Proof Theory Seminar
Assignment 4

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Homework courtesy of Derek Dreyer

Out: June 4, 2012
Due: June 18, 2012

In Derek Dreyer's lectures we have seen a proof of strong normalization for the simply typed λ-calculus:

$$
\begin{array}{llcl}
\text{Types} & \phi & ::= & b \mid \phi_1 \to \phi_2 \\
\text{Terms} & M & ::= & x \mid \lambda x.M_1 \mid M_1\ M_2 \\
\text{Elim. Context} & \mathcal{E} & ::= & \bullet \mid \mathcal{E}\,M \\
\text{Reduction} & & & \dfrac{M \hookrightarrow M'}{\lambda x.M \hookrightarrow \lambda x.M'} \qquad \dfrac{M_1 \hookrightarrow M_1'}{M_1 M_2 \hookrightarrow M_1' M_2} \\
 & & & \dfrac{M_2 \hookrightarrow M_2'}{M_1 M_2 \hookrightarrow M_1 M_2'} \qquad \dfrac{}{(\lambda x.M)\ N \ \hookrightarrow\ M[N/x]}
\end{array}
$$

In this assignment, you will have to extend the proofs to include conjuctions, i.e., *pairs*:

$$
\begin{array}{llcl}
\text{Types} & \phi & ::= & \ldots \mid \phi_1 \times \phi_2 \\
\text{Terms} & M & ::= & \ldots \mid \langle M_1, M_2 \rangle \mid \pi_1 M \mid \pi_2 M \\
\text{Elim. Context} & \mathcal{E} & ::= & \ldots \mid \pi_i\,\mathcal{E} \\
\text{Reduction} & & & \dfrac{M_1 \hookrightarrow M_1'}{\langle M_1, M_2 \rangle \hookrightarrow \langle M_1', M_2 \rangle} \qquad \dfrac{M_2 \hookrightarrow M_2'}{\langle M_1, M_2 \rangle \hookrightarrow \langle M_1, M_2' \rangle} \\
 & & & \dfrac{M \hookrightarrow M'}{\pi_i M \hookrightarrow \pi_i M'} \qquad \dfrac{}{\pi_i \langle M_1, M_2 \rangle \ \hookrightarrow\ M_i}
\end{array}
$$

More precisely, you have to:

1. Define the logical relation case $L[\phi_1 \times \phi_2]$
2. Prove the new case of Lemma 1:

$$L[\phi] \subseteq \mathrm{SN}$$

3. Prove the new case of Lemma 3:

$$\mathcal{E}[x] \in \mathrm{SN} \implies \mathcal{E}[x] \in L[\phi]$$

4. Prove the new case of Lemma 2:

$$\mathcal{E}[M[N/x]] \in L[\phi] \text{ and } N \in \mathrm{SN} \implies \mathcal{E}[(\lambda x.M)\ N] \in L[\phi]$$

5. State and prove a new lemma (call it Lemma 2a), which is like Lemma 2 but where the β-expansion occurring at the head of the elimination context $\mathcal{E}$ concerns a *pair* $\langle M_1, M_2 \rangle$ instead of a function $\lambda x.M$. This is an entirely new lemma, which you will need in one of the new cases of the Fundamental Theorem (step 6 below). You will have to prove it in its entirety (all cases, not just for pair types) from scratch, but the proof should be relatively similar to the one for Lemma 2.

6. Prove the new cases of the Fundamental Theorem of Logical Relations:

$$\Gamma \vdash M : \phi \wedge \gamma \in L[\Gamma] \implies \gamma M \in L[\phi]$$