On Various Negative Translations

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Abstract. Several proof translations of classical mathematics into intuitionistic mathematics have been proposed in the literature over the past century. These are normally referred to as *negative translations* or *double-negation translations*. Among those, the most commonly cited are translations due to Kolmogorov, Gödel, Gentzen, Kuroda and Krivine (in chronological order). In this paper we propose a framework for explaining how these different translations are related to each other. More precisely, we define a notion of a (modular) simplification starting from Kolmogorov translation, which leads to a partial order between different negative translations. In this derived ordering, Kuroda and Krivine are minimal elements. Two new minimal translations are introduced, with Gödel and Gentzen translations sitting in between Kolmogorov and one of these new translations.

1 Introduction

With the discovery of paradoxes and inconsistencies in the early formalisation of set theory, mathematicians started to worry about the logical foundations of mathematics. Proofs by contradiction, which concluded the existence of a mathematical object without actually constructing it, were immediately thought by some to be the source of the problem. Mathematicians were then segregated between those who thought classical reasoning should be allowed as long as it was finitistically justified (e.g. Hilbert) and those who thought proofs in mathematics should avoid non-constructive arguments (e.g. Brouwer). Constructivism and intuitionistic logic were born.

It was soon discovered, however, that the consistency of arithmetic based on intuitionistic logic (Heyting arithmetic) is equivalent to the consistency of arithmetic based on classical logic (Peano arithmetic). Therefore, if one accepts that intuitionistic arithmetic is consistent, then one must also accept that classical arithmetic is consistent. That was achieved via a simple translation of classical into intuitionistic logic which preserves the statement $0 = 1$. So any proof of $0 = 1$ in Peano arithmetic (if ever one is found) can be effectively translated into a proof of $0 = 1$ in Heyting arithmetic.

The first such translation is due to Kolmogorov [19] in 1925. He observed that placing a double negation $\neg\neg$ in front of every subformula turns a classically valid formula into an intuitionistically valid one. Formally, defining

$$\begin{array}{rclrcl}
(A \wedge B)^{Ko} & :\equiv & \neg\neg(A^{Ko} \wedge B^{Ko}) & P^{Ko} & :\equiv & \neg\neg P, \text{ for } P \text{ atomic} \\
(A \vee B)^{Ko} & :\equiv & \neg\neg(A^{Ko} \vee B^{Ko}) & (\forall x A)^{Ko} & :\equiv & \neg\neg\forall x A^{Ko} \\
(A \rightarrow B)^{Ko} & :\equiv & \neg\neg(A^{Ko} \rightarrow B^{Ko}) & (\exists x A)^{Ko} & :\equiv & \neg\neg\exists x A^{Ko},
\end{array}$$

one can show that A is provable classically if and only if A^{Ko} is provable intuitionistically. Kolmogorov's translation, however, was apparently not known to Gödel and Gentzen who both came up with similar translations [9, 10, 12] a few years later. Gentzen's translation (nowadays known as Gödel-Gentzen negative translation [4, 16, 27]) simply places a double negation in front of atomic formulas, disjunctions, and existential quantifiers, i.e.

$$\begin{array}{rclrcl}
(A \wedge B)^{GG} & :\equiv & A^{GG} \wedge B^{GG} & P^{GG} & :\equiv & \neg\neg P, \text{ for } P \text{ atomic} \\
(A \vee B)^{GG} & :\equiv & \neg\neg(A^{GG} \vee B^{GG}) & (\forall x A)^{GG} & :\equiv & \forall x A^{GG} \\
(A \rightarrow B)^{GG} & :\equiv & A^{GG} \rightarrow B^{GG} & (\exists x A)^{GG} & :\equiv & \neg\neg\exists x A^{GG}.
\end{array}$$

As with Kolmogorov's translation, we also have that $\mathsf{CL} \vdash A$ if and only if $\mathsf{IL} \vdash A^{GG}$, where CL and IL stand for classical and intuitionistic logic, respectively. Gödel's suggested translation was in fact somewhere in between Kolmogorov's and Gentzen's, as it also placed a double negation in front of the clause for implication, i.e.

$$(A \rightarrow B)^{GG} :\equiv \neg(A^{GG} \wedge \neg B^{GG}) \Leftrightarrow_{\mathsf{IL}} \neg\neg(A^{GG} \rightarrow B^{GG}).$$

In the 1950's, Kuroda revisited the issue of negative translations [21], and proposed a different (somewhat simpler) translation:

$$\begin{array}{rclrcl}
(A \wedge B)_{Ku} & :\equiv & A_{Ku} \wedge B_{Ku} & P_{Ku} & :\equiv & P, \text{ for } P \text{ atomic} \\
(A \vee B)_{Ku} & :\equiv & A_{Ku} \vee B_{Ku} & (\forall x A)_{Ku} & :\equiv & \forall x \neg\neg A_{Ku} \\
(A \rightarrow B)_{Ku} & :\equiv & A_{Ku} \rightarrow B_{Ku} & (\exists x A)_{Ku} & :\equiv & \exists x A_{Ku}.
\end{array}$$

Let $A^{Ku} :\equiv \neg\neg A_{Ku}$. Similarly to Kolmogov, Gödel and Gentzen, Kuroda showed that $\mathsf{CL} \vdash A$ if and only if $\mathsf{IL} \vdash A^{Ku}$. In particular, if A does not contain universal quantifiers then $\mathsf{CL} \vdash A$ iff $\mathsf{IL} \vdash \neg\neg A$, since $(\cdot)_{Ku}$ is the identity mapping on formulas not containing universal quantifiers. Finally, relatively recently, following the work of Krivine [20], yet another different translation was developed[1], namely

$$\begin{array}{rclrcl}
(A \wedge B)_{Kr} & :\equiv & A_{Kr} \vee B_{Kr} & P_{Kr} & :\equiv & \neg P, \text{ for } P \text{ atomic} \\
(A \vee B)_{Kr} & :\equiv & A_{Kr} \wedge B_{Kr} & (\forall x A)_{Kr} & :\equiv & \exists x A_{Kr} \\
(A \rightarrow B)_{Kr} & :\equiv & \neg A_{Kr} \wedge B_{Kr} & (\exists x A)_{Kr} & :\equiv & \neg\exists x \neg A_{Kr}.
\end{array}$$

[1] Throughout the paper this translation is going to be called "Krivine negative translation" as currently done in the literature (see [28, 18]) even though it should be better called Streicher-Reus translation. Although inspired by the Krivine's work in [20] it is the syntactical translation studied by Streicher and Reus [29] in a version presented in [3, 28] we are using here.

Letting $A^{Kr} :\equiv \neg A_{Kr}$, we also have that $\mathsf{CL} \vdash A$ if and only if $\mathsf{IL} \vdash A^{Kr}$.

It is also known that all these translations lead to intuitionistically equivalent formulas, in the sense that A^{Ko}, A^{GG}, A^{Ku} and A^{Kr} are all provably intuitionistically equivalent. As such, one could say that they are all essentially the same. On the other hand, it is obvious that they are intrinsically different. The goal of the present paper is to explain the precise sense in which Gödel-Gentzen, Kuroda and Krivine translations are systematic simplifications of Kolmogorov's original translation, and show that, in a precise sense, the latter two are optimal (modular) translations of classical logic into intuitionistic logic. Gödel-Gentzen translation is in between Kolmogorov's and a new optimal variant we discuss in Section 5 below.

For more comprehensive surveys on the different negative translations, with more historical background, see [17, 18, 23, 30, 31].

Note. Due to space restriction all proofs have been omitted. For all proofs see the full version of the paper at the authors webpages.

1.1 Some useful results

Our considerations on the different negative translations is based on the fact that formulas with various negations can be simplified to intuitionistically equivalent formulas with fewer negations. The cases when this is (or isn't) possible are outlined in the following lemma.

Lemma 1. *The following equivalences are provable in* IL*:*

1. $\neg\neg(\neg\neg A \wedge \neg\neg B) \leftrightarrow \neg\neg(A \wedge B)$
2. $\neg\neg(\neg\neg A \vee \neg\neg B) \leftrightarrow \neg\neg(A \vee B)$
3. $\neg\neg(\neg\neg A \to \neg\neg B) \leftrightarrow \neg\neg(A \to B)$
4. $\neg\neg\exists x\neg\neg A \leftrightarrow \neg\neg\exists x A$
5. $\neg\neg(\neg A \wedge \neg B) \leftrightarrow \neg(A \vee B)$
6. $\neg\neg(\neg A \vee \neg B) \leftrightarrow \neg(A \wedge B)$
7. $\neg\neg(\neg A \to \neg B) \leftrightarrow \neg(\neg A \wedge B)$
8. $\neg\neg\forall x\neg A \leftrightarrow \neg\exists x A$
9. $\neg\neg(\neg\neg A \wedge \neg\neg B) \leftrightarrow (\neg\neg A \wedge \neg\neg B)$
10. $\neg\neg(\neg\neg A \vee \neg\neg B) \leftrightarrow (\neg\neg\neg A \to \neg\neg B)$
11. $\neg\neg(\neg\neg A \to \neg\neg B) \leftrightarrow (\neg\neg A \to \neg\neg B)$
12. $\neg\neg\forall x\neg\neg A \leftrightarrow \forall x\neg\neg A$
13. $\neg(\neg\neg A \wedge \neg\neg B) \leftrightarrow (\neg\neg A \to \neg B)$
14. $\neg(\neg\neg A \vee \neg\neg B) \leftrightarrow (\neg A \wedge \neg B)$
15. $\neg(\neg\neg A \to \neg\neg B) \leftrightarrow (\neg\neg A \wedge \neg B)$
16. $\neg\exists x\neg\neg A \leftrightarrow \forall x\neg A$.

The following equivalences are provable in CL *but not in* IL*:*

17. $\neg\neg\forall x\neg\neg A \leftrightarrow \neg\neg\forall x A$
18. $\neg\neg\exists x\neg A \leftrightarrow \neg\forall x A$
19. $\neg\neg(\neg\neg A \vee \neg\neg B) \leftrightarrow (\neg\neg A \vee \neg\neg B)$
20. $\neg\neg\exists x\neg\neg A \leftrightarrow \exists x\neg\neg A$
21. $\neg\forall x\neg\neg A \leftrightarrow \exists x\neg A$
22. $\neg(\neg\neg A \wedge \neg\neg B) \leftrightarrow (\neg A \vee \neg B)$.

1.2 Logical framework

In the language of classical logic CL and intuitionistic logic IL, we consider as primitive the constants $\bot$, $\top$, the connectives $\wedge$, $\vee$, $\to$ and the quantifiers $\forall$ and $\exists$. We write $\neg A$ as an abbreviation for $A \to \bot$. Note that CL can be formulated using a proper subset of the symbols we consider as primitive. It would be sufficient, for instance, to consider the fragment $\{\bot, \to, \vee, \exists\}$ or $\{\bot, \to, \wedge, \forall\}$ (as adopted by Schwichtenberg in [26]). Our choice of dealing directly with the full set $\{\bot, \top, \to, \wedge, \vee, \forall, \exists\}$ in the classical framework has two main reasons: First, it emphasises which symbols are treated in a similar or different manner in classical and intuitionistic logic; second, in some embeddings of CL into IL we are going to analyse, the translations of certain formulas are syntactically different to the derived translations we would obtain considering just a subset of primitive symbols. In fact, usually when we choose to work with a subset of the logical connectives in classical logic, we are implicitly committing ourselves to one of the particular negative translations.

2 Modular Translations

Let us first observe that all negative translations mentioned above are in general not optimal – in the sense of introducing the least number of negations in order to turn a classically valid formula into an intuitionistically valid one. For instance, Kuroda translation of a purely universal formula $\forall x P(x)$ is $\neg\neg\forall x \neg\neg P(x)$, whereas Gödel-Gentzen would give the optimal translation $\forall x \neg\neg P(x)$. On the other hand, for purely existential formulas $\exists x P(x)$ we have that Kuroda gives the optimal translation, whereas Gödel-Gentzen introduces unnecessary negations. The important property of all these translations, however, is that they are *modular*, i.e. except for a single non-modular step applied to the whole formula, the translation of a formula is based on the translation of its immediate sub-formulas. The following definition makes this precise.

Definition 1 (Modular negative translations). *We say that a translation $(\cdot)^{Tr}$ from CL to IL is* modular *if there are formula constructors $I^{Tr}_{\Box}(\cdot,\cdot)$ for $\Box \in \{\wedge, \vee, \to\}$, $I^{Tr}_{Q}(\cdot)$ for $Q \in \{\forall, \exists\}$, $I^{Tr}_{at}(\cdot)$ and $I^{Tr}_{\vdash}(\cdot)$ called translation of connectives, quantifiers, atomic formulas and the provability sign, respectively, such that for each formula A of CL:*

$$A^{Tr} \equiv I^{Tr}_{\vdash}(A_{Tr})$$

where $(\cdot)_{Tr}$ is defined inductively as:

$$\begin{array}{rlrl}
(A \wedge B)_{Tr} & :\equiv I^{Tr}_{\wedge}(A_{Tr}, B_{Tr}) & P_{Tr} & :\equiv I^{Tr}_{at}(P), \text{ for } P \text{ atomic} \\
(A \vee B)_{Tr} & :\equiv I^{Tr}_{\vee}(A_{Tr}, B_{Tr}) & (\forall x A)_{Tr} & :\equiv I^{Tr}_{\forall}(A_{Tr}) \\
(A \to B)_{Tr} & :\equiv I^{Tr}_{\to}(A_{Tr}, B_{Tr}) & (\exists x A)_{Tr} & :\equiv I^{Tr}_{\exists}(A_{Tr}).
\end{array}$$

A modular translation is called a negative translation *if (i) $A \leftrightarrow_{\mathsf{CL}} I^{Tr}_{\vdash}(A_{Tr})$ and (ii) $\mathsf{IL} \vdash I^{Tr}_{\vdash}(A_{Tr})$ whenever $\mathsf{CL} \vdash A$.*[2]

[2] A negative translation is usually assumed to satisfy a third condition (iii) $I^{Tr}_{\vdash}(A_{Tr}) \leftrightarrow_{\mathsf{IL}} B$ for some B constructed from doubly negated atomic formulas by means of $\forall, \wedge, \to, \bot$; ensuring that all negative translations are equivalent (see [30]).

For instance, Krivine negative translation is a modular translation with

$$\begin{array}{ll} I_{\wedge}^{Kr}(A,B) :\equiv A \vee B & I_{at}^{Kr}(P) :\equiv \neg P, \text{ for } P \text{ atomic} \\ I_{\vee}^{Kr}(A,B) :\equiv A \wedge B & I_{\forall}^{Kr}(A) :\equiv \exists x A \\ I_{\to}^{Kr}(A,B) :\equiv \neg A \wedge B & I_{\exists}^{Kr}(A) :\equiv \neg \forall x \neg A \end{array}$$

and $I_{\vdash}^{Kr}(A) :\equiv \neg A$. Similarly, one can easily see how Kolmogorov, Gödel-Gentzen, and Kuroda translations are also modular translations.

Definition 2 (Relating modular translations). *We define a relation $\sim$ between modular translations as follows: Given translations T_1 and T_2 we define $T_1 \sim T_2$ if the following equivalences are intuitionistically valid:*

$$\begin{array}{ll} I_{\Box}^{T_1}(A,B) \leftrightarrow_{\mathsf{IL}} I_{\Box}^{T_2}(A,B) & I_{at}^{T_1}(P) \leftrightarrow_{\mathsf{IL}} I_{at}^{T_2}(P) \\ I_{Q}^{T_1}(A) \quad\;\; \leftrightarrow_{\mathsf{IL}} I_{Q}^{T_2}(A) & I_{\vdash}^{T_1}(A) \leftrightarrow_{\mathsf{IL}} I_{\vdash}^{T_2}(A), \end{array}$$

for all formulas A, B, and atomic formulas P, $\Box \in \{\wedge, \vee, \to\}$ and $Q \in \{\forall, \exists\}$.

In other words, two modular translations are related via $\sim$ if the corresponding translations of connectives, quantifiers, atoms and provability are equivalent formulas in IL. It is immediate that $\sim$ is an *equivalence relation*. In what follows we say that two modular translations are the *same* if they are in the same equivalent class for the relation $\sim$ (i.e. they are the same mod $\sim$). When two translations are not the same (in the previous sense), we say they are *different*. Two different translations T_1 and T_2 from CL to IL are said to be *equivalent* if for each formula A, the two translations of A, namely A^{T_1} and A^{T_2}, are equivalent formulas in IL. For instance, changing the clause for $\exists x A$ in the Gödel-Gentzen translation to $(\exists x A)^{GG} :\equiv \neg \forall x \neg A^{GG}$ does not change the interpretation, since intuitionistically we have that $\neg \forall x \neg A$ is equivalent to $\neg\neg\exists x A$. So, these would be just two ways of writing the same translation. On the other hand, Kuroda translation is different from Gödel-Gentzen's since, for instance, we do not normally have that $\forall x A$ is equivalent to $\forall x \neg\neg A$ intuitionistically.

3 Simplifications

Noticing that Kuroda and Gödel-Gentzen negative translations could be reached (in a modular way) from Kolmogorov translation via equivalences in IL, arose the idea of looking for a general strategy covering the standard negative translations.

Thus, our goal is to show that the different negative translations are obtained via a systematic simplification of Kolmogorov translation. For that, we need the concept of "simplification" we define below. Intuitively, the idea of a simplification is to transform formulas into intuitionistically equivalent formulas with less negations *preserving the modularity of the translation.*

Definition 3 (Simplification from inside/outside). *A* simplification from inside *is a set of transformations (at most one for each connective and quantifier)* *of the following form:*

$$\neg\neg(NA \,\Box\, NB) \stackrel{r}{\Rightarrow} N(N_1 A \,\Box^r N_2 B)$$
$$\neg\neg Q x NA \quad \stackrel{r}{\Rightarrow} N(Q^r x N_1 A),$$

where $\Box, \Box^r \in \{\wedge, \vee, \rightarrow\}$*, and* $Q, Q^r \in \{\forall, \exists\}$*,* N *stands for a single or a double negation (same choice in all the set of transformations), and* N_1 *and* N_2 *are negations (possible none and not necessarily the same in all transformations) such that*

(i) both sides are equivalent formulas in IL *and*
(ii) the number of negations on right side is strictly less than on left side.

A simplification from outside *is defined in a similar way replacing the shape of the transformation before by*

$$N(\neg\neg A \,\Box\, \neg\neg B) \stackrel{r}{\Rightarrow} N_1 NA \,\Box^r N_2 NB$$
$$NQx\neg\neg A \quad \stackrel{r}{\Rightarrow} Q^r x N_1 NA.$$

Intuitively, in the first case we are moving negations N *outwards* over the outer double negation $\neg\neg$, whereas in the second case we are moving N *inwards* over the inner $\neg\neg$. The moving of negations is done so that we reduce the number of negations while keeping the modularity of the translation.

Definition 4 (Maximal simplification). *A simplification is* maximal *if*

(i) it is not properly included in any other simplification, i.e. including new transformations for other connectives prevents the new set of being a simplification, and
(ii) it is not possible to replace $\Box^r$*,* Q^r*,* N_1 *and* N_2 *so as to reduce the number of negations on the right side of any transformation.*

Intuitively, a simplification being maximal means that we can not get ride of more negations.

Proposition 1. *Let* r_1 *and* r_2 *be the set of transformations:*

$$\begin{array}{llll}
\neg\neg(\neg\neg A \wedge \neg\neg B) & \stackrel{r_1}{\Rightarrow} \neg\neg(A \wedge B) & \neg\neg(\neg A \wedge \neg B) & \stackrel{r_2}{\Rightarrow} \neg(A \vee B) \\
\neg\neg(\neg\neg A \vee \neg\neg B) & \stackrel{r_1}{\Rightarrow} \neg\neg(A \vee B) & \neg\neg(\neg A \vee \neg B) & \stackrel{r_2}{\Rightarrow} \neg(A \wedge B) \\
\neg\neg(\neg\neg A \rightarrow \neg\neg B) & \stackrel{r_1}{\Rightarrow} \neg\neg(A \rightarrow B) & \neg\neg(\neg A \rightarrow \neg B) & \stackrel{r_2}{\Rightarrow} \neg(\neg A \wedge B) \\
\neg\neg\exists x\neg\neg A & \stackrel{r_1}{\Rightarrow} \neg\neg\exists x A, & \neg\neg\forall x\neg A & \stackrel{r_2}{\Rightarrow} \neg\exists x A,
\end{array}$$

respectively. The sets r_1 *and* r_2 *are maximal simplifications from inside.*

Proposition 2. *Let* r_3 *and* r_4 *be the set of transformations:*

$$\begin{array}{llll}
\neg\neg(\neg\neg A \wedge \neg\neg B) & \stackrel{r_3}{\Rightarrow} \neg\neg A \wedge \neg\neg B & \neg(\neg\neg A \wedge \neg\neg B) & \stackrel{r_4}{\Rightarrow} \neg\neg A \rightarrow \neg B \\
\neg\neg(\neg\neg A \vee \neg\neg B) & \stackrel{r_3}{\Rightarrow} \neg\neg\neg A \rightarrow \neg\neg B & \neg(\neg\neg A \vee \neg\neg B) & \stackrel{r_4}{\Rightarrow} \neg A \wedge \neg B \\
\neg\neg(\neg\neg A \rightarrow \neg\neg B) & \stackrel{r_3}{\Rightarrow} \neg\neg A \rightarrow \neg\neg B & \neg(\neg\neg A \rightarrow \neg\neg B) & \stackrel{r_4}{\Rightarrow} \neg\neg A \wedge \neg B \\
\neg\neg\forall x\neg\neg A & \stackrel{r_3}{\Rightarrow} \forall x\neg\neg A, & \neg\exists x\neg\neg A & \stackrel{r_4}{\Rightarrow} \forall x\neg A,
\end{array}$$

respectively. The sets r_3 *and* r_4 *are maximal simplifications from outside.*

Proposition 3. *The simplifications r_1, r_2, r_3 and r_4 are the only maximal simplifications.*

4 Kolmogorov Simplified

Definition 3 identifies a class of transformations which can be applied to Kolmogorov negative translation without spoiling the modularity property of the translation. We now present standard ways of simplifying Kolmogorov translation via the maximal (or proper subsets of the maximal) simplifications introduced above.

Definition 5 (Simplification path). Applying a simplification to a formula A *consists in changing the formula through successive steps, applying in each step a transformation allowed by the simplification (i.e. transforming a subformula having the shape of the left-hand side of the transformation by the corresponding right-hand side), till no longer be possible to simplify the expression via that simplification. We call the path of formulas starting in A we obtain this way a* simplification path.

Note that every step in a simplification path acts over a particular connective or quantifier and all formulas in a simplification path are equivalent formulas in IL. The process of applying a simplification is not unique and can lead to different formulas. Nevertheless, all simplification paths are obviously finite since in each step the number of negations is decreasing. From now on, we consider that all simplification paths start with formulas in Kolmogorov form (i.e. formulas of the form A^{Ko}).

Definition 6 (Length of simplification path). *The length of a simplification path P, denoted $s(P)$, is the number of steps in P, or equivalently the number of nodes in P minus one, where by node we refer to each formula in P.*

Clearly, it is not true that two simplification paths with the same length lead to the same formula, i.e. have the same final node. For instance, consider applying simplification r_1 to the formula below in two different ways:

$$\begin{tikzcd}
& \neg\neg(\neg\neg(\neg\neg A \wedge \neg\neg B) \wedge \neg\neg\exists x \neg\neg A) \arrow[dl, dash] \arrow[dr, dash] & \\
\neg\neg(\neg\neg(\neg\neg A \wedge \neg\neg B) \wedge \neg\neg\exists x A) \arrow[d, dash] & & \neg\neg(\neg\neg(A \wedge B) \wedge \neg\neg\exists x \neg\neg A) \arrow[d, dash] \\
\neg\neg((\neg\neg A \wedge \neg\neg B) \wedge \exists x A) & & \neg\neg((A \wedge B) \wedge \exists x \neg\neg A)
\end{tikzcd}$$

Nevertheless, we prove that if a simplification is maximal or is a subset of a maximal simplification then the length of the longest paths is determined by the initial formula and, moreover, all the paths with longest length lead to the same formula. In other words, we have a kind of confluence property for longest paths. First some definitions and auxiliary results.

Notation. In order to simplify the formulation of Lemmas 2 and 3 we use the following abbreviations

- *Removing the double negations from inside* over $\Box$ or Q, with $\Box \in \{\wedge, \vee, \rightarrow\}$ and $Q \in \{\forall, \exists\}$, stands for replacing $\neg\neg(\neg\neg A \Box \neg\neg B)$ by $\neg\neg(A \Box B)$, or $\neg\neg Qx\neg\neg A$ by $\neg\neg QxA$.
- *Removing the double negation from outside* over $\Box \in \{\wedge, \rightarrow\}$ or Q consists in replacing $\neg\neg(\neg\neg A \Box \neg\neg B)$ by $\neg\neg A \Box \neg\neg B$, or replacing $\neg\neg Qx\neg\neg A$ by $Qx\neg\neg A$.
- *Removing the double negation from outside* over $\vee$ consists in replacing $\neg\neg(\neg\neg A \vee \neg\neg B)$ by $\neg\neg\neg A \rightarrow \neg\neg B$.
- *Removing single negations (from inside or outside)* over $\Box \in \{\vee, \rightarrow\}$ in the formula $\neg\neg(\neg\neg A \Box \neg\neg B)$ consists in transforming the double negations in single negations, replacing $\Box$ by $\wedge$ and in the case $\Box \equiv \rightarrow$ adding a negation before A. *Removing a single negation (from inside or outside)* over a quantifier symbol Q in the formula $\neg\neg Qx\neg\neg A$ consists in replacing the double negations by single negations and replacing Q by its dual.
- *Removing a single negation from inside (respectively outside)* over $\wedge$ in the formula $\neg\neg(\neg\neg A \wedge \neg\neg B)$ consists in replacing this formula by $\neg(\neg A \vee \neg B)$ (or replacing this formula by $\neg(\neg\neg A \rightarrow \neg B)$ respectively).

We denote by $\#^A_\Box$ and $\#^A_Q$ the number of symbols $\Box$ and Q respectively, occurring in the formula A. For the sake of counting symbols, the negation symbols $\neg$ introduced by the translations are considered as primitive, and hence do not change the value of $\#^A_\rightarrow$. For example $(\#^A_\rightarrow) = (\#^{A^{Ko}}_\rightarrow)$.

Lemma 2. *For the simplification r_1 and for any formula A^{Ko} there is a simplification path P_{r_1} from A^{Ko} such that*

$$s(P_{r_1}) = (\#^{A^{Ko}}_\wedge) + (\#^{A^{Ko}}_\vee) + (\#^{A^{Ko}}_\rightarrow) + (\#^{A^{Ko}}_\exists)$$

and the formula in the last node can be obtained from A^{Ko} locating in this formula all the occurrences of conjunctions, disjunctions, implications and existential quantifications and removing at once all the double negations from inside these connectives and quantifiers.

Any simplification r'_1 obtained from r_1 by removing one or more transformations admits a similar result discounting and disregarding the logical symbols in the left-hand side of the transformations removed.

The (omitted) proof above in fact provides an algorithm to construct a simplification path for the simplification r with $r \equiv r_1$ or $r \equiv r'_1$. The simplification path from A^{Ko} constructed this way is called *standard path for r*.

Lemma 3. *For the simplifications r_2, r_3, r_4 and for any formula A^{Ko}, there are simplification paths P_{r_2}, P_{r_3}, P_{r_4} such that*

$s(P_{r_2}) = (\#_{\wedge}^{A^{Ko}}) + (\#_{\vee}^{A^{Ko}}) + (\#_{\rightarrow}^{A^{Ko}}) + (\#_{\forall}^{A^{Ko}})$,
$s(P_{r_3}) = (\#_{\wedge}^{A^{Ko}}) + (\#_{\vee}^{A^{Ko}}) + (\#_{\rightarrow}^{A^{Ko}}) + (\#_{\forall}^{A^{Ko}})$ *and*
$s(P_{r_4}) = (\#_{\wedge}^{A^{Ko}}) + (\#_{\vee}^{A^{Ko}}) + (\#_{\rightarrow}^{A^{Ko}}) + (\#_{\exists}^{A^{Ko}})$.

Moreover, in P_{r_2} the last node can be obtained from A^{Ko} removing at once the single negations from inside all the conjunctions, disjunctions, implications and universal quantifications; the formula in the last node in P_{r_3} can be obtained from A^{Ko} by removing at once the double negations from outside the conjunctions, disjunctions, implications and universal quantifications; and the formula in the last node of P_{r_4} can be obtained from A^{Ko} by removing at once the single negations from outside the conjunctions, disjunctions, implications and existential quantifications.

The result can be adapted in the expected way to simplifications obtained from r_2, r_3 or r_4 by removing one or more transformations.

Again, the proof above provides algorithms to construct simplification paths for the simplifications r_2, r_3, r_4 and its subsets. The simplification paths from A^{Ko} constructed via these algorithms are called *standard paths*.

Lemma 4. *If the simplification is a subset of a maximal one, in each step of a simplification path we act over a connective or a quantifier already occurring in the initial formula, and we never act twice over the same connective or quantifier.*

Note that, in the previous lemma, the hypothesis of considering just subsets of maximal simplifications is essential. In the example below we present a (non maximal) simplification from inside that contradicts the lemma. Consider the simplification:

$$\neg\neg(\neg A \wedge \neg B) \Rightarrow \neg(A \vee \neg\neg B)$$
$$\neg\neg(\neg A \vee \neg B) \Rightarrow \neg(A \wedge B).$$

From $\neg\neg(\neg\neg A \wedge \neg\neg(\neg\neg B \wedge \neg\neg C))$ we can construct the following two paths:

$$\begin{tikzcd}[column sep=tiny]
& \neg\neg(\neg\neg A \wedge \neg\neg(\neg\neg B \wedge \neg\neg C)) \arrow[dl, dash] \arrow[dr, dash] & \\
\neg(\neg A \vee \neg\neg\neg(\neg\neg B \wedge \neg\neg C)) \arrow[dr, dash] & & \neg\neg(\neg\neg A \wedge \neg(\neg B \vee \neg\neg\neg C)) \arrow[dl, dash] \\
& \neg(\neg A \vee \neg\neg(\neg B \vee \neg\neg\neg C)) \arrow[d, dash] & \\
& \neg(\neg A \vee \neg(B \wedge \neg\neg C)) &
\end{tikzcd}$$

The two corollaries below are now immediate:

Corollary 1. *For each formula A^{Ko} and each simplification that is a subset of r_1, r_2, r_3 or r_4, any simplification path from A^{Ko} has length smaller or equal to the length of the corresponding standard path.*

Corollary 2. *If the simplification is a subset of a maximal one, two simplification paths with the longest length lead to the same formula.*

The result above justifies the next definition:

Definition 7. *Let r be a subset of a maximal simplification and A^{Ko} a formula in Kolmogorov form. We denote by $r(A^{Ko})$ the formula in the last node of a simplification path with longest length.*

5 Standard Translations

Simplifying the Kolmogorov negative translation via the maximal simplifications r_1 and r_2 we obtain exactly Kuroda and Krivine negative translations.

Proposition 4. $r_1(A^{Ko}) \equiv A^{Ku}$ *and* $r_2(A^{Ko}) \equiv A^{Kr}$.

This study concerning maximal simplifications led us not only to the two standard negative translations above but also to the discovery of two new minimal modular embeddings from CL to IL. Consider the translations described below:

$$\begin{array}{llll} (A \wedge B)^G & :\equiv A^G \wedge B^G & P^G & :\equiv \neg\neg P, \text{ for } P \text{ atomic} \\ (A \vee B)^G & :\equiv \neg A^G \to B^G & (\forall x A)^G & :\equiv \forall x A^G \\ (A \to B)^G & :\equiv A^G \to B^G & (\exists x A)^G & :\equiv \neg\neg\exists x A^G \end{array}$$

which is like the $(\cdot)^{GG}$-translation except for the $\vee$-clause where only one negation (rather than two) is introduced, and

$$\begin{array}{llll} (A \wedge B)_E & :\equiv \neg A_E \to B_E & P_E & :\equiv \neg P, \text{ for } P \text{ atomic} \\ (A \vee B)_E & :\equiv A_E \wedge B_E & (\forall x A)_E & :\equiv \neg\forall x \neg A_E \\ (A \to B)_E & :\equiv \neg A_E \wedge B_E & (\exists x A)_E & :\equiv \forall x A_E \end{array}$$

with $A^E :\equiv \neg A_E$, which is similar to Krivine except that negations are introduced in the $\{\wedge, \forall\}$-clauses whereas Krivine introduces negations on the $\exists$-clause.

Immediately as a corollary of the next proposition, we have that the translations $(\cdot)^G$ and $(\cdot)^E$ are embeddings from CL to IL, different but equivalent to the standard embeddings considered previously.

Proposition 5. $r_3(A^{Ko}) \equiv A^G$ *and* $r_4(A^{Ko}) \equiv A^E$.

Let r_3' be the (non-maximal) simplification we obtain from r_3 by removing the transformation $\neg\neg(\neg\neg A \vee \neg\neg B) \Rightarrow \neg\neg\neg A \to \neg\neg B$. We can easily prove that $r_3'(A^{Ko}) \equiv A^{GG}$. Thus, Gödel-Gentzen negative translation is strictly in between Kolmogorov and the $(\cdot)^G$-translation.

6 Final remarks

We conclude with a few remarks on two other negative translations, some related work and other avenues for further research.

6.1 On non-modular negative translations

Working with *modular* translations brings various benefits. For instance, we can prove properties of the translation by a simple induction on the structure of the formulas, and when applying the translation to concrete proofs this can be done in a modular fashion. On the other hand, if we allow a translation to be non-modular, we can of course construct simpler embeddings, i.e. we can simplify Kolmogorov negative translation even more, getting ride of more negations.

For example, consider the simplification r_3 followed by one more transformation $\neg\neg\exists x\neg\neg A \Rightarrow \neg\forall x\neg A$ to be applied, whenever possible, at the end of the simplification path. As such we could first simplify $\neg\neg(\neg\neg A \wedge \neg\neg\exists x\neg\neg B)$ using r_3 to the formula $\neg\neg A \wedge \neg\neg\exists x\neg\neg B$ and then apply the final simplification to obtain $\neg\neg A \wedge \neg\forall x\neg B$. Although non-modular, these kind of procedures also give rise to translations of classical into intuitionistic logic.

Avigad [2] presented a more sophisticated non-modular translation that results from a fragment of r_1, avoiding unnecessary negations. More precisely, Avigad's M-translation is defined as:

$$
\begin{array}{llll}
(A \wedge B)^M :\equiv \neg(\sim A \vee \sim B)^M & P^M & :\equiv P, \text{ for } P \text{ atomic} \\
(A \vee B)^M :\equiv A^M \vee B^M & \bar{P}^M & :\equiv \neg P \\
(\forall x A)^M :\equiv \neg(\exists x \sim A)^M & (\exists x A)^M & :\equiv \exists x A^M,
\end{array}
$$

where in classical logic we consider the negations of atomic formulas $\bar{P}$ as primitive and the formula $\sim A$ is obtained from A replacing $\wedge$, $\forall$, P respectively by $\vee$, $\exists$ and $\bar{P}$ and conversely. Avigad showed that

(1) $\vdash_{\mathsf{IL}} \neg A^M \leftrightarrow \neg A^S$
(2) If $\vdash_{\mathsf{CL}} A$ then $\vdash_{\mathsf{IL}} \neg(\sim A)^M$,

where A^S stands for any of the standard equivalent translations mentioned before such as Gödel-Gentzen, Kolmogorov, Kuroda or Krivine negative translation.

Lemma 5. $\neg(\sim A)^M \leftrightarrow_{\mathsf{IL}} \neg\neg A^M$

Although translation $(\cdot)^M$, as presented by Avigad, is not modular, notice that it can be equivalently written in a modular way as

$$
\begin{array}{llll}
(A \wedge B)^{M'} :\equiv \neg\neg A^{M'} \wedge \neg\neg B^{M'} & P^{M'} & :\equiv P, \text{ for } P \text{ atomic} \\
(A \vee B)^{M'} :\equiv A^{M'} \vee B^{M'} & \bar{P}^{M'} & :\equiv \neg P \\
(\forall x A)^{M'} :\equiv \forall x \neg\neg A^{M'} & (\exists x A)^{M'} & :\equiv \exists x A^{M'},
\end{array}
$$

since $(\forall x A)^M :\equiv \neg(\exists x \sim A)^M :\equiv \neg\exists x((\sim A)^M) \leftrightarrow_{\mathsf{IL}} \forall x \neg(\sim A)^M \overset{\text{L5}}{\leftrightarrow}_{\mathsf{IL}} \forall x \neg\neg A^M$ and

$$
\begin{aligned}
(A \wedge B)^M &:\equiv \neg(\sim A \vee \sim B)^M :\equiv \neg((\sim A)^M \vee (\sim B)^M) \\
&\leftrightarrow_{\mathsf{IL}} \neg(\sim A)^M \wedge \neg(\sim B)^M \overset{\text{L5}}{\leftrightarrow}_{\mathsf{IL}} \neg\neg A^M \wedge \neg\neg B^M.
\end{aligned}
$$

The translation $(\cdot)^{M'}$ can be obtained from Kolmogorov negative translation via a non-maximal simplification, more precisely the simplification r_1 (corresponding to Kuroda translation) without the transformation $\neg\neg(\neg\neg A \wedge \neg\neg B) \overset{r_1}{\Rightarrow} \neg\neg(A \wedge B)$.

Avigad's translation $(\cdot)^M$ is a *non-modular* simplification of $(\cdot)^{M'}$ since for universal quantifications, for conjunctions and for provability we replace $\neg\neg A^M$ by $\neg(\sim A)^M$ which, although equivalent, has possibly less negations, as we see in the (omitted) proof of Lemma 5. Moreover, as pointed by Avigad in [2], we can simplify the translation $(\cdot)^M$ even further defining $(A \wedge B)^M$ as being $A^M \wedge B^M$. The corresponding modular version in this case is exactly Kuroda negative translation.

6.2 On Gödel-Gentzen negative translation

Although nowadays it is common to name the translation $(\cdot)^{GG}$, presented in Section 1, by *Gödel-Gentzen negative translation*, a few remarks should be made at this point. The translations due to Gödel and Gentzen ([12] and [10], respectively) where introduced in the context of number theory translating an atomic formula P into P itself. Later Kleene [17] considered the translation of the pure logical part, observing that double-negating atomic formulas was necessary, since one does not have stability $\neg\neg P \to P$ in general.

Rigorously, Gentzen's original formulation instead of double negating disjunctions and existential quantifiers used the following intuitionistic equivalent definitions $(A \vee B)^{GG} :\equiv \neg(\neg A^{GG} \wedge \neg B^{GG})$ and $\exists x A^{GG} :\equiv \neg\forall x \neg A^{GG}$, since, as such, one can then work in the $\{\exists, \vee\}$-free fragment of intuitionistic logic.

Moreover, as pointed in Section 1 already, Gödel's original double-negation translation differs from Gentzen's negative translation in the way implication is treated. We can easily see, however, that Gödel's negative translation can be obtained from Kolmogorov negative translation via the non-maximal simplification consisting in r_3' without the transformation $\neg\neg(\neg\neg A \to \neg\neg B) \Rightarrow \neg\neg A \to \neg\neg B$, being, therefore, more expensive in term of negations than Gentzen's negative translation. Another non-maximal simplification, more precisely r_3' without the transformation $\neg\neg(\neg\neg A \wedge \neg\neg B) \Rightarrow \neg\neg A \wedge \neg\neg B$, leads to Aczel's $(\cdot)^N$ variant [1].

Finally, we observe that sometimes in Kolmogorov or Gödel-Gentzen negative translations, $\bot$ is transformed differently from the other atomic formulas, not into $\neg\neg\bot$ but into $\bot$ itself. This change is easily adapted to our framework, considering in the modular definition of a translation an extra operator $I_\bot^{Tr}(\bot)$ and defining $\bot_{Tr} :\equiv I_\bot^{Tr}(\bot)$. Note that the translations where $I_\bot^{Tr}(\bot) :\equiv \bot$ are the same as the ones with $I_\bot^{Tr}(\bot) :\equiv \neg\neg\bot$, since $\bot \leftrightarrow \neg\neg\bot$ in IL.

6.3 On intuitionistic versus minimal logic

More than translating CL into IL, it is well known that some negative translations produce embeddings of CL into minimal logic ML (i.e. intuitionistic logic without ex-falso-quodlibet). More precisely

$$\mathsf{CL} \vdash A \text{ iff } \mathsf{ML} \vdash A^*,$$

where $* \in \{Ko, GG\}$, for instance. But for Kuroda negative translation we just have $\mathsf{CL} \vdash A$ iff $\mathsf{IL} \vdash A^{Ku}$ (see [31]). In our framework, this appears as no surprise since the direct implication in the transformation

$$\neg\neg(\neg\neg A \to \neg\neg B) \stackrel{r_1}{\Rightarrow} \neg\neg(A \to B)$$

is valid in IL but not in ML. All the other equivalences in Lemma 1 are provable in minimal logic. We observe, however, that a small change in Kuroda negative translation produces an embedding in ML. More precisely, if we change in r_1 the clause for implication to

$$\neg\neg(\neg\neg A \to \neg\neg B) \stackrel{\tilde{r}_1}{\Rightarrow} \neg\neg(A \to \neg\neg B)$$

we obtain a non-maximal simplification (in IL) which corresponds to a modular translation $(\cdot)^{\tilde{Ku}}$ between Kolmogorov and Kuroda negative translations. Since $\neg\neg(\neg\neg A \to \neg\neg B) \leftrightarrow_{\mathsf{ML}} \neg\neg(A \to \neg\neg B)$ the simplification $\tilde{r}_1$ is maximal in ML. Therefore, the negative translation $(\cdot)^{\tilde{Ku}}$ that inserts $\neg\neg$ in (i) the beginning of the formula, (ii) after each universal quantifier, and (iii) in front of the conclusion of each implication is such that $\mathsf{CL} \vdash A$ iff $\mathsf{ML} \vdash A^{\tilde{Ku}}$.

6.4 Other related work

Strong monads. Part of the present study could have been developed in a more general context. Let T be a (logical operator having the properties of a) strong monad and consider the translation $(\cdot)^T$ that inserts T in the beginning of each subformula. Assuming that $(\mathsf{T}A)^T \leftrightarrow \mathsf{T}A^T$ what we obtain is a translation of $\mathsf{ML}+(\mathsf{T}A \to A)$ into ML. We name such embedding *Kolmogorov* T*-translation*. It can be seen that all the transformations in simplifications $\tilde{r}_1$ and r'_3 remain valid equivalences in ML when we replace $\neg\neg$ by any strong monad T. Thus, from Kolmogorov T-translation we can obtain, by means of the previous simplifications, the corresponding Kuroda (ML variant) and Gödel-Gentzen T-translations. As particular cases we have

- $\mathsf{T}A :\equiv \neg\neg A$ (recovering the standard double-negation translations),
- $\mathsf{T}B :\equiv (B \to A) \to A$ (corresponding to Friedman A-translations [7]),
- $\mathsf{T}A :\equiv \neg A \to A$ or $\mathsf{T}A :\equiv (A \to R) \to A$ (Peirce translations [6]).

As references on these more general embeddings see [1, 6].

Semantical approaches. In this paper we did not discuss semantical approaches to the negative translations. Some considerations concerning conversions between Heyting and Boolean algebras whose valuation of formulas is related via negative translations can be found in [13, 25] and a more abstract treatment of negative translations in terms of categorical logic can be found in [15].

CPS **transformations**. There is a close connection between negative translations and *continuation passing style* (CPS) transformations. In the literature [8, 14, 29], we can find various CPS-translations from $\lambda\mu$-calculus into λ-calculus that correspond (at the type level) to the standard negative translations. Since the CPS technique captures evaluation ordering for the source language (such as call-by-name, call-by-value, call-by-need) it would be interesting to see if our simplifications linking the standard negative translations can be expressed and are meaningful at the calculus reduction strategy level. See also Chapters 9 and 10 in [24].

Linear logic. Although not addressed in this paper, the refined framework of linear logic with its exponentials can be useful in the study of the negative translations. It would be interesting to analyse our simplifications through the refined lens of Linear Logic. For related references see [11, 5, 22].

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