:

$$\text{top} \circ \mu = \sigma \circ \text{top} \qquad \text{and} \qquad \text{bot} \circ \mu = \sigma \circ \text{bot}\,. \tag{5}$$

Specifically we have

$$\begin{array}{ll}
\mu\left(\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right) := \begin{bmatrix}0\\ 0\end{bmatrix}\begin{bmatrix}1 & 0 & 2\\ 2 & 0 & 1\end{bmatrix} & \mu\left(\begin{bmatrix}1 & 0 & 2\\ 2 & 0 & 1\end{bmatrix}\right) := \begin{bmatrix}0\\ 0\end{bmatrix}\begin{bmatrix}2 & 0 & 1 & 0\\ 0 & 1 & 0 & 2\end{bmatrix}\\
\mu\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) := \begin{bmatrix}0\\ 0\end{bmatrix}\begin{bmatrix}2 & 0 & 1\\ 1 & 0 & 2\end{bmatrix} & \mu\left(\begin{bmatrix}2 & 0 & 1\\ 1 & 0 & 2\end{bmatrix}\right) := \begin{bmatrix}0\\ 0\end{bmatrix}\begin{bmatrix}0 & 1 & 0 & 2\\ 2 & 0 & 1 & 0\end{bmatrix}\\
\mu\left(\begin{bmatrix}0 & 2\\ 2 & 0\end{bmatrix}\right) := \begin{bmatrix}0\\ 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} & \mu\left(\begin{bmatrix}2 & 0 & 1 & 0\\ 0 & 1 & 0 & 2\end{bmatrix}\right) := \begin{bmatrix}0\\ 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 2\\ 2 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\\
\mu\left(\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\right) := \begin{bmatrix}0\\ 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} & \mu\left(\begin{bmatrix}0 & 1 & 0 & 2\\ 2 & 0 & 1 & 0\end{bmatrix}\right) := \begin{bmatrix}0\\ 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}
\end{array}$$

and

$$\mu\left(\begin{bmatrix}0\\ 0\end{bmatrix}\right) := \begin{bmatrix}0\\ 0\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} \qquad \mu\left(\begin{bmatrix}1\\ 1\end{bmatrix}\right) := \begin{bmatrix}0\\ 0\end{bmatrix}\begin{bmatrix}2\\ 2\end{bmatrix} \qquad \mu\left(\begin{bmatrix}2\\ 2\end{bmatrix}\right) := \begin{bmatrix}0\\ 0\end{bmatrix}.$$

For later use we remark that the morphism $\iota : \Sigma^* \to \Delta^*$, defined by $\iota(0) = a_0, \iota(1) = a_2, \iota(2) = a_8$ satisfies $\text{top} \circ \iota = \sigma$, while $\text{bot} \circ \iota$ and $\text{tl} \circ \sigma$ agree on Σ^ω. It follows that

$$\text{top}(\iota(W_\infty)) = W_\infty \quad \text{and} \quad \text{bot}(\iota(W_\infty)) = \text{tl}(W_\infty)\,. \tag{6}$$

Associated with the morphism μ we have its *incidence matrix* $M(\mu) \in \mathbb{N}^{11\times 11}$, where $M(\mu)_{i,j} := |\mu(a_i)|_j$ is the number of occurrences of a_j in $\mu(a_i)$ for all $i, j \in \{0, \ldots, 10\}$. It is straightforward that $M(\mu)^n_{i,j} = |\mu^n(a_i)|_j$ for all n and all $i, j \in \{0, \ldots, 10\}$. Matrix $M(\mu)$ admits a block decomposition

$$M(\mu) = \begin{pmatrix}B_1 & B_2\\ 0 & B_3\end{pmatrix},$$

where B_1 is the restriction of $M(\mu)$ to the set of mismatch symbols Δ_0 and B_3 is the restriction to the set of match symbols Δ_1. By direct calculation one see that both B_1 and B_3 are primitive and have respective spectral radii $\rho' \approx 1.395$ and $\rho \approx 1.839$. [1]

We hence have that for all $n \in \mathbb{N}$ and $i \in \{0, \ldots, 7\}$,

$$\frac{\sum_{j=0}^{7} |\mu^n(a_i)|_j}{|\mu^n(a_i)|} = \frac{\sum_{j=0}^{7} M(\mu)^n_{i,j}}{\sum_{j=0}^{10} M(\mu)^n_{i,j}} \le \frac{\sum_{j=0}^{7} (B_1^n)_{i,j}}{\sum_{j=0}^{2}\sum_{k=0}^{n-1} (B_1^k B_2 B_3^{n-k})_{i,j}} \ll \left(\frac{\rho'}{\rho}\right)^n. \tag{7}$$

We deduce that the frequency of mismatch symbols in $\mu^n(a_0)$ converges to 0 as n tends to infinity. The above reasoning shows, *inter alia* that for all $a \in \Delta$ the sequence $|\mu^n(a)| = \Theta(\rho^n)$ and hence there exists a constant κ such that $|\mu^n(a)| \le \kappa|\mu^n(b)|$ for all $a, b \in \Delta$ and all n sufficiently large.

[1] The inequality $\rho' < \rho$ implies that the Tribonacci morphism has pure discrete spectrum (see [22, Section 6.3] and the references therein).

5.2 Matching Polynomials

Define eval : $\Sigma^* \to \mathbb{Z}[x]$ by $\text{eval}(u_0 \dots u_n) := \sum_{i=0}^{n} u_i x^i$. For all $i \in \{0, \dots, 7\}$ and $n \in \mathbb{N}$ define the *matching polynomial* $P_{i,n}(x) \in \mathbb{Z}[x]$ by

$$P_{i,n} := \text{eval}(\text{top}(\mu^n(a_i))) - \text{eval}(\text{bot}(\mu^n(a_i))) \,. \tag{8}$$

By inspection, the only common root of $P_{i,0}(x)$ for $i \in \{0, \dots, 7\}$ is $x = 1$.

Observe that for all $n \in \mathbb{N}$ and $i \in \{0, 1, 2, 3\}$ we have

$$\text{top}(\mu^n(a_{2i})) = \text{bot}(\mu^n(a_{2i+1})) \quad \text{and} \quad \text{bot}(\mu^n(a_{2i})) = \text{top}(\mu^n(a_{2i+1}))\,,$$

and hence $P_{2i,n}(x) = -P_{2i+1,n}(x)$. As a consequence we can focus our attention on the even-index polynomials $P_{0,n}$, $P_{2,n}$, $P_{4,n}$, and $P_{6,n}$. Indeed, writing $\boldsymbol{P}_n := (P_{0,n}, P_{2,n}, P_{4,n}, P_{6,n}) \in \mathbb{Z}[x]^4$ and

$$M_n := (-1) \cdot \begin{pmatrix} 0 & 0 & -x^{t_n} & 0 \\ x^{t_n} & 0 & 0 & 0 \\ 0 & 0 & 0 & x^{t_n} \\ x^{t_n} + x^{t_{n+1}} & x^{t_n + t_{n+1}} & 0 & 0 \end{pmatrix},$$

then we have $\boldsymbol{P}_{n+1} = M_n \boldsymbol{P}_n$ for all $n \in \mathbb{N}$. But now, since $\boldsymbol{P}_0(\beta) \neq \boldsymbol{0}$ for all $\beta \neq 1$ and $\det(M_n) = -x^{t_{n+1} + 4t_n}$, it follows that $\boldsymbol{P}_n(\beta) \neq \boldsymbol{0}$ for all $\beta \in \mathbb{C} \setminus \{0, 1\}$.

5.3 Putting Things Together

▶ **Theorem 5.** *The Tribonacci word* $\boldsymbol{u} := W_\infty$ *is echoing.*

Proof. We refer to Definition 3. Let $c, \varepsilon_1 > 0$ be given and write $\boldsymbol{w} := \iota(u_0 u_1 \dots u_{c-1}) \in \Delta^*$. It follows from (6) that $\text{top}(\boldsymbol{w})$ is a prefix of $\boldsymbol{u}$ and $\text{bot}(\boldsymbol{w})$ is a prefix of $\text{tl}(\boldsymbol{u})$. Given $n_0 \in \mathbb{N}$, write

$$\mu^{n_0}(\boldsymbol{w}) = \boldsymbol{w}_0 a_{i_1} \boldsymbol{w}_1 \cdots \boldsymbol{w}_{d-1} a_{i_d} \boldsymbol{w}_d \,, \tag{9}$$

where $\boldsymbol{w}_0, \dots, \boldsymbol{w}_d \in \Delta_1^*$ are sequences of match symbols and $a_{i_1}, \dots, a_{i_d} \in \Delta_0$ are mismatch symbols. For $n_0 \in \mathbb{N}$ sufficiently large it holds that $\mu^{n_0}(\boldsymbol{w})$ contains at least two occurrences of every mismatch symbol and the proportion of mismatch symbols in $\mu^{n_0}(\boldsymbol{w})$ is at most ε_1/κ, for κ as in Section 5.1.

For all $n \in \mathbb{N}$, referring to Equation (9), we have

$$\mu^{n+n_0}(\boldsymbol{w}) = \mu^n(\boldsymbol{w}_0) \underbrace{\mu^n(a_{i_1})}_{I_{1,n}} \mu^n(\boldsymbol{w}_1) \cdots \mu^n(\boldsymbol{w}_{d-1}) \underbrace{\mu^n(a_{i_d})}_{I_{d,n}} \mu^n(\boldsymbol{w}_d) \,. \tag{10}$$

The data to show the echoing property are as follows. For all $n \in \mathbb{N}$ define $r_n := |\sigma^n(0)| = t_{n+n_0}$, $s_n := |\text{top}(\mu^{n+n_0}(\boldsymbol{w}))|$, and we take d as in (9). For all $j \in \{1, \dots, d\}$ we define $I_{j,n} \subseteq \mathbb{N}$ to be the interval of positions in $\mu^{n+n_0}(a_0)$ corresponding to the shortest suffix of $\mu^n(a_{i_j})$ that contains all mismatch symbols that occur therein (see (10)).

From Equation (5) we have $\text{top}(\mu^{n+n_0}(\boldsymbol{w})) = \langle u_0, \dots, u_{s_n} \rangle$ and $\text{bot}(\mu^{n+n_0}(\boldsymbol{w})) = \langle u_{r_n}, \dots, u_{s_n + r_n} \rangle$. Since $\text{top}(\boldsymbol{w})$ contains at least c occurrences of the letter 0, we get that $s_n \geq c r_n$, establishing the Long-Overlap Property.

By construction, the intervals $I_{1,n}, \dots, I_{d,n}$ contain all indices where the above two strings differ and they have total length at most $\varepsilon_1 s_n$, establishing the Short-Intervals Property.

By definition of the matching polynomials, it holds that $P_{i_j,n}(\beta^{-1})$ is the product of $\sum_{i\in I_{j,n}}(u_i - u_{i+r_n})\beta^{-1}$ and a power of β for all $j \in \{1,\ldots,d\}$. We have shown in Section 5.2 that the vector $(P_{0,n}(\beta^{-1}),\ldots,P_{7,n}(\beta^{-1}))$ is non-zero for all n. Since all mismatch symbols occur at least twice among $a_{i_1},\ldots,a_{i_d}$, we have that $P_{i_j,n}(\beta^{-1})$ is non-zero for at least two different choices of $j \in \{1,\ldots,d\}$. It follows that $\sum_{i\in I_{j,n}}(u_i - u_{i+r_n})\beta^{-1}$ is non-zero for at least two different values of j, establishing the Non-Vanishing Property.

Finally, note that there is at least one letter between each pair of intervals $I_{j,n}$ in (10). Hence the inequality $|\mu(a^n)| \leq \kappa|\mu^n(b)|$ for $a,b \in \Delta$ implies that $d(\{0\},I_{1,n}) \gg s_n$ and $d(I_{j,n},I_{j+1,n}) \gg s_n$ for all $j \in \{0,\ldots,d-1\}$, which implies the Expanding-Gaps Property. ◀

6 Transcendence Results

6.1 Transcendence for Echoing Words

▶ **Theorem 6.** *Let Σ be a finite set of algebraic numbers and let $\boldsymbol{u} \in \Sigma^\omega$ be an echoing word. Then for any algebraic number β such that $|\beta| > 1$, the sum $\alpha := \sum_{n=0}^{\infty} \frac{u_n}{\beta^n}$ is transcendental.*

Proof. Suppose for a contradiction that α is algebraic. By scaling, we can assume without loss of generality that Σ consists solely of algebraic integers. Let K be the field generated over $\mathbb{Q}$ by $\{\beta\}\cup\Sigma$ and write $S \subseteq M(K)$ for the set comprising all infinite places of K and those finite places of K arising from prime-ideal divisors of elements of $\{\beta\}\cup\Sigma$. Let $v_0 \in S$ be the place corresponding to the inclusion of K in $\mathbb{C}$. Recall that $|a|_{v_0} = |a|^{1/[K:\mathbb{Q}]}$, where $|a|$ denotes the usual absolute value on $\mathbb{C}$.

Applying the definition of echoing sequence (as given in Definition 3) for values of c and ε_1 to be specified later, we obtain $d \geq 2$ such that for all $n \in \mathbb{N}$ there are $r_n, s_n \in \mathbb{N}$ and intervals $\{0\} < I_{1,n} < \cdots < I_{d,n} < \{s_n+1\}$, of total length ℓ_n, satisfying Items 1–4 of Definition 3.

For $n \in \mathbb{N}$, define $\boldsymbol{a}_n = (a_{1,n},\ldots,a_{d+3,n}) \in (\mathcal{O}_S)^{d+3}$ by

$$a_{1,n} := \beta^{r_n},\ a_{2,n} := \sum_{i=0}^{r_n} u_i\beta^{r_n-i},\ a_{3,n} := 1,\ a_{j+3,n} := \sum_{i\in I_{n,j}} (u_{i+r_n} - u_i)\beta^{-i}\ (j = 1,\ldots,d)$$

The Non-Vanishing Property (Definition 3(4)) implies that for all $n \in \mathbb{N}$ we have $a_{j+3,n} \neq 0$ for at least two elements $j \in \{1,\ldots,d\}$. By passing to a subsequence we henceforth assume without loss of generality that there exists $J \subseteq \{1,\ldots,d\}$, of cardinality at least two, such that for all j and n, $a_{j+3,n} \neq 0$ if and only if $j \in J$.

▷ Claim 7. If $F(x_1,\ldots,x_{d+3}) = \sum_{i\in\{1,2,3\}\cup J}\alpha_i x_i$ is a linear form with coefficients in K such that $F(\boldsymbol{a}_n) = 0$ for infinitely many n, then $\alpha_j = 0$ for all $j \in J$.

The proof of the claim is as follows. For all $n \in \mathbb{N}$ we have $F(\boldsymbol{a}_n) = P_n(\beta)$ for the polynomial $P_n(x) := P_{0,n}(x) + \sum_{j\in J} P_{j,n}(x)$, where

$$P_{0,n}(x) := \alpha_1 x^{r_n} + \alpha_2 \sum_{i=0}^{r_n} u_i x^{r_n-i} + \alpha_3 \quad \text{and} \quad P_{j,n}(x) := \alpha_j \sum_{i\in I_{j,n}} (u_i - u_{i+r_n})x^{-i}\,.$$

Polynomial P_n has at most $r_n + \ell_n$ monomials. From Proposition 2 and the property $d(\{0\},I_{1,n}) = \omega(\log(r_n+\ell_n))$ (see Definition 3(3)), we deduce that $\sum_{j\in J} P_{j,n}(\beta) = 0$ for infinitely many n. Now $\sum_{j\in J} P_{j,n}(x)$ comprises at most ℓ_n monomials. Thus Proposition 2 and the assumption that $d(I_{j,n}, I_{j+1,n}) = \omega(\log \ell_n)$ for all $j \in \{1, \ldots, d-1\}$, entail that for infinitely many n we have $P_{j,n}(\beta) = 0$ for all $j \in J$. But $P_{j,n}(\beta) = \alpha_j a_{j,n}$ and so, since $a_{j,n} \neq 0$ for all $j \in J$, we have $\alpha_j = 0$ for all $j \in J$. This concludes the proof of the claim.

Consider the linear form $L_0(x_1, \ldots, x_{d+3}) := \alpha x_1 - x_2 - \alpha x_3 - \sum_{j \in J} x_{j+3}$. Then there exists $c_1 > 1$ such that for all n,

$$\begin{aligned} 0 < |L_0(\boldsymbol{a}_n)| &= \left|\beta^{r_n}\alpha - \textstyle\sum_{i=0}^{r_n} u_i \beta^{r_n - i} + \alpha + \sum_{j \in J} \sum_{i \in I_{j,n}} (u_i - u_{i+r_n})\beta^i\right| \\ &= \left|\textstyle\sum_{i=s_n+1}^{\infty} (u_i - u_{i+r_n})\beta^{-i}\right| < c_1 |\beta|^{-s_n}, \end{aligned} \tag{11}$$

where the left-hand inequality follows from an application of Claim 7 to L_0. Consider a linear form $L(x_1, \ldots, x_{d+3})$ with the following properties: (i) L has coefficients in K; (ii) L has support $\{x_i : i \in I \cup J\}$ for some $I \subseteq \{1, 2, 3\}$; (iii) $0 < |L(\boldsymbol{a}_n)| < c_1 |\beta|^{-s_n}$ for all $n \in \mathbb{N}$; (iv) the set I is minimal with respect to set inclusion among linear forms satisfying (i)–(iii). We have just exhibited a form, namely L_0, that satisfies Conditions (i)–(iii), so L is well-defined.

Let $c_2 \geq 2$ be an upper bound of the set of numbers $\{|\gamma|_v : \gamma \in \{\beta\} \cup A \cup A - A, \, v \in S\}$. Then for $v \in S$, by the assumption that $s_n \geq c r_n$ we have

$$|a_{2,n}|_v \leq \sum_{i=0}^{r_n} c_2^{i+1} \leq c_2^{r_n+2} \leq c_2^{(c^{-1}s_n+2)}. \tag{12}$$

We moreover have

$$\prod_{j \in J} \prod_{v \in S} |a_{j+3,n}|_v \leq \prod_{j \in J} \prod_{v \in S} \sum_{i=0}^{|I_{j,n}|} c_2^{i+1} \leq \prod_{j \in J} c_2^{|S|(2+|I_{j,n}|)} \leq c_2^{|S|(2d+\varepsilon_1 s_n)}, \tag{13}$$

where we use the assumption that the intervals $I_{1,n}, \ldots, I_{d,n}$ have total length $\ell_n \leq \varepsilon_1 s_n$. We also have $\prod_{v \in S} |a_{1,n}|_v = \prod_{v \in S} |\beta^{r_n}|_v = 1$ by the product formula and, obviously, $\prod_{v \in S} |a_{3,n}|_v = 1$.

Pick $i_0 \in I \cup J$. Then, combining (12) and (13) and the bound $|L(\boldsymbol{a}_n)| < c_1|\beta|^{-s_n}$, we have

$$|L(\boldsymbol{a}_n)|_{v_0} \cdot \prod_{\substack{(i,v) \in (I \cup J) \times S \\ (i,v) \neq (i_0, v_0)}} |a_{i,n}|_v \leq c_2^{\left(s_n(\varepsilon_1 + c^{-1}) + 2d + 2\right)|S|} \cdot (c_1 |\beta|^{-s_n})^{1/[K:\mathbb{Q}]}. \tag{14}$$

For c sufficiently large, ε_1 sufficiently small, and all but finitely many n, the right-hand side of (14) is less than $|\beta|^{-s_n/2[K:\mathbb{Q}]}$. On the other hand, there exists a constant $c_3 > 0$ such that the height of $\boldsymbol{a}_n$ satisfies the bound $H(\boldsymbol{a}_n) \leq |\beta|^{c_3 s_n}$ for all n. Thus there exists $\varepsilon > 0$ such that the right-hand side of (14) is at most $H(\boldsymbol{a}_n)^{-\varepsilon}$ for infinitely many n.

Given (14), we can apply Theorem 1 to obtain a non-zero linear form $L'(x_1, \ldots, x_{3+d})$ that has coefficients in K and support in $\{x_i : i \in I \cup J\}$, such that for infinitely many $n \in \mathbb{N}$ we have both $0 < |L(\boldsymbol{a}_n)| < c_1|\beta|^{-s_n}$ and $L'(\boldsymbol{a}_n) = 0$. By Claim 7, the support of L' is in fact contained in I. Hence, by subtracting a suitable multiple of L' from L we obtain a linear form $L''(x_1, \ldots, x_{3+d})$ with strictly fewer coefficients than L such that $0 < |L''(\boldsymbol{a}_n)| < c_1|\beta|^{-s_n}$ for infinitely many $n \in \mathbb{N}$. But this contradicts the minimality of the support of L. ◀

6.2 Transcendence for Sturmian Words and the Tribonacci Word

Combining the transcendence result for echoing words (Theorem 6) with the fact that Sturmian words and the Tribonacci word are echoing (Theorem 4 and Theorem 5), we obtain:

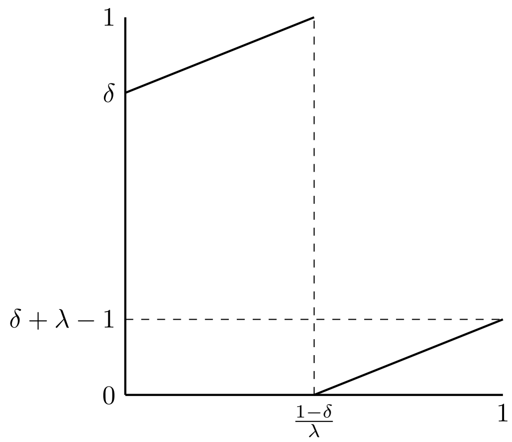

Figure 1 A plot of $f_{\lambda,\delta} : I \to I$

▶ **Theorem 8.** *Let β be an algebraic number with $|\beta| > 1$.*
1. *Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ be Sturmian words with the same slope such that $\boldsymbol{u}_i$ is not a suffix of $\boldsymbol{u}_j$ for all $i \neq j$. Then $\{1, S_\beta(\boldsymbol{u}_1), \ldots, S_\beta(\boldsymbol{u}_k)\}$ is linearly independent over $\overline{\mathbb{Q}}$.*
2. *Let $\boldsymbol{u}$ be the Tribonacci word. Then $S_\beta(\boldsymbol{u})$ is transcendental.*

7 Application to Limit Sets of Contracted Rotations

Let $0 < \lambda, \delta < 1$ be real numbers such that $\lambda + \delta > 1$. We call the map $f = f_{\lambda,\delta} : I \to I$ given by $f(x) := \{\lambda x + \delta\}$ a *contracted rotation* with slope λ and *offset* δ. Associated with f we have the map $F = F_{\lambda,\delta} : \mathbb{R} \to \mathbb{R}$, given by $F(x) = \lambda\{x\} + \delta + \lfloor x \rfloor$. We call F a *lifting* of f: it is characterised by the properties that $F(x+1) = F(x) + 1$ and $\{F(x)\} = f(\{x\})$ for all $x \in \mathbb{R}$. The *rotation number* $\theta = \theta_{\lambda,\delta}$ of f is defined by

$$\theta := \lim_{n\to\infty} \frac{F^n(x_0)}{n},$$

where the limit exists and is independent of the initial point $x_0 \in \mathbb{R}$.

If the rotation number θ is irrational then the restriction of f to the *limit set* $\bigcap_{n\geq 0} f^n(I)$ is topologically conjugated to the rotation map $R = R_\theta : I \to I$ with $R(y) = \{y + \theta\}$. The closure of the limit set is a Cantor set $C = C_{\lambda,\delta}$, that is, C is compact, nowhere dense, and has no isolated points. On the other hand, if θ is rational then the limit set C is the unique periodic orbit of f. For each choice of slope $0 < \lambda < 1$ and irrational rotation number $0 < \theta < 1$, there exists a unique offset δ such that $\delta + \lambda > 1$ and the map f has rotation number θ. It is known that such δ must be transcendental if λ is algebraic [14].

The main result of this section is as follows:

▶ **Theorem 9.** *Let $0 < \lambda, \theta < 1$ be such that λ is algebraic and θ is irrational. Let δ be the unique offset such that the contracted rotation $f_{\lambda,\delta}$ has rotation number θ. Then every element of the Cantor set $C_{\lambda,\delta}$ other than 0 and 1 is transcendental.*

A special case of Theorem 9, in which λ is assumed to be the reciprocal of an integer, was proven in [8, Theorem 1.2]. In their discussion of the latter result the authors conjecture the truth of Theorem 9, i.e., the more general case in which λ may be algebraic. As noted in [8], while $C_{\lambda,\delta}$ is homeomorphic to the Cantor ternary set, it is a longstanding open problem, formulated by Mahler [19], whether the Cantor ternary set contains irrational algebraic elements.

Proof of Theorem 9. For a real number $0 < x < 1$ define

$$\xi_x := \sum_{n \geq 1} (\lceil x + (n+1)\theta \rceil - \lceil x + n\theta \rceil) \lambda^n$$

$$\xi'_x := \sum_{n \geq 1} (\lfloor x + (n+1)\theta \rfloor - \lfloor x + n\theta \rfloor) \lambda^n .$$

Note that for all x the binary sequence $\langle \lceil x + (n+1)\theta \rceil - \lceil x + n\theta \rceil : n \in \mathbb{N} \rangle$ is the coding of $-x - \theta$ by $1 - \theta$ (as defined in Section 4) and hence is Sturmian of slope $1 - \theta$. Similarly, the binary sequence $\langle \lfloor x + (n+1)\theta \rfloor - \lfloor x + n\theta \rfloor : n \in \mathbb{N} \rangle$ is the coding of $x + \theta$ by θ and hence is Sturmian of slope θ. Thus for all x, both ξ_x and ξ'_x are Sturmian numbers.

It is shown in [8, Lemma 4.2][2] that for every element of $y \in C_{\lambda,\delta} \setminus \{0, 1\}$, either there exists $z \in \mathbb{Z}$ and $0 < x < 1$ with $x \notin \mathbb{Z}\theta + \mathbb{Z}$ such that

$$y = z + \xi_0 - \xi_{-x}$$

or else there exists a strictly positive integer m and $\gamma \in \mathbb{Q}(\beta)$ such that

$$y = \gamma + (1 - \beta^{-m})\, \xi'_0 .$$

In either case, transcendence of y follows from Theorem 8. ◀

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[2] The proof of the lemma is stated for β an integer but carries over without change for β algebraic.

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