Proof Theory Seminar
Assignment 1

Instructor: Deepak Garg

Out: April 19, 2012
Due: April 30, 2012

Please submit your solutions on the due date, either in written form on paper during class or typeset in LaTeX through email *before* class. For help with LaTeX macros, contact the instructor.

Problem 1: Proofs

Give natural deduction proofs for the following hypothetical judgments in intuitionistic logic. If the judgment does not have a natural deduction proof, just say "No proof". The notation $\vdash \varphi$ means $\cdot \vdash \varphi\ \mathsf{true}$, where $\cdot$ is the empty set, and $\neg\varphi$ denotes $\varphi \supset \bot$.

1. $\vdash (\varphi_1 \supset (\varphi_2 \supset \varphi_3)) \supset ((\varphi_1 \supset \varphi_2) \supset (\varphi_1 \supset \varphi_3))$
2. $\vdash ((\varphi_1 \supset \varphi_2) \vee (\varphi_1 \supset \varphi_3)) \supset (\varphi_1 \supset (\varphi_2 \vee \varphi_3))$
3. $\vdash (\varphi_1 \supset (\varphi_2 \vee \varphi_3)) \supset ((\varphi_1 \supset \varphi_2) \vee (\varphi_1 \supset \varphi_3))$
4. $\vdash \varphi \vee \neg\varphi$
5. $\vdash (\top \supset \varphi) \supset \varphi$

Problem 2: Substitution

Recall from class the following metatheoretic structural property of natural deduction.

- (Substitution) If $\mathcal{D} :: \Gamma \vdash \varphi\ \mathsf{true}$ and $\mathcal{E} :: \Gamma, \varphi\ \mathsf{true} \vdash \psi\ \mathsf{true}$, then there exists $\mathcal{F} :: \Gamma \vdash \psi\ \mathsf{true}$.

The proof of this property is by induction on the given derivation $\mathcal{E}$. In class we saw some of the inductive cases of this proof. Now, write down the inductive cases where $\mathcal{E}$ ends in the rules ($\bot$E) and ($\vee$E).

Problem 3: Alternative formulation of substitution

The substitution property can also formalized as the following stronger statement.

- (Substitution') If $\Gamma' \subseteq \Gamma$, $\mathcal{D} :: \Gamma' \vdash \varphi\ \mathsf{true}$ and $\mathcal{E} :: \Gamma, \varphi\ \mathsf{true} \vdash \psi\ \mathsf{true}$, then there exists $\mathcal{F} :: \Gamma \vdash \psi\ \mathsf{true}$.

This property can also be proved by induction on the derivation of $\mathcal{E}$. Show the cases of this proof corresponding to the rules (hyp), ($\wedge$I), and ($\supset$I).

Hint: With this formulation of substitution, you will have to use the weakening theorem in the case of (hyp). It will not appear in the case of ($\supset$I), as it did for the earlier formulation.

Problem 4: Local Contraction and Expansion

Consider an (incorrect) natural deduction system for intuitionistic logic in which the introduction rule for implication ($\supset$I) is *replaced* by the following new rule:

$$\frac{\Gamma \vdash \varphi_2 \ \mathsf{true}}{\Gamma \vdash \varphi_1 \supset \varphi_2 \ \mathsf{true}} \supset\text{I'}$$

1. State which of the following two properties breaks for the implication connective: local soundness or local completeness. Explain why this is the case and show why the other property (which does not break) still holds.

2. What new elimination rule ($\supset$E') could be used to replace ($\supset$E) so as to restore both local soundness and local completeness for implication? (Hint: Remember the golden rule – only information that goes into the premises of the introduction rule can be produced in the conclusion of the elimination rule.)

Problem 5: Admissible Properties

This problem was removed from the homework.

Problem 6: Classical Logic

We discussed in class that classical logic is obtained by adding any *one* of the following two rules to intuitionistic logic.

$$\frac{}{\Gamma \vdash \varphi \vee \neg\varphi}\text{EM} \qquad\qquad \frac{\Gamma \vdash \neg\neg\varphi}{\Gamma \vdash \varphi}\text{DNE}$$

We said in class that these rules are equivalent, in the sense that each can be derived from the other within natural deduction for intuitionistic logic. We also showed in class how to derive (DNE) from (EM). This problem asks you to do the converse: Show a derivation of rule (EM) in natural deduction extended with the rule (DNE).

Hint: First prove the following theorem in natural deduction: $\Gamma \vdash \neg\neg(\varphi \vee \neg\varphi)$.