

A unifying type-theory for higher-order (amortized) cost analysis (Technical Appendix)

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1 Development for dlPCF's embedding

1.1 Syntax

Expressions	$e ::= v \mid e_1 e_2 \mid \langle\langle e_1, e_2 \rangle\rangle \mid \text{let}\langle\langle x, y \rangle\rangle = e_1 \text{ in } e_2 \mid \langle e, e \rangle \mid \text{fst}(e) \mid \text{snd}(e) \mid \text{inl}(e) \mid \text{inr}(e) \mid \text{case } e \text{ of } e; e \mid \text{let! } x = e_1 \text{ in } e_2 \mid e \square \mid e :: e \mid e; x.e$
Values	$v ::= x \mid c \mid \lambda x.e \mid \langle\langle v_1, v_2 \rangle\rangle \mid \langle v, v \rangle \mid \text{inl}(e) \mid \text{inr}(e) \mid \text{nil} \mid !e \mid \Lambda.e \mid \text{ret } e \mid \text{bind } x = e_1 \text{ in } e_2 \mid \uparrow^I \mid \text{release } x = e_1 \text{ in } e_2 \mid \text{store } e$ (No value forms for $[I] \tau$)
Index	$I ::= N \mid i \mid I + I \mid I - I \mid \sum_{i < I} I \mid \bigoplus_a^{I, I} I \mid \lambda_{si}. I \mid I I$
Sort	$S ::= \mathbb{N} \mid \mathbb{R}^+ \mid S \rightarrow S$
Kind	$K ::= \text{Type} \mid S \rightarrow K$
Types	$\tau ::= \mathbf{1} \mid \mathbf{b} \mid \tau_1 \multimap \tau_2 \mid \tau_1 \otimes \tau_2 \mid \tau_1 \& \tau_2 \mid \tau_1 \oplus \tau_2 \mid !_{a < I} \tau \mid [I] \tau \mid \mathbb{M} I \tau \mid \alpha \mid \forall \alpha : K . \tau \mid \forall i : S . \tau \mid \lambda_{ti}. \tau \mid \tau I \mid L^I \tau \mid \exists i : S . \tau \mid c \Rightarrow \tau \mid c \& \tau$
Constraints	$c ::= I = I \mid I < I \mid c \wedge c$
Lin. context for term variables	$\Gamma ::= . \mid \Gamma, x : \tau$
Bounded Lin. context for term variables	$\Omega ::= . \mid \Omega, x :_{a < I} \tau$
Unres. context for sort variables	$\Theta ::= . \mid \Theta, i : S$
Unres. context for type variables	$\Psi ::= . \mid \Psi, \alpha : K$

Definition 1 (Bounded sum of context for dlPCF). $\sum_{a < I} . = .$

$$\sum_{a < I} \Gamma, x : [b < J] \tau = (\sum_{a < I} \Gamma), x : [c < \sum_{a < I} J] \sigma$$

where

$$\tau = \sigma[(\sum_{d < a} J[d/a] + b)/c]$$

Definition 2 (Bounded sum of multiplicity context). $\sum_{a < I} . = .$

$$\sum_{a < I} \Omega, x :_{b < J} \tau = (\sum_{a < I} \Omega), x :_{c < \sum_{a < I} J} \sigma$$

where

$$\tau = \sigma[(\sum_{d < a} J[d/a] + b)/c]$$

Definition 3 (Binary sum of context for dlPCF).

$$\Gamma_1 \oplus \Gamma_2 \triangleq \begin{cases} \Gamma_2 & \Gamma_1 = . \\ (\Gamma'_1 \oplus \Gamma_2/x), x : [c < I + J] \tau & \Gamma_1 = \Gamma'_1, x : [a < I] \tau[a/c] \wedge (x : [b < J] \tau[I + b/c]) \in \Gamma_2 \\ (\Gamma'_1 \oplus \Gamma_2), x :_{a < I} \tau & \Gamma_1 = \Gamma'_1, x : [a < I] \tau \wedge (x : [-] -) \notin \Gamma_2 \end{cases}$$

Definition 4 (Binary sum of multiplicity context).

$$\Omega_1 \oplus \Omega_2 \triangleq \begin{cases} \Omega_2 & \Omega_1 = . \\ (\Omega'_1 \oplus \Omega_2/x), x :_{c < I + J} \tau & \Omega_1 = \Omega'_1, x :_{a < I} \tau[a/c] \wedge (x :_{b < J} \tau[I + b/c]) \in \Omega_2 \\ (\Omega'_1 \oplus \Omega_2), x :_{a < I} \tau & \Omega_1 = \Omega'_1, x :_{a < I} \tau \wedge (x :_{-} -) \notin \Omega_2 \end{cases}$$

Definition 5 (Binary sum of affine context).

$$\Gamma_1 \oplus \Gamma_2 \triangleq \begin{cases} \Gamma_2 & \Gamma_1 = . \\ (\Gamma'_1 \oplus \Gamma_2), x : \tau & \Gamma_1 = \Gamma'_1, x : \tau \wedge (x : -) \notin \Gamma_2 \end{cases}$$

1.2 Typesystem

Typing $\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau$

$$\begin{array}{c}
\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma, x : \tau \vdash x : \tau} \text{T-var1} \qquad \frac{\Theta, \Delta \models I \geq 1}{\Psi; \Theta; \Delta; \Omega, x :_{a < I} \tau; \Gamma \vdash x : \tau[0/a]} \text{T-var2} \\
\\
\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash () : \mathbf{1}} \text{T-unit} \qquad \frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash c : \mathbf{b}} \text{T-base} \\
\\
\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{nil} : L^0 \tau} \text{T-nil} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e_1 : \tau \quad \Psi; \Theta; \Delta; \Omega_2; \Gamma_2 \vdash e_2 : L^n \tau \quad \Theta; \Delta \vdash n : \mathbb{N}}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash e_1 :: e_2 : L^{n+1} \tau} \text{T-cons} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e : L^n \tau \quad \Psi; \Theta; \Delta, n = 0; \Omega_2; \Gamma_2 \vdash e_1 : \tau' \quad \Psi; \Theta; \Delta, n > 0; \Omega_2; \Gamma_2, h : \tau, t : L^{n-1} \tau \vdash e_2 : \tau'}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash \text{match } e \text{ with } | \text{nil} \mapsto e_1 \mid h :: t \mapsto e_2 : \tau'} \text{T-match} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau[n/s] \quad \Theta; \Delta \vdash n : S}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \exists s : S . \tau} \text{T-existI} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : \exists s : S . \tau \quad \Psi; \Theta, s : S; \Delta; \Omega; \Gamma_2, x : \tau \vdash e' : \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash e; x.e' : \tau'} \text{T-existE} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma, x : \tau_1 \vdash e : \tau_2}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \lambda x. e : (\tau_1 \multimap \tau_2)} \text{T-lam} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e_1 : (\tau_1 \multimap \tau_2) \quad \Psi; \Theta; \Delta; \Omega_2; \Gamma_2 \vdash e_2 : \tau_1}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash e_1 e_2 : \tau_2} \text{T-app} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \quad \Psi; \Theta; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau'} \text{T-sub} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e_1 : \tau_1 \quad \Psi; \Theta; \Delta; \Omega_2; \Gamma_2 \vdash e_2 : \tau_1}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash \langle\langle e_1, e_2 \rangle\rangle : (\tau_1 \otimes \tau_2)} \text{T-tensorI} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e : (\tau_1 \otimes \tau_2) \quad \Psi; \Theta; \Delta; \Omega_2; \Gamma_2, x : \tau_1, y : \tau_2 \vdash e' : \tau}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash \text{let}\langle\langle x, y \rangle\rangle = e \text{ in } e' : \tau} \text{T-tensorE} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e_1 : \tau_1 \quad \Psi; \Theta; \Delta; \Omega; \Gamma \vdash e_2 : \tau_2}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \langle e_1, e_2 \rangle : (\tau_1 \& \tau_2)} \text{T-withI} \qquad \frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\tau_1 \& \tau_2)}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{fst}(e) : \tau_1} \text{T-fst} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\tau_1 \& \tau_2)}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{snd}(e) : \tau_2} \text{T-snd}
\end{array}$$

$$\begin{array}{c}
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau_1}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{inl}(e) : \tau_1 \oplus \tau_2} \text{T-inl} \qquad \frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau_2}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{inr}(e) : \tau_1 \oplus \tau_2} \text{T-inr} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e : (\tau_1 \oplus \tau_2) \quad \Psi; \Theta; \Delta; \Omega_2; \Gamma_2, x : \tau_1 \vdash e_1 : \tau \quad \Psi; \Theta; \Delta; \Omega_2; \Gamma_2, y : \tau_2 \vdash e_2 : \tau}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash \text{case } e \text{ of } e_1; e_2 : \tau} \text{T-case} \\
\\
\frac{\Psi; \Theta, a; \Delta, a < I; \Omega; . \vdash e : \tau}{\Psi; \Theta; \Delta; \sum_{a < I} \Omega; . \vdash !e : !_{a < I} \tau} \text{T-subExpI} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e : (!_{a < I} \tau) \quad \Psi; \Theta; \Delta; \Omega_2, x :_{a < I} \tau; \Gamma_2 \vdash e' : \tau'}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash \text{let } !x = e \text{ in } e' : \tau'} \text{T-subExpE} \\
\\
\frac{\Psi, \alpha : K; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \Lambda.e : (\forall \alpha : K . \tau)} \text{T-tabs} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\forall \alpha : K . \tau) \quad \Psi; \Theta; \Delta \vdash \tau' : K}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e [] : (\tau[\tau'/\alpha])} \text{T-tapp} \\
\\
\frac{\Psi; \Theta, i : S; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \Lambda.e : (\forall i : S . \tau)} \text{T-iabs} \qquad \frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\forall i : S . \tau) \quad \Theta; \Delta \vdash I : S}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e [] : (\tau[I/i])} \text{T-iapp} \\
\\
\frac{\Psi; \Theta, b; \Delta, b < L; \Omega, x :_{a < I} \tau[(b+1 + \binom{b+1, a}{b} I)/b]; . \vdash e : \tau \quad L \geq \binom{0, 1}{b} I}{\Psi; \Theta; \Delta; \sum_{b < L} \Omega; . \vdash \text{fix } x.e : \tau[0/b]} \text{T-fix} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \quad \Psi; \Theta; \Delta \models \Gamma' \sqsubseteq \Gamma \quad \Psi; \Theta; \Delta \models \Omega' \sqsubseteq \Omega}{\Psi; \Theta; \Delta; \Omega'; \Gamma' \vdash e : \tau} \text{T-weaken} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{ret } e : \mathbb{M} 0 \tau} \text{T-ret} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e_1 : \mathbb{M} I_1 \tau_1 \quad \Psi; \Theta; \Delta; \Omega_2; \Gamma_2, x : \tau_1 \vdash e_2 : \mathbb{M} I_2 \tau_2 \quad \Theta; \Delta \vdash I_1 : \mathbb{R}^+ \quad \Theta; \Delta \vdash I_2 : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash \text{bind } x = e_1 \text{ in } e_2 : \mathbb{M}(I_1 + I_2) \tau_2} \text{T-bind} \\
\\
\frac{\Theta; \Delta \vdash I : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \uparrow^I : \mathbb{M} I \mathbf{1}} \text{T-tick} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e_1 : [I_1] \tau_1 \quad \Psi; \Theta; \Delta; \Omega_2; \Gamma_2, x : \tau_1 \vdash e_2 : \mathbb{M}(I_1 + I_2) \tau_2 \quad \Theta; \Delta \vdash I_1 : \mathbb{R}^+ \quad \Theta; \Delta \vdash I_2 : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash \text{release } x = e_1 \text{ in } e_2 : \mathbb{M} I_2 \tau_2} \text{T-release}
\end{array}$$

$$\begin{array}{c}
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \quad \Theta; \Delta \vdash I : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{store } e : \mathbb{M} I ([I] \tau)} \text{T-store} \qquad \frac{\Psi; \Theta; \Delta, c; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \Lambda. e : (c \Rightarrow \tau)} \text{T-CI} \\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (c \Rightarrow \tau) \quad \Theta; \Delta \models c}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e [] : \tau} \text{T-CE} \qquad \frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \quad \Theta; \Delta \models c}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (c \& \tau)} \text{T-CAndI} \\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (c \& \tau) \quad \Psi; \Theta; \Delta, c; \Omega; \Gamma, x : \tau \vdash e' : \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{clet } x = e \text{ in } e' : \tau'} \text{T-CAndE}
\end{array}$$

Figure 1: Typing rules for λ -Amor

$$\begin{array}{c}
\frac{}{\Psi; \Theta; \Delta \vdash \tau <: \tau} \text{sub-refl} \qquad \frac{\Psi; \Theta; \Delta \vdash \tau'_1 <: \tau_1 \quad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau'_2}{\Psi; \Theta; \Delta \vdash \tau_1 \multimap \tau_2 <: \tau'_1 \multimap \tau'_2} \text{sub-arrow} \\
\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau'_1 \quad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau'_2}{\Psi; \Theta; \Delta \vdash \tau_1 \otimes \tau_2 <: \tau'_1 \otimes \tau'_2} \text{sub-tensor} \\
\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau'_1 \quad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau'_2}{\Psi; \Theta; \Delta \vdash \tau_1 \& \tau_2 <: \tau'_1 \& \tau'_2} \text{sub-with} \\
\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau'_1 \quad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau'_2}{\Psi; \Theta; \Delta \vdash \tau_1 \oplus \tau_2 <: \tau'_1 \oplus \tau'_2} \text{sub-sum} \\
\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau' \quad \Theta; \Delta \models n' \leq n}{\Psi; \Theta; \Delta \vdash [n] \tau <: [n'] \tau'} \text{sub-potential} \\
\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau' \quad \Theta; \Delta \models n \leq n'}{\Psi; \Theta; \Delta \vdash \mathbb{M} n \tau <: \mathbb{M} n' \tau'} \text{sub-monad} \\
\frac{\Psi; \Theta, a; \Delta, a < J \vdash \tau <: \tau' \quad \Theta, a; \Delta \models J \leq I}{\Psi; \Theta; \Delta \vdash !_{a < J} \tau <: !_{a < J} \tau'} \text{sub-subExp} \\
\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta \vdash L^n \tau <: L^n \tau'} \text{sub-list} \qquad \frac{\Psi; \Theta, s; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta \vdash \exists s. \tau <: \exists s. \tau'} \text{sub-exist} \\
\frac{\Psi, \alpha : K; \Psi; \Theta; \Delta \vdash \tau_1 <: \tau_2}{\Psi; \Theta; \Delta \vdash \forall \alpha : K. \tau_1 <: \forall \alpha. \tau_2} \text{sub-typePoly} \qquad \frac{\Psi; \Theta, i : S; \Delta \vdash \tau_1 <: \tau_2}{\Psi; \Theta; \Delta \vdash \forall i : S. \tau_1 <: \forall i. \tau_2} \text{sub-indexPoly}
\end{array}$$

$$\begin{array}{c}
\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau_2 \quad \Theta; \Delta \models c_2 \implies c_1}{\Psi; \Theta; \Delta \vdash c_1 \Rightarrow \tau_1 <: c_2 \Rightarrow \tau_2} \text{sub-constraint} \\
\\
\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau_2 \quad \Theta; \Delta \models c_1 \implies c_2}{\Psi; \Theta; \Delta \vdash c_1 \& \tau_1 <: c_2 \& \tau_2} \text{sub-CAnd} \\
\\
\frac{\Theta; \Delta \vdash k : \mathbb{R}^+ \quad \Theta; \Delta \vdash k' : \mathbb{R}^+}{\Psi; \Theta; \Delta \vdash [k](\tau_1 \multimap \tau_2) <: ([k'] \tau_1 \multimap [k' + k] \tau_2)} \text{sub-potArrow} \\
\\
\frac{}{\Psi; \Theta; \Delta \vdash \tau <: [0] \tau} \text{sub-potZero} \quad \frac{\Psi; \Theta, i : S; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta \vdash \lambda_{s i} : S . \tau <: \lambda_{t i} : S . \tau'} \text{sub-familyAbs} \\
\\
\frac{\Theta \vdash I : S}{\Psi; \Theta; \Delta \vdash (\lambda_{t i} : S . \tau) I <: \tau[I/i]} \text{sub-familyApp1} \\
\\
\frac{\Theta \vdash I : S}{\Psi; \Theta; \Delta \vdash \tau[I/i] <: (\lambda_{t i} : S . \tau) I} \text{sub-familyApp2} \\
\\
\frac{}{\Psi; \Theta; \Delta \vdash [\sum_{a < I} K] !_{a < I} \tau <: !_{a < I} [K] \tau} \text{sub-bSum}
\end{array}$$

Figure 2: Subtyping

$$\begin{array}{c}
\frac{}{\Psi; \Theta; \Delta \vdash \Gamma \sqsubseteq .} \text{dlpcf-subBase} \\
\\
\frac{x : [a < J] \tau' \in \Gamma_1 \quad \Psi; \Theta, a; \Delta, a < I \vdash \tau' \sqsubseteq \tau \quad \Psi; \Theta; \Delta \vdash I \leq J \quad \Psi; \Theta; \Delta \vdash \Gamma_1/x \sqsubseteq \Gamma_2}{\Theta; \Delta \vdash \Gamma_1 \sqsubseteq \Gamma_2, x : [a < I] \tau} \text{dlpcf-subInd}
\end{array}$$

Figure 3: Γ Subtyping for dlPCF

$$\begin{array}{c}
\frac{}{\Psi; \Theta; \Delta \vdash \Omega \sqsubseteq .} \text{sub-mBase} \\
\\
\frac{x :_{a < J} \tau' \in \Omega_1 \quad \Psi; \Theta, a; \Delta, a < I \vdash \tau' <: \tau \quad \Theta; \Delta \vdash I \leq J \quad \Psi; \Theta; \Delta \vdash \Omega_1/x \sqsubseteq \Omega_2}{\Psi; \Theta; \Delta \vdash \Omega_1 \sqsubseteq \Omega_2, x :_{a < I} \tau} \text{sub-mInd}
\end{array}$$

Figure 4: Ω Subtyping

$$\frac{}{\Psi; \Theta; \Delta \vdash \Gamma \sqsubseteq .} \text{ sub-lBase}$$

$$\frac{x : \tau' \in \Gamma_1 \quad \Psi; \Theta; \Delta \vdash \tau' <: \tau \quad \Psi; \Theta; \Delta \vdash \Gamma_1/x \sqsubseteq \Gamma_2}{\Psi; \Theta; \Delta \vdash \Gamma_1 \sqsubseteq \Gamma_2, x : \tau} \text{ sub-lBase}$$

Figure 5: Γ Subtyping

$$\frac{}{\Theta, i : S; \Delta \vdash i : S} \text{ S-var} \quad \frac{}{\Theta; \Delta \vdash N : \mathbb{N}} \text{ S-nat} \quad \frac{}{\Theta; \Delta \vdash R : \mathbb{R}^+} \text{ S-real} \quad \frac{\Theta; \Delta \vdash i : \mathbb{N}}{\Theta \vdash i : \mathbb{R}^+} \text{ S-real1}$$

$$\frac{\Theta; \Delta \vdash I_1 : \mathbb{N} \quad \Theta; \Delta \vdash I_2 : \mathbb{N}}{\Theta; \Delta \vdash I_1 + I_2 : \mathbb{N}} \text{ S-add-Nat} \quad \frac{\Theta; \Delta \vdash I_1 : \mathbb{R}^+ \quad \Theta; \Delta \vdash I_2 : \mathbb{R}^+}{\Theta; \Delta \vdash I_1 + I_2 : \mathbb{R}^+} \text{ S-add-Real}$$

$$\frac{\Theta; \Delta \vdash I_1 : \mathbb{N} \quad \Theta; \Delta \vdash I_2 : \mathbb{N} \quad \Theta; \Delta \models I_1 \geq I_2}{\Theta; \Delta \vdash I_1 - I_2 : \mathbb{N}} \text{ S-minus-Nat}$$

$$\frac{\Theta; \Delta \vdash I_1 : \mathbb{R}^+ \quad \Theta; \Delta \vdash I_2 : \mathbb{R}^+ \quad \Theta; \Delta \models I_1 \geq I_2}{\Theta; \Delta \vdash I_1 - I_2 : \mathbb{R}^+} \text{ S-minus-Real}$$

$$\frac{\Theta; \Delta \vdash I_1 : \mathbb{N} \quad \Theta; \Delta, a : \mathbb{N} \vdash I_2 : \mathbb{N}}{\Theta; \Delta \vdash \sum_{a < I_1} I_2 : \mathbb{N}} \text{ S-bSum-Nat}$$

$$\frac{\Theta; \Delta \vdash I_1 : \mathbb{N} \quad \Theta; \Delta, a : \mathbb{N} \vdash I_2 : \mathbb{R}^+}{\Theta; \Delta \vdash \sum_{a < I_1} I_2 : \mathbb{R}^+} \text{ S-bSum-Real}$$

$$\frac{\Theta; \Delta \vdash I_1 : \mathbb{N} \quad \Theta; \Delta \vdash I_2 : \mathbb{N} \quad \Theta; \Delta, a : \mathbb{N} \vdash I_3 : \mathbb{N}}{\Theta \vdash \bigotimes_{a}^{I_1, I_2} I_3 : \mathbb{N}} \text{ S-forest}$$

$$\frac{\Theta, i : S; \Delta \vdash I : S'}{\Theta; \Delta \vdash \lambda_s i. I : S \rightarrow S'} \text{ S-family}$$

Figure 6: Typing rules for sorts

$$\begin{array}{c}
\frac{}{\Psi; \Theta; \Delta \vdash \mathbf{1} : Type} \text{K-unit} \qquad \frac{}{\Psi; \Theta; \Delta \vdash \mathbf{b} : Type} \text{K-base} \\
\\
\frac{\Psi; \Theta; \Delta \vdash \tau_1 : K \quad \Psi; \Theta; \Delta \vdash \tau_2 : K}{\Psi; \Theta; \Delta \vdash \tau_1 \multimap \tau_2 : K} \text{K-arrow} \\
\\
\frac{\Psi; \Theta; \Delta \vdash \tau_1 : K \quad \Psi; \Theta; \Delta \vdash \tau_2 : K}{\Psi; \Theta; \Delta \vdash \tau_1 \otimes \tau_2 : K} \text{K-tensor} \\
\\
\frac{\Psi; \Theta; \Delta \vdash \tau_1 : K \quad \Psi; \Theta; \Delta \vdash \tau_2 : K}{\Psi; \Theta; \Delta \vdash \tau_1 \& \tau_2 : K} \text{K-with} \qquad \frac{\Psi; \Theta; \Delta \vdash \tau_1 : K \quad \Psi; \Theta; \Delta \vdash \tau_2 : K}{\Psi; \Theta; \Delta \vdash \tau_1 \oplus \tau_2 : K} \text{K-or} \\
\\
\frac{\Psi; \Theta, a : S; \Delta, a < I \vdash \tau : K \quad \Theta \vdash I : \mathbb{N}}{\Psi; \Theta; \Delta \vdash !_{a < I} \tau : K} \text{K-subExp} \\
\\
\frac{\Psi; \Theta; \Delta \vdash \tau : K \quad \Theta; \Delta \vdash I : \mathbb{R}^+}{\Psi; \Theta; \Delta \vdash [I] \tau : K} \text{K-lab} \qquad \frac{\Psi; \Theta; \Delta \vdash \tau : K \quad \Theta; \Delta \vdash I : \mathbb{R}^+}{\Psi; \Theta; \Delta \vdash \mathbb{M} I \tau : K} \text{K-monad} \\
\\
\frac{\Psi, \alpha : K'; \Theta; \Delta \vdash \tau : K}{\Psi; \Theta; \Delta \vdash \forall \alpha. \tau : K} \text{K-tabs} \qquad \frac{\Psi; \Theta, i : S; \Delta \vdash \tau : K}{\Psi; \Theta; \Delta \vdash \forall i. \tau : K} \text{K-iabs} \\
\\
\frac{\Psi; \Theta; \Delta, c \vdash \tau : K}{\Psi; \Theta; \Delta \vdash c \Rightarrow \tau : K} \text{K-constraint} \qquad \frac{\Psi; \Theta; \Delta \vdash \tau : K \quad \Theta; \Delta \models c}{\Psi; \Theta; \Delta \vdash c \& \tau : K} \text{K-consAnd} \\
\\
\frac{\Psi; \Theta, i : S; \Delta \vdash \tau : K}{\Psi; \Theta; \Delta \vdash \lambda_t i. \tau : S \rightarrow K} \text{K-family} \qquad \frac{\Psi; \Theta; \Delta \vdash \tau : S \rightarrow K \quad \Theta; \Delta \vdash I : S}{\Psi; \Theta; \Delta \vdash \tau \ I : K} \text{K-iapp}
\end{array}$$

Figure 7: Kind rules for types

1.3 Semantics

Pure reduction, $e \Downarrow_t v$ Forcing reduction, $e \Downarrow_t^c v$

$$\begin{array}{c}
\frac{e_1 \Downarrow_{t_1} v \quad e_2 \Downarrow_{t_2} l}{e_1 :: e_2 \Downarrow_{t_1+t_2+1} v :: l} \text{E-cons} \qquad \frac{e_1 \Downarrow_{t_1} \text{nil} \quad e_2 \Downarrow_{t_2} v}{\text{match } e_1 \text{ with } | \text{nil} \mapsto e_2 | h :: t \mapsto e_3 \Downarrow_{t_1+t_2+1} v} \text{E-matchNil} \\
\\
\frac{e_1 \Downarrow_{t_1} v_h :: l \quad e_3[v_h/h][l/t] \Downarrow_{t_2} v}{\text{match } e_1 \text{ with } | \text{nil} \mapsto e_2 | h :: t \mapsto e_3 \Downarrow_{t_1+t_2+1} v} \text{E-matchCons} \\
\\
\frac{e_1 \Downarrow_{t_1} v \quad e_2[v/x] \Downarrow_{t_2} v'}{e_1; x.e_2 \Downarrow_{t_1+t_2+1} v'} \text{E-exist} \qquad \frac{e_1 \Downarrow_{t_1} \lambda x.e' \quad e'[e_2/x] \Downarrow_{t_2} v'}{e_1 e_2 \Downarrow_{t_1+t_2+1} v'} \text{E-app} \\
\\
\frac{e_1 \Downarrow_{t_1} v_1 \quad e_2 \Downarrow_{t_2} v_2}{\langle\langle e_1, e_2 \rangle\rangle \Downarrow_{t_1+t_2+1} \langle\langle v_1, v_2 \rangle\rangle} \text{E-TI} \qquad \frac{e \Downarrow_{t_1} \langle\langle v_1, v_2 \rangle\rangle \quad e'[v_1/x][v_2/y] \Downarrow_{t_2} v}{\text{let } \langle\langle x, y \rangle\rangle = e \text{ in } e' \Downarrow_{t_1+t_2+1} v} \text{E-TE} \\
\\
\frac{e_1 \Downarrow_{t_1} v_1 \quad e_2 \Downarrow_{t_2} v_2}{\langle e_1, e_2 \rangle \Downarrow_{t_1+t_2+1} \langle v_1, v_2 \rangle} \text{E-WI} \qquad \frac{e \Downarrow_t \langle v_1, v_2 \rangle}{\text{fst}(e) \Downarrow_{t+1} v_1} \text{E-fst} \qquad \frac{e \Downarrow_t \langle v_1, v_2 \rangle}{\text{snd}(e) \Downarrow_{t+1} v_2} \text{E-snd} \\
\\
\frac{e \Downarrow_t v}{\text{inl}(e) \Downarrow_{t+1} \text{inl}(v)} \text{E-inl} \qquad \frac{e \Downarrow_t v}{\text{inr}(e) \Downarrow_{t+1} \text{inr}(v)} \text{E-inr} \qquad \frac{e \Downarrow_{t_1} \text{inl}(v) \quad e'[v/x] \Downarrow_{t_2} v'}{\text{case } e \text{ of } e'; e'' \Downarrow_{t_1+t_2+1} \text{inl}(v')} \text{E-case1} \\
\\
\frac{e \Downarrow_{t_1} \text{inr}(v) \quad e''[v/y] \Downarrow_{t_2} v''}{\text{case } e \text{ of } e'; e'' \Downarrow_{t_1+t_2+1} \text{inl}(v'')} \text{E-case2} \qquad \frac{}{!e \Downarrow_0 !e} \text{E-expI} \\
\\
\frac{e \Downarrow_{t_1} !e'' \quad e'[e''/x] \Downarrow_{t_2} v}{\text{let } !x = e \text{ in } e' \Downarrow_{t_1+t_2+1} v} \text{E-expE} \qquad \frac{e[\text{fix } x.e'/x] \Downarrow_t v}{\text{fix } x.e \Downarrow_{t+1} v} \text{E-fix} \\
\\
\frac{v \in \{(), x, \text{nil}, \lambda y.e, \Lambda.e, \text{ret } e, \text{bind } x = e_1 \text{ in } e_2, \uparrow^\kappa, \text{release } x = e_1 \text{ in } e_2, \text{store } e\}}{v \Downarrow_0 v} \text{E-val}
\end{array}$$

$$\begin{array}{c}
\frac{e \Downarrow_{t_1} \Lambda.e' \quad e' \Downarrow_{t_2} v}{e \square \Downarrow_{t_1+t_2+1} v} \text{E-tapp} \qquad \frac{e \Downarrow_{t_1} \Lambda.e' \quad e' \Downarrow_{t_2} v}{e \square \Downarrow_{t_1+t_2+1} v} \text{E-iapp} \\
\\
\frac{e \Downarrow_{t_1} \Lambda.e' \quad e' \Downarrow_{t_1} v}{e \square \Downarrow_{t_1+t_2+1} v} \text{E-CE} \qquad \frac{e_1 \Downarrow_{t_1} v \quad e_2[v/x] \Downarrow_{t_2} v'}{\text{clet } x = e_1 \text{ in } e_2 \Downarrow_{t_1+t_2+1} v'} \text{E-CandE} \qquad \frac{e \Downarrow_t v}{\text{ret } e \Downarrow_{t+1}^0 v} \text{E-return} \\
\\
\frac{e_1 \Downarrow_{t_1} v_1 \quad v_1 \Downarrow_{t_2}^{c_1} v'_1 \quad e_2[v'_1/x] \Downarrow_{t_3} v_2 \quad v_2 \Downarrow_{t_4}^{c_2} v'_2}{\text{bind } x = e_1 \text{ in } e_2 \Downarrow_{t_1+t_2+t_3+t_4+1}^{c_1+c_2} v'_2} \text{E-bind} \qquad \frac{}{\uparrow^\kappa \Downarrow_1^\kappa ()} \text{E-tick} \\
\\
\frac{e_1 \Downarrow_{t_1} v_1 \quad e_2[v_1/x] \Downarrow_{t_2} v_2 \quad v_2 \Downarrow_{t_3}^c v'_2}{\text{release } x = e_1 \text{ in } e_2 \Downarrow_{t_1+t_2+t_3+1}^c v'_2} \text{E-release} \qquad \frac{e \Downarrow_t v}{\text{store } e \Downarrow_{t+1}^0 v} \text{E-store}
\end{array}$$

Figure 8: Evaluation rules: pure and forcing

1.4 Model

Definition 6 (Value and expression relation).

$$\begin{aligned}
\llbracket \mathbf{1} \rrbracket &\triangleq \{(p, T, ())\} \\
\llbracket \mathbf{b} \rrbracket &\triangleq \{(p, T, v) \mid v \in \llbracket \mathbf{b} \rrbracket\} \\
\llbracket L^0 \tau \rrbracket &\triangleq \{(p, T, nil)\} \\
\llbracket L^{s+1} \tau \rrbracket &\triangleq \{(p, T, v :: l) \mid \exists p_1, p_2. p_1 + p_2 \leq p \wedge (p_1, T, v) \in \llbracket \tau \rrbracket \wedge (p_2, T, l) \in \llbracket L^s \tau \rrbracket\} \\
\llbracket \tau_1 \otimes \tau_2 \rrbracket &\triangleq \{(p, T, \langle\langle v_1, v_2 \rangle\rangle) \mid \exists p_1, p_2. p_1 + p_2 \leq p \wedge (p_1, T, v_1) \in \llbracket \tau_1 \rrbracket \wedge (p_2, T, v_2) \in \llbracket \tau_2 \rrbracket\} \\
\llbracket \tau_1 \& \tau_2 \rrbracket &\triangleq \{(p, T, \langle v_1, v_2 \rangle) \mid (p, T, v_1) \in \llbracket \tau_1 \rrbracket \wedge (p, T, v_2) \in \llbracket \tau_2 \rrbracket\} \\
\llbracket \tau_1 \oplus \tau_2 \rrbracket &\triangleq \{(p, T, \text{inl}(v)) \mid (p, T, v) \in \llbracket \tau_1 \rrbracket\} \cup \{(p, T, \text{inr}(v)) \mid (p, T, v) \in \llbracket \tau_2 \rrbracket\} \\
\llbracket \tau_1 \multimap \tau_2 \rrbracket &\triangleq \{(p, T, \lambda x. e) \mid \forall p', e', T' < T. (p', T', e') \in \llbracket \tau_1 \rrbracket_{\mathcal{E}} \implies (p + p', T', e[e'/x]) \in \llbracket \tau_2 \rrbracket_{\mathcal{E}}\} \\
\llbracket !_{a < I} \tau \rrbracket &\triangleq \{(p, T, !e) \mid \exists p_0, \dots, p_{I-1}. p_0 + \dots + p_{I-1} \leq p \wedge \forall 0 \leq i < I. (p_i, T, e) \in \llbracket \tau[i/a] \rrbracket_{\mathcal{E}}\} \\
\llbracket [n] \tau \rrbracket &\triangleq \{(p, T, v) \mid \exists p'. p' + n \leq p \wedge (p', T, v) \in \llbracket \tau \rrbracket\} \\
\llbracket \mathbb{M} n \tau \rrbracket &\triangleq \{(p, T, v) \mid \forall n', T' < T. v \Downarrow_{T'}^{n'} v' \implies \exists p'. n' + p' \leq p + n \wedge (p', T - T', v') \in \llbracket \tau \rrbracket\} \\
\llbracket \forall \alpha. \tau \rrbracket &\triangleq \{(p, T, \Lambda. e) \mid \forall \tau', T' < T. (p, T', e) \in \llbracket \tau[\tau'/\alpha] \rrbracket_{\mathcal{E}}\} \\
\llbracket \forall i. \tau \rrbracket &\triangleq \{(p, T, \Lambda. e) \mid \forall I T' < T. (p, T', e) \in \llbracket \tau[I/i] \rrbracket_{\mathcal{E}}\} \\
\llbracket c \Rightarrow \tau \rrbracket &\triangleq \{(p, T, \Lambda. e) \mid \forall T' < T. \models c \implies (p, T', e) \in \llbracket \tau \rrbracket_{\mathcal{E}}\} \\
\llbracket c \& \tau \rrbracket &\triangleq \{(p, T, v) \mid \models c \wedge (p, T, v) \in \llbracket \tau \rrbracket\} \\
\llbracket \exists s. \tau \rrbracket &\triangleq \{(p, T, v) \mid \exists s'. (p, T, v) \in \llbracket \tau[s'/s] \rrbracket\} \\
\llbracket \lambda_i. \tau \rrbracket &\triangleq f \text{ where } \forall I. f I = \llbracket \tau[I/i] \rrbracket \\
\llbracket \tau I \rrbracket &\triangleq \llbracket \tau \rrbracket I \\
\llbracket \tau \rrbracket_{\mathcal{E}} &\triangleq \{(p, T, e) \mid \forall v, T' < T. e \Downarrow_{T'} v \implies (p, T - T', v) \in \llbracket \tau \rrbracket\}
\end{aligned}$$

Definition 7 (Interpretation of typing contexts).

$$\begin{aligned}
\llbracket \Gamma \rrbracket_{\mathcal{E}} &= \{(p, T, \gamma) \mid \exists f : \text{Vars} \rightarrow \text{Pots}. \\
&\quad (\forall x \in \text{dom}(\Gamma). (f(x), T, \gamma(x)) \in \llbracket \Gamma(x) \rrbracket_{\mathcal{E}}) \wedge (\sum_{x \in \text{dom}(\Gamma)} f(x) \leq p)\} \\
\llbracket \Omega \rrbracket_{\mathcal{E}} &= \{(p, T, \delta) \mid \exists f : \text{Vars} \rightarrow \text{Indices} \rightarrow \text{Pots}. \\
&\quad (\forall (x :_{a < I} \tau) \in \Omega. \forall 0 \leq i < I. (f x i, T, \delta(x)) \in \llbracket \tau[i/a] \rrbracket_{\mathcal{E}}) \wedge \\
&\quad (\sum_{x :_{a < I} \tau \in \Omega} \sum_{0 \leq i < I} f x i) \leq p\}
\end{aligned}$$

Definition 8 (Type and index substitutions). $\sigma : \text{TypeVar} \rightarrow \text{Type}$, $\iota : \text{IndexVar} \rightarrow \text{Index}$

Lemma 9 (Value monotonicity lemma). $\forall p, p', v, \tau, T', T$.

$$(p, T, v) \in \llbracket \tau \rrbracket \wedge p \leq p' \wedge T' \leq T \implies (p', T', v) \in \llbracket \tau \rrbracket$$

Proof. Proof by induction on τ □

Lemma 10 (Expression monotonicity lemma). $\forall p, p', v, \tau, T', T$.

$$(p, T, e) \in \llbracket \tau \rrbracket_{\mathcal{E}} \wedge p \leq p' \wedge T' \leq T \implies (p', T', e) \in \llbracket \tau \rrbracket_{\mathcal{E}}$$

Proof. From Definition 6 and Lemma 69 □

Lemma 11 (Lemma for substitution). $\forall p, \delta, I, \Omega$.

$$\begin{aligned}
(p, \delta) \in \llbracket \sum_{a < I} \Omega \rrbracket &\implies \exists p_0, \dots, p_{I-1}. \\
p_0 + \dots + p_{I-1} &\leq p \wedge \forall 0 \leq i < I. (p_i, \delta) \in \llbracket \Omega[i/a] \rrbracket
\end{aligned}$$

Proof. Given: $(p, \delta) \in \llbracket \sum_{a < I} \Omega \rrbracket$

When $\Omega = \cdot$

The proof is trivial simply choose p_i as 0 and we are done

When $\Omega(a) = x_0 :_{b < J_0(a)} \tau_0(a), \dots, x_n :_{b < J_n(a)} \tau_n(a)$

Therefore from Definition 2 and Definition 7 we have

$\exists f : \mathcal{Vars} \rightarrow \mathcal{Indices} \rightarrow \mathcal{Pots}$.

$$(\forall (x_j :_{c < \sum_{a < I} J_j} \sigma) \in (\sum_{a < I} \Omega). \forall 0 \leq i < \sum_{a < I} J_j. (f \ x \ i, \delta(x_j)) \in \llbracket \sigma[i/c] \rrbracket) \wedge$$

$$(\sum_{x_j :_{c < \sum_{a < I} J_j} \sigma \in (\sum_{a < I} \Omega) \sum_{0 \leq i < \sum_{a < I} J_j} f \ x_j \ i) \leq p \quad (\text{SM0})$$

To prove the desired, for each $i \in [0, I - 1]$ we choose

p_i as $\sum_{x_j :_{b < J_j(i)} \tau_j(i) \in (\Omega(i)) \sum_{0 \leq k < J_j(i)} f \ x_j \ (k + \sum_{d < i} J_j[d/i])$

and we need to prove

1. $p_0 + \dots + p_{I-1} \leq p$:

It suffices to prove that

$$\sum_{0 \leq i < I} \sum_{x_j :_{b < J_j(i)} \tau_j(i) \in \text{dom}(\Omega(i)) \sum_{0 \leq k < J_j(i)} f \ x_j \ (k + \sum_{d < i} J_j(i)[d/i]) \leq p$$

We know that $\text{dom}(\sum_{a < I} \Omega) = \text{dom}(\Omega)$ and from (SM0) we get the desired

2. $\forall 0 \leq i < I. (p_i, \delta) \in \llbracket \Omega[i/a] \rrbracket$:

This means given some $0 \leq i < I$, from Definition 7 it suffices to prove that

$\exists f' : \mathcal{Vars} \rightarrow \mathcal{Indices} \rightarrow \mathcal{Pots}$.

$$(\forall (x_j :_{b < J_j(i)} \tau_j(i)) \in \Omega[i/a]. \forall 0 \leq k < J_j(i). (f' \ x_j \ k, \delta(x_j)) \in \llbracket \tau_j(i)[k/b] \rrbracket) \wedge$$

$$(\sum_{x_j :_{b < J_j(i)} \tau_j(i) \in \Omega[i/a] \sum_{0 \leq k < J_j(i)} f' \ x \ k) \leq p_i$$

We choose f' s.t

$$\forall x_j :_{b < J_j(i)} \tau_j(i) \in (\Omega[i/a]). \forall 0 \leq k < J_j(i). f' \ x_j \ k = f \ x_j \ (k + \sum_{d < i} J_j[d/i]),$$

And we need to prove:

- (a) $\forall (x_j :_{b < J_j(i)} \tau_j(i)) \in \Omega[i/a]. \forall 0 \leq k < J_j(i). (f' \ x_j \ k, \delta(x_j)) \in \llbracket \tau_j(i)[k/b] \rrbracket$:

This means given some $(x_j :_{b < J_j(i)} \tau_j(i)) \in \Omega[i/a]$ and some $0 \leq k < J_j(i)$ and it suffices to prove that

$$(f' \ x_j \ k, \delta(x_j)) \in \llbracket \tau_j(i)[k/b] \rrbracket$$

This means we need to prove that

$$(f \ x_j \ (k + \sum_{d < i} J_j[d/i]), \delta(x_j)) \in \llbracket \tau_j(i)[k/b] \rrbracket \quad (\text{SM1})$$

Instantiating (SM0) with the given x_j and $(k + \sum_{d < i} J_j[d/i])$ we get

$$(f \ x_j \ (k + \sum_{d < i} J_j[d/i]), \delta(x_j)) \in \llbracket \sigma[(k + \sum_{d < i} J_j[d/i])/c] \rrbracket$$

And from Definition 2 we get the desired

- (b) $(\sum_{x_j :_{b < J_j(i)} \tau_j(i) \in \Omega[i/a] \sum_{0 \leq k < J_j(i)} f' \ x \ k) \leq p_i$:

It suffices to prove that

$$(\sum_{x_j :_{b < J_j(i)} \tau_j(i) \in \Omega[i/a] \sum_{0 \leq k < J_j(i)} f \ x \ (k + \sum_{d < i} J_j[d/i])) \leq p_i$$

Since we know that p_i is $\sum_{x_j :_{b < J_j(i)} \tau_j(i) \in (\Omega(i)) \sum_{0 \leq k < J_j(i)} f \ x_j \ (k + \sum_{d < i} J_j[d/i])$ therefore we are done

□

Theorem 12 (Fundamental theorem). $\forall \Psi, \Theta, \Delta, \Omega, \Gamma, e, \tau \in Type.$

$$\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \wedge (p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}} \wedge (p_m, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}} \wedge \cdot \models \Delta \iota \implies (p_l + p_m, T, e \gamma \delta) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}.$$

Proof. Proof by induction on the typing judgment

1. T-var1:

$$\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma, x : \tau \vdash x : \tau} \text{T-var1}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma, x : \tau \sigma \iota \rrbracket_{\mathcal{E}}$ and $(p_m, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l + p_m, T, x \delta \gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$

Since we are given that $(p_l, T, \gamma) \in \llbracket \Gamma, x : \tau \sigma \iota \rrbracket_{\mathcal{E}}$ therefore from Definition 7 we know that $\exists f. (f(x), T, \gamma(x)) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$ where $f(x) \leq p_l$

Therefore from Lemma 70 we get $(p_l + p_m, T, x \delta \gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$

2. T-var2:

$$\frac{\Theta, \Delta \models I \geq 1}{\Psi; \Theta; \Delta; \Omega, x :_{a < I} \tau; \Gamma \vdash x : \tau[0/a]} \text{T-var2}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma, \sigma \iota \rrbracket_{\mathcal{E}}$ and $(p_m, T, \delta) \in \llbracket (\Omega, x :_{a < I} \tau) \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l + p_m, x \delta \gamma) \in \llbracket \tau[0/a] \sigma \iota \rrbracket_{\mathcal{E}}$

Since we are given that $(p_m, T, \delta) \in \llbracket (\Omega, x :_{a < I} \tau) \sigma \iota \rrbracket_{\mathcal{E}}$ therefore from Definition 7 we know that

$\exists f : \mathcal{Vars} \rightarrow \mathcal{Indices} \rightarrow \mathcal{Pots}.$

$((f \ x \ 0, T, \delta(x)) \in \llbracket \tau[0/a] \sigma \iota \rrbracket_{\mathcal{E}})$ where $(f \ x \ 0) \leq p_m$

Therefore from Lemma 70 we get $(p_l + p_m, T, x \delta \gamma) \in \llbracket \tau[0/a] \sigma \iota \rrbracket_{\mathcal{E}}$

3. T-unit:

$$\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash () : \mathbf{1}} \text{T-unit}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}, (p_m, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$ and $\models \Delta \iota$

To prove: $(p_l + p_m, T, () \delta \gamma) \in \llbracket \mathbf{1} \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$\forall T' < T. () \Downarrow_{T'} () \implies (p_m + p_l, T - T', ()) \in \llbracket \mathbf{1} \rrbracket$$

This means given $() \Downarrow_0 ()$ it suffices to prove that

$$(p_l + p_m, T, ()) \in \llbracket \mathbf{1} \rrbracket$$

We get this directly from Definition 6

4. T-base:

$$\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash c : \mathbf{b}} \text{T-base}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma, \sigma\iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket \Omega, \sigma\iota \rrbracket_{\mathcal{E}}$ and $\models \Delta \iota$

To prove: $(p_l + p_m, T, c) \in \llbracket \mathbf{b} \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$\forall v, T' < T . c \Downarrow_{T'} v \implies (p_m + p_l, T - T', c) \in \llbracket \mathbf{b} \rrbracket$$

This means given some $v, T' < T$ s.t $c \Downarrow_{T'} v$. Also from E-val we know that $T' = 0$ therefore it suffices to prove that

$$(p_l + p_m, T, v) \in \llbracket \mathbf{b} \rrbracket$$

From (E-val) we know that $v = c$ therefore it suffices to prove that

$$(p_l + p_m, T, c) \in \llbracket \mathbf{b} \rrbracket$$

We get this directly from Definition 6

5. T-nil:

$$\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash nil : L^0 \tau} \text{T-nil}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma, \sigma\iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket \Omega, \sigma\iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l + p_m, T, nil \delta\gamma) \in \llbracket L^0 \tau \sigma\iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall T' < T, v'. nil \Downarrow_{T'} v' \implies (p_l + p_m, T - T', v') \in \llbracket L^0 \tau \sigma\iota \rrbracket$$

This means given some $T' < T, v'$ s.t $nil \Downarrow_{T'} v'$ it suffices to prove that

$$(p_l + p_m, T - T', v') \in \llbracket L^0 \tau \sigma\iota \rrbracket$$

From (E-val) we know that $T' = 0$ and $v' = nil$, therefore it suffices to prove that

$$(p_l + p_m, T, nil) \in \llbracket L^0 \tau \sigma\iota \rrbracket$$

We get this directly from Definition 66

6. T-cons:

$$\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e_1 : \tau \quad \Psi; \Theta; \Delta; \Omega_2; \Gamma_2 \vdash e_2 : L^n \tau \quad \Theta \vdash n : \mathbb{N}}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash e_1 :: e_2 : L^{n+1} \tau} \text{T-cons}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma_1 \oplus \Gamma_2) \sigma\iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket (\Omega) \sigma\iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l + p_m, T, (e_1 :: e_2) \delta\gamma) \in \llbracket L^{n+1} \tau \sigma\iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$\forall v', t < T . (e_1 :: e_2) \delta\gamma \Downarrow_t v' \implies (p_l + p_m, T - t, v') \in \llbracket L^{n+1} \tau \sigma\iota \rrbracket$$

This means given some $v', t < T$ s.t $(e_1 :: e_2) \delta\gamma \Downarrow_t v'$, it suffices to prove that $(p_l + p_m, T - t, v') \in \llbracket L^{n+1} \tau \sigma\iota \rrbracket$

From (E-cons) we know that $\exists v_f, l. v' = v_f :: l$

Therefore from Definition 6 it suffices to prove that

$$\exists p_1, p_2. p_1 + p_2 \leq p_l + p_m \wedge (p_1, T - t, v_f) \in \llbracket \tau \sigma\iota \rrbracket \wedge (p_2, T - t, l) \in \llbracket L^n \tau \sigma\iota \rrbracket \quad (\text{F-C0})$$

From Definition 7 and Definition 5 we know that $\exists p_{l1}, p_{l2}. p_{l1} + p_{l2} = p_l$ s.t

$$(p_{l1}, T, \gamma) \in \llbracket (\Gamma_1) \sigma\iota \rrbracket_{\mathcal{E}} \text{ and } (p_{l2}, T, \gamma) \in \llbracket (\Gamma_2) \sigma\iota \rrbracket_{\mathcal{E}}$$

Similarly from Definition 7 and Definition 4 we also know that

$$\exists p_{m1}, p_{m2}. p_{m1} + p_{m2} = p_m \text{ s.t}$$

$$(p_{m1}, T, \delta) \in \llbracket (\Omega_1) \sigma\iota \rrbracket_{\mathcal{E}} \text{ and } (p_{m2}, T, \delta) \in \llbracket (\Omega_2) \sigma\iota \rrbracket_{\mathcal{E}}$$

IH1:

$$(p_{l1} + p_{m1}, T, e_1 \delta\gamma) \in \llbracket \tau \sigma\iota \rrbracket_{\mathcal{E}}$$

Therefore from Definition 6 we have

$$\forall t1 < T. e_1 \delta\gamma \Downarrow_{t1} v_f \implies (p_{l1} + p_{m1}, T - t1, v_f) \in \llbracket \tau \rrbracket$$

Since we are given that $(e_1 :: e_2) \delta\gamma \Downarrow_t v_f :: l$ therefore fom E-cons we also know that $\exists t1 < t. e_1 \delta\gamma \Downarrow_{t1} v_f$

$$\text{Therefore we have } (p_{l1} + p_{m1}, T - t1, v_f) \in \llbracket \tau \sigma\iota \rrbracket \quad (\text{F-C1})$$

IH2:

$$(p_{l2} + p_{m2}, T, e_2 \delta\gamma) \in \llbracket L^n \tau \sigma\iota \rrbracket_{\mathcal{E}}$$

Therefore from Definition 6 we have

$$\forall t2 < T. e_2 \delta\gamma \Downarrow_{t2} l \implies (p_{l2} + p_{m2}, T - t2, l) \in \llbracket L^n \tau \sigma\iota \rrbracket$$

Since we are given that $(e_1 :: e_2) \delta\gamma \Downarrow_t v_f :: l$ therefore fom E-cons we also know that $\exists t2 < t - t1. e_2 \delta\gamma \Downarrow_{t2} l$

Since $t2 < t - t1 < t < T$, therefore we have

$$(p_{l2} + p_{m2}, T - t2, l) \in \llbracket L^n \tau \sigma\iota \rrbracket \quad (\text{F-C2})$$

In order to prove (F-C0) we choose p_1 as $p_{l1} + p_{m1}$ and p_2 as $p_{l2} + p_{m2}$, we get the desired from (F-C1), (F-C2) and Lemma 69

7. T-match:

$$\frac{\Psi; \Theta; \Delta; \Omega_2; \Gamma_1 \vdash e : L^n \tau \quad \Psi; \Theta; \Delta, n = 0; \Omega_2; \Gamma_2 \vdash e_1 : \tau' \quad \Psi; \Theta; \Delta, n > 0; \Omega_2; \Gamma_2, h : \tau, t : L^{n-1} \tau \vdash e_2 : \tau'}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash \text{match } e \text{ with } |nil \mapsto e_1 \mid h :: t \mapsto e_2 : \tau'} \quad \text{T-match}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma_1 \oplus \Gamma_2) \sigma\iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket \Omega \sigma\iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l + p_m, T, (\text{match } e \text{ with } |nil \mapsto e_1 \mid h :: t \mapsto e_2) \delta\gamma) \in \llbracket \tau' \sigma\iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$\forall t < T, v_f. (\text{match } e \text{ with } | \text{nil} \mapsto e_1 \mid h :: t \mapsto e_2) \delta\gamma \Downarrow_t v_f \implies (p_l + p_m, T - t, v_f) \in \llbracket \tau' \sigma \iota \rrbracket$$

This means given some $t < T, v_f$ s.t. $(\text{match } e \text{ with } | \text{nil} \mapsto e_1 \mid h :: t \mapsto e_2) \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_l + p_m, T - t, v_f) \in \llbracket \tau' \sigma \iota \rrbracket \quad (\text{F-M0})$$

From Definition 7 and Definition 5 we know that $\exists p_{l1}, p_{l2}. p_{l1} + p_{l2} = p_l$ s.t.

$$(p_{l1}, T, \gamma) \in \llbracket (\Gamma_1) \sigma \iota \rrbracket_{\mathcal{E}} \text{ and } (p_{l2}, T, \gamma) \in \llbracket (\Gamma_2) \sigma \iota \rrbracket_{\mathcal{E}}$$

Similarly from Definition 7 and Definition 4 we also know that

$$\exists p_{m1}, p_{m2}. p_{m1} + p_{m2} = p_m \text{ s.t.}$$

$$(p_{m1}, T, \delta) \in \llbracket (\Omega_1) \sigma \iota \rrbracket_{\mathcal{E}} \text{ and } (p_{m2}, T, \delta) \in \llbracket (\Omega_2) \sigma \iota \rrbracket_{\mathcal{E}}$$

IH1

$$(p_{l1} + p_{m1}, T, e \delta\gamma) \in \llbracket L^n \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 6 we have

$$\forall t' < T. e \delta\gamma \Downarrow_{t'} v_1 \implies (p_{l1} + p_{m1}, T - t', v_1) \in \llbracket L^n \tau \sigma \iota \rrbracket$$

Since we know that $(\text{match } e \text{ with } | \text{nil} \mapsto e_1 \mid h :: t \mapsto e_2) \delta\gamma \Downarrow_t v_f$ therefore from E-match we know that $\exists t' < t, v_1. e \delta\gamma \Downarrow_{t'} v_1$.

$$\text{Since } t' < t < T, \text{ therefore we have } (p_{l1} + p_{m1}, T - t', v_1) \in \llbracket L^n \tau \sigma \iota \rrbracket$$

2 cases arise:

(a) $v_1 = \text{nil}$:

In this case we know that $n = 0$ therefore

IH2

$$(p_{l2} + p_{m2}, T, e_1 \delta\gamma) \in \llbracket \tau' \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 6 we have

$$\forall t_1 < T. e_1 \delta\gamma \Downarrow_{t_1} v_f \implies (p_{l2} + p_{m2}, T - t_1, v_f) \in \llbracket \tau' \sigma \iota \rrbracket$$

Since we know that $(\text{match } e \text{ with } | \text{nil} \mapsto e_1 \mid h :: t \mapsto e_2) \delta\gamma \Downarrow_t v_f$ therefore from E-match we know that $\exists t_1 < t. e_1 \delta\gamma \Downarrow_{t_1} v_f$.

Since $t_1 < t < T$ therefore we have

$$(p_{l2} + p_{m2}, T - t_1, v_f) \in \llbracket \tau' \sigma \iota \rrbracket$$

And from Lemma 69 we get

$$(p_{l2} + p_{m2} + p_{l1} + p_{m1}, T - t, v_f) \in \llbracket \tau' \sigma \iota \rrbracket_{\mathcal{E}}$$

And finally since $p_l = p_{l1} + p_{l2}$ and $p_m = p_{m1} + p_{m2}$ therefore we get

$$(p_l + p_m, T - t, v_f) \in \llbracket \tau' \sigma \iota \rrbracket_{\mathcal{E}}$$

And we are done

(b) $v_1 = v :: l$:

In this case we know that $n > 0$ therefore

IH2

$$(p_{l2} + p_{m2} + p_{l1} + p_{m1}, T, e_2 \delta\gamma) \in \llbracket \tau' \sigma\iota \rrbracket_{\mathcal{E}}$$

where

$$\gamma' = \gamma \cup \{h \mapsto v\} \cup \{t \mapsto l\} \text{ and}$$

This means from Definition 6 we have

$$\forall t_2 < T . e_2 \delta\gamma' \Downarrow_{t_2} v_f \implies (p_{l2} + p_{m2} + p_{l1} + p_{m1}, T - t_2, v_f) \in \llbracket \tau' \sigma\iota \rrbracket$$

Since we know that (match e with $|nil \mapsto e_1 \mid h :: t \mapsto e_2$) $\delta\gamma \Downarrow_t v_f$ therefore from E-match we know that $\exists t_2 < t . e_2 \delta\gamma' \Downarrow v_f$.

Since $t_2 < t < T$ therefore we have

$$(p_{l2} + p_{m2} + p_{l1} + p_{m1}, T - t_2, v_f) \in \llbracket \tau' \sigma\iota \rrbracket$$

From Lemma 69 we get

$$(p_{l2} + p_{m2} + p_{l1} + p_{m1}, T - t, v_f) \in \llbracket \tau' \sigma\iota \rrbracket$$

And finally since $p_l = p_{l1} + p_{l2}$ and $p_m = p_{m1} + p_{m2}$ therefore we get

$$(p_l + p_m, T - t, v_f) \in \llbracket \tau' \sigma\iota \rrbracket_{\mathcal{E}}$$

And we are done

8. T-existI:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau[n/s] \quad \Theta \vdash n : S}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \exists s : S . \tau} \text{T-existI}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma\iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket \Omega \sigma\iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l + p_m, T, e \delta\gamma) \in \llbracket \exists s . \tau \sigma\iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$\forall t < T, v_f . e \delta\gamma \Downarrow_t v_f \implies (p_l + p_m, T - t, v_f \delta\gamma) \in \llbracket \exists s . \tau \sigma\iota \rrbracket$$

This means given some $t < T, v_f$ s.t $e \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_l + p_m, T - t, v_f) \in \llbracket \exists s . \tau \sigma\iota \rrbracket$$

From Definition 6 it suffices to prove that

$$\exists s' . (p_l + p_m, T - t, v_f) \in \llbracket \tau[s'/s] \sigma\iota \rrbracket \quad (\text{F-E0})$$

IH: $(p_l + p_m, T, e \delta\gamma) \in \llbracket \tau[n/s] \sigma\iota \rrbracket_{\mathcal{E}}$

This means from Definition 6 we have

$$\forall t' < T . e \delta\gamma \Downarrow_{t'} v_f \implies (p_l + p_m, T - t', v_f) \in \llbracket \tau[n/s] \sigma\iota \rrbracket$$

Since we are given that $e \delta\gamma \Downarrow_t v_f$ therefore we get

$$(p_l + p_m, T - t, v_f) \in \llbracket \tau[n/s] \sigma\iota \rrbracket \quad (\text{F-E1})$$

To prove (F-E0) we choose s' as n and we get the desired from (F-E1)

9. T-existsE:

$$\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e : \exists s.\tau \quad \Psi; \Theta, s; \Delta; \Omega_2; \Gamma_2, x : \tau \vdash e' : \tau' \quad \Psi; \Theta; \Delta \vdash \tau' : K}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash e; x.e' : \tau'} \text{T-existE}$$

Given: $(p_l, T, \gamma) \in [(\Gamma_1 \oplus \Gamma_2) \sigma \iota]_{\mathcal{E}}$, $(p_m, T, \delta) \in [(\Omega) \sigma \iota]_{\mathcal{E}}$

To prove: $(p_l + p_m, T, (e; x.e') \delta \gamma) \in [\tau' \sigma \iota]_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$\forall t < T, v_f. (e; x.e') \delta \gamma \Downarrow_t v_f \implies (p_l + p_m, T - t, v_f) \in [\tau' \sigma \iota]$$

This means given some $t < T, v_f$ s.t $(e; x.e') \delta \gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_l + p_m, T - t, v_f) \in [\tau' \sigma \iota] \quad (\text{F-EE0})$$

From Definition 7 and Definition 5 we know that $\exists p_{l1}, p_{l2}. p_{l1} + p_{l2} = p_l$ s.t

$$(p_{l1}, T, \gamma) \in [(\Gamma_1) \sigma \iota]_{\mathcal{E}} \text{ and } (p_{l2}, T, \gamma) \in [(\Gamma_2) \sigma \iota]_{\mathcal{E}}$$

Similarly from Definition 7 and Definition 4 we also know that

$$\exists p_{m1}, p_{m2}. p_{m1} + p_{m2} = p_m \text{ s.t}$$

$$(p_{m1}, T, \delta) \in [(\Omega_1) \sigma \iota]_{\mathcal{E}} \text{ and } (p_{m2}, T, \delta) \in [(\Omega_2) \sigma \iota]_{\mathcal{E}}$$

IH1

$$(p_{l1} + p_{m1}, T, e \delta \gamma) \in [\exists s.\tau \sigma \iota]_{\mathcal{E}}$$

This means from Definition 6 we have

$$\forall t_1 < T. e \delta \gamma \Downarrow_{t_1} v_1 \implies (p_{l1}, T - t_1, v_1) \in [\exists s.\tau \sigma \iota]_{\mathcal{E}}$$

Since we know that $(e; x.e') \delta \gamma \Downarrow_t v_f$ therefore from E-existE we know that $\exists t_1 < t, v_1. e \delta \gamma \Downarrow_{t_1} v_1$. Therefore we have

$$(p_{l1} + p_{m1}, T - t_1, v_1) \in [\exists s.\tau \sigma \iota]_{\mathcal{E}}$$

Therefore from Definition 6 we have

$$\exists s'. (p_{l1} + p_{m1}, T - t_1, v_1) \in [\tau[s'/s] \sigma \iota]_{\mathcal{E}} \quad (\text{F-EE1})$$

IH2

$$(p_{l2} + p_{m2} + p_{l1} + p_{m1}, T, e' \delta' \gamma) \in [\tau' \sigma \iota']_{\mathcal{E}}$$

where

$$\delta' = \delta \cup \{x \mapsto e_1\} \text{ and } \iota' = \iota \cup \{s \mapsto s'\}$$

This means from Definition 6 we have

$$\forall t_2 < T. e' \delta' \gamma \Downarrow_{t_2} v_f \implies (p_{l2} + p_{m2} + p_{l1} + p_{m1}, T - t_2, v_f) \in [\tau' \sigma \iota']_{\mathcal{E}}$$

Since we know that $(e; x.e') \delta \gamma \Downarrow_t v_f$ therefore from E-existE we know that $\exists t_2 < t. e' \delta' \gamma \Downarrow_{t_2} v_f$.

Since $t_2 < t < T$ therefore we have

$$(p_{l2} + p_{m2} + p_{l1} + p_{m1}, T - t_2, v_f) \in \llbracket \tau' \sigma \iota \rrbracket$$

Since $p_l = p_{l1} + p_{l2}$ and $p_m = p_{m1} + p_{m2}$ therefore we get

$$(p_l + p_m, T - t_2, v_f) \in \llbracket \tau' \sigma \iota \rrbracket$$

And finally from Lemma 69 and since we have $\Psi; \Theta; \Delta \vdash \tau' : K$ therefore we also have

$$(p_l + p_m, T - t, v_f) \in \llbracket \tau' \sigma \iota \rrbracket$$

And we are done.

10. T-lam:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma, x : \tau_1 \vdash e : \tau_2}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \lambda x.e : (\tau_1 \multimap \tau_2)} \text{T-lam}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma, \sigma \iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$ and $\models \Delta \iota$

To prove: $(p_l + p_m, T, (\lambda x.e) \delta \gamma) \in \llbracket (\tau_1 \multimap \tau_2) \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$\forall t < T, v_f. (\lambda x.e) \delta \gamma \Downarrow_t v_f \implies (p_l + p_m, T - t, v_f) \in \llbracket (\tau_1 \multimap \tau_2) \sigma \iota \rrbracket$$

This means given some $t < T, v_f$ s.t $(\lambda x.e) \delta \gamma \Downarrow_t v_f$. From E-val we know that $t = 0$ and $v_f = (\lambda x.e) \delta \gamma$. Therefore we have

$$(p_l + p_m, T, (\lambda x.e) \delta \gamma) \in \llbracket (\tau_1 \multimap \tau_2) \sigma \iota \rrbracket$$

From Definition 6 it suffices to prove that

$$\forall p', e', T' < T. (p', T', e') \in \llbracket \tau_1 \sigma \iota \rrbracket_{\mathcal{E}} \implies (p_l + p_m + p', T', e[e'/x]) \in \llbracket \tau_2 \sigma \iota \rrbracket_{\mathcal{E}}$$

This means given some $p', e', T' < T$ s.t $(p', T', e') \in \llbracket \tau_1 \sigma \iota \rrbracket_{\mathcal{E}}$ it suffices to prove that

$$(p_l + p_m + p', T', e[e'/x]) \in \llbracket \tau_2 \sigma \iota \rrbracket_{\mathcal{E}} \quad (\text{F-L1})$$

From IH we know that

$$(p_l + p' + p_m, T, e \delta \gamma') \in \llbracket \tau_2 \sigma \iota \rrbracket_{\mathcal{E}}$$

where

$$\gamma' = \gamma \cup \{x \mapsto e'\}$$

Therefore from Lemma 70 we get the desired

11. T-app:

$$\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e_1 : (\tau_1 \multimap \tau_2) \quad \Psi; \Theta; \Delta; \Omega_2; \Gamma_2 \vdash e_2 : \tau_1}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash e_1 e_2 : \tau_2} \text{T-app}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma_1 \oplus \Gamma_2) \sigma \iota \rrbracket_{\mathcal{E}}$, $(p_m, \delta) \in \llbracket (\Omega_1 \oplus \Omega_2) \sigma \iota \rrbracket_{\mathcal{E}}$ and $\models \Delta \iota$

To prove: $(p_l + p_m, T, e_1 e_2 \delta \gamma) \in \llbracket \tau_2 \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$\forall t < T, v_f. (e_1 e_2) \delta \gamma \Downarrow_t v_f \implies (p_m + p_l, T - t, v_f) \in \llbracket \tau_2 \sigma \iota \rrbracket$$

This means given some $t < T, v_f$ s.t $(e_1 e_2) \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_m + p_l, T - t, v_f) \in \llbracket \tau_2 \sigma \iota \rrbracket \quad (\text{F-A0})$$

From Definition 7 and Definition 5 we know that $\exists p_{l1}, p_{l2}. p_{l1} + p_{l2} = p_l$ s.t

$$(p_{l1}, T, \gamma) \in \llbracket (\Gamma_1) \sigma \iota \rrbracket_{\mathcal{E}} \text{ and } (p_{l2}, T, \gamma) \in \llbracket (\Gamma_2) \sigma \iota \rrbracket_{\mathcal{E}}$$

Similarly from Definition 7 and Definition 4 we also know that $\exists p_{m1}, p_{m2}. p_{m1} + p_{m2} = p_m$ s.t

$$(p_{m1}, T, \delta) \in \llbracket (\Omega_1) \sigma \iota \rrbracket_{\mathcal{E}} \text{ and } (p_{m2}, T, \delta) \in \llbracket (\Omega_2) \sigma \iota \rrbracket_{\mathcal{E}}$$

IH1

$$(p_{l1} + p_{m1}, T, e_1 \delta\gamma) \in \llbracket (\tau_1 \multimap \tau_2) \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 6 we have

$$\forall t_1 < T . e_1 \Downarrow_{t_1} \lambda x. e \implies (p_{l1} + p_{m1}, T - t_1, \lambda x. e) \in \llbracket (\tau_1 \multimap \tau_2) \sigma \iota \rrbracket$$

Since we know that $(e_1 e_2) \delta\gamma \Downarrow_t v_f$ therefore from E-app we know that $\exists t_1 < t. e_1 \Downarrow_{t_1} \lambda x. e$, therefore we have

$$(p_{l1} + p_{m1}, T - t_1, \lambda x. e) \in \llbracket (\tau_1 \multimap \tau_2) \sigma \iota \rrbracket$$

Therefore from Definition 6 we have

$$\forall p', e_1, T_1 < T - t_1. (p', T_1, e'_1) \in \llbracket \tau_1 \sigma \iota \rrbracket_{\mathcal{E}} \implies (p_{l1} + p_{m1} + p', T_1, e[e'_1/x]) \in \llbracket \tau_2 \sigma \iota \rrbracket_{\mathcal{E}} \quad (\text{F-A1})$$

IH2

$$(p_{l2} + p_{m2}, T - t_1 - 1, e_2 \delta\gamma) \in \llbracket \tau_1 \sigma \iota \rrbracket_{\mathcal{E}} \quad (\text{F-A2})$$

Instantiating (F-A1) with $p_{l2} + p_{m2}$ and $e_2 \delta\gamma$ we get

$$(p_{l1} + p_{m1} + p_{l2} + p_{m2}, T - t_1 - 1, e[e_2 \delta\gamma/x]) \in \llbracket \tau_2 \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 6 we have

$$\forall t_2 < T - t_1 - 1. e[e_2 \delta\gamma/x] \Downarrow_{t_2} v_f \implies (p_l + p_m, T - t_1 - 1 - t_2, v_f) \in \llbracket \tau_2 \sigma \iota \rrbracket$$

Since we know that $(e_1 e_2) \delta\gamma \Downarrow_t v_f$ therefore from E-app we know that $\exists t_2. e[e_2 \delta\gamma/x] \Downarrow_{t_2} v_f$, where $t_2 = t - t_1 - 1$, therefore we have

$$(p_l + p_m, T - t_1 - t_2 - 1, v_f) \in \llbracket \tau_2 \sigma \iota \rrbracket$$

Since from E-app we know that $t = t_1 + t_2 + 1$, this proves (F-A0)

12. T-sub:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \quad \Theta; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau'} \text{ T-sub}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma) \sigma \iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket (\Omega) \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l + p_m, T, e \delta\gamma) \in \llbracket \tau' \sigma \iota \rrbracket_{\mathcal{E}}$

IH $(p_l + p_m, T, e \delta\gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$

We get the desired directly from IH and Lemma 73

13. T-weaken:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \quad \Psi; \Theta \models \Gamma' <: \Gamma \quad \Psi; \Theta \models \Omega' <: \Omega}{\Psi; \Theta; \Delta; \Omega'; \Gamma' \vdash e : \tau} \text{T-weaken}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma')\sigma\iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket (\Omega')\sigma\iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l + p_m, T, e \delta\gamma) \in \llbracket \tau \sigma\iota \rrbracket_{\mathcal{E}}$

Since we are given that $(p_l, T, \gamma) \in \llbracket (\Gamma')\sigma\iota \rrbracket_{\mathcal{E}}$ therefore from Lemma 15 we also have $(p_l, T, \gamma) \in \llbracket (\Gamma)\sigma\iota \rrbracket_{\mathcal{E}}$

Similarly since we are given that $(p_m, T, \delta) \in \llbracket (\Omega')\sigma\iota \rrbracket_{\mathcal{E}}$ therefore from Lemma 16 we also have $(p_m, T, \delta) \in \llbracket (\Omega)\sigma\iota \rrbracket_{\mathcal{E}}$

IH:

$(p_l + p_m, T, e \delta\gamma) \in \llbracket \tau \sigma\iota \rrbracket_{\mathcal{E}}$

We get the desired directly from IH

14. T-tensorI:

$$\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e_1 : \tau_1 \quad \Psi; \Theta; \Delta; \Omega_2; \Gamma_2 \vdash e_2 : \tau_2}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash \langle\langle e_1, e_2 \rangle\rangle : (\tau_1 \otimes \tau_2)} \text{T-tensorI}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma_1 \oplus \Gamma_2)\sigma\iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket (\Omega_1 \oplus \Omega_2)\sigma\iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l + p_m, T, \langle\langle e_1, e_2 \rangle\rangle \delta\gamma) \in \llbracket (\tau_1 \otimes \tau_2)\sigma\iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$\forall t < T. \langle\langle e_1, e_2 \rangle\rangle \delta\gamma \downarrow_t \langle\langle v_{f1}, v_{f2} \rangle\rangle \implies (p_l + p_m, T - t, \langle\langle v_{f1}, v_{f2} \rangle\rangle) \in \llbracket (\tau_1 \otimes \tau_2)\sigma\iota \rrbracket$$

This means given some $t < T$ s.t $\langle\langle e_1, e_2 \rangle\rangle \delta\gamma \downarrow_t \langle\langle v_{f1}, v_{f2} \rangle\rangle$ it suffices to prove that

$$(p_l + p_m, T - t, \langle\langle v_{f1}, v_{f2} \rangle\rangle) \in \llbracket (\tau_1 \otimes \tau_2)\sigma\iota \rrbracket \quad (\text{F-TI0})$$

From Definition 7 and Definition 5 we know that $\exists p_{l1}, p_{l2}. p_{l1} + p_{l2} = p_l$ s.t

$$(p_{l1}, T, \gamma) \in \llbracket (\Gamma_1)\sigma\iota \rrbracket_{\mathcal{E}} \text{ and } (p_{l2}, T, \gamma) \in \llbracket (\Gamma_2)\sigma\iota \rrbracket_{\mathcal{E}}$$

Similarly from Definition 7 and Definition 4 we also know that $\exists p_{m1}, p_{m2}. p_{m1} + p_{m2} = p_m$ s.t

$$(p_{m1}, T, \delta) \in \llbracket (\Omega_1)\sigma\iota \rrbracket_{\mathcal{E}} \text{ and } (p_{m2}, T, \delta) \in \llbracket (\Omega_2)\sigma\iota \rrbracket_{\mathcal{E}}$$

IH1:

$$(p_{l1} + p_{m1}, T, e_1 \delta\gamma) \in \llbracket \tau_1 \sigma\iota \rrbracket_{\mathcal{E}}$$

Therefore from Definition 6 we have

$$\forall t_1 < T. e_1 \delta\gamma \downarrow_{t_1} v_{f1} \implies (p_{l1} + p_{m1}, T - t_1, v_{f1}) \in \llbracket \tau_1 \sigma\iota \rrbracket$$

Since we are given that $\langle\langle e_1, e_2 \rangle\rangle \delta\gamma \downarrow_t \langle\langle v_{f1}, v_{f2} \rangle\rangle$ therefore fom E-TI we know that $\exists t_1 < t. e_1 \delta\gamma \downarrow_{t_1} v_{f1}$

Hence we have $(p_{l1} + p_{m1}, T - t_1, v_{f1}) \in \llbracket \tau_1 \sigma \iota \rrbracket$ (F-TI1)

IH2:

$(p_{l2} + p_{m2}, T, e_2 \delta\gamma) \in \llbracket \tau_2 \sigma \iota \rrbracket_{\mathcal{E}}$

Therefore from Definition 6 we have

$\forall t_2 < T . e_2 \delta\gamma \Downarrow_{t_2} v_{f2} \implies (p_{l2} + p_{m2}, T - t_2, v_{f2}) \in \llbracket \tau_2 \sigma \iota \rrbracket$

Since we are given that $\langle\langle e_1, e_2 \rangle\rangle \delta\gamma \Downarrow_t \langle\langle v_{f1}, v_{f2} \rangle\rangle$ therefore fom E-TI we also know that $\exists t_2 < t . e_2 \delta\gamma \Downarrow_{t_2} v_{f2}$

Since $t_2 < t < T$ therefore we have

$(p_{l2} + p_{m2}, T - t_2, v_{f2}) \in \llbracket \tau_2 \sigma \iota \rrbracket$ (F-TI2)

Applying Lemma 69 on (F-TI1) and (F-TI2) and by using Definition 66 we get the desired.

15. T-tensorE:

$$\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e : (\tau_1 \otimes \tau_2) \quad \Psi; \Theta; \Delta; \Omega_2; \Gamma_2, x : \tau_1, y : \tau_2 \vdash e' : \tau}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash \text{let}\langle\langle x, y \rangle\rangle = e \text{ in } e' : \tau} \text{T-tensorE}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma_1 \oplus \Gamma_2) \sigma \iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l + p_m, T, (\text{let}\langle\langle x, y \rangle\rangle = e \text{ in } e') \delta\gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall t < T, v_f . (\text{let}\langle\langle x, y \rangle\rangle = e \text{ in } e') \delta\gamma \Downarrow_t v_f \implies (p_l + p_m, T - t, v_f) \in \llbracket \tau \sigma \iota \rrbracket$

This means given some $t < T, v_f$ s.t $(\text{let}\langle\langle x, y \rangle\rangle = e \text{ in } e') \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$(p_l + p_m, T - t, v_f) \in \llbracket \tau \sigma \iota \rrbracket$ (F-TE0)

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2} . p_{l1} + p_{l2} = p_l$ s.t

$(p_{l1}, T, \gamma) \in \llbracket (\Gamma_1) \sigma \iota \rrbracket_{\mathcal{E}}$ and $(p_{l2}, T, \gamma) \in \llbracket (\Gamma_2) \sigma \iota \rrbracket_{\mathcal{E}}$

Similarly from Definition 7 and Definition 4 we also know that $\exists p_{m1}, p_{m2} . p_{m1} + p_{m2} = p_m$ s.t

$(p_{m1}, T, \delta) \in \llbracket (\Omega_1) \sigma \iota \rrbracket_{\mathcal{E}}$ and $(p_{m2}, T, \delta) \in \llbracket (\Omega_2) \sigma \iota \rrbracket_{\mathcal{E}}$

IH1

$(p_{l1} + p_{m1}, T, e \delta\gamma) \in \llbracket (\tau_1 \otimes \tau_2) \sigma \iota \rrbracket_{\mathcal{E}}$

This means from Definition 66 we have

$\forall t_1 < T . e \delta\gamma \Downarrow_{t_1} \langle\langle v_1, v_2 \rangle\rangle \delta\gamma \implies (p_{l1} + p_{m1}, T - t_1, \langle\langle v_1, v_2 \rangle\rangle) \in \llbracket (\tau_1 \otimes \tau_2) \sigma \iota \rrbracket$

Since we know that $(\text{let}\langle\langle x, y \rangle\rangle = e \text{ in } e') \delta\gamma \Downarrow_t v_f$ therefore from E-subExpE we know that $\exists t_1 < t, v_1, v_2 . e \delta\gamma \Downarrow_{t_1} \langle\langle v_1, v_2 \rangle\rangle$. Therefore we have

$(p_{l1} + p_{m1}, T - t_1, \langle\langle v_1, v_2 \rangle\rangle) \in \llbracket (\tau_1 \otimes \tau_2) \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 66 we know that

$$\exists p_1, p_2. p_1 + p_2 \leq p_{l_1} + p_{m_1} \wedge (p_1, T, v_1) \in \llbracket \tau_1 \sigma \iota \rrbracket \wedge (p_2, T, v_2) \in \llbracket \tau_2 \sigma \iota \rrbracket \quad (\text{F-TE1})$$

IH2

$$(p_{l_2} + p_{m_2} + p_1 + p_2, T, e' \delta \gamma') \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

where

$$\gamma' = \gamma \cup \{x \mapsto v_1\} \cup \{y \mapsto v_2\}$$

This means from Definition 66 we have

$$\forall t_2 < T. e' \delta \gamma' \Downarrow_{t_2} v_f \implies (p_{l_2} + p_{m_2} + p_1 + p_2, T - t_2, v_f) \in \llbracket \tau \sigma \iota \rrbracket$$

Since we know that $(\text{let}\langle\langle x, y \rangle\rangle = e \text{ in } e') \delta \gamma \Downarrow_t v_f$ therefore from E-TE we know that $\exists t_2 < t. e' \delta \gamma' \Downarrow_{t_2} v_f$. Therefore we have

$$(p_{l_2} + p_{m_2} + p_1 + p_2, T - t_2, v_f) \in \llbracket \tau \sigma \iota \rrbracket$$

From Lemma 69 we get

$$(p_l + p_m, T - t, v_f) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

And we are done

16. T-withI:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e_1 : \tau_1 \quad \Psi; \Theta; \Delta; \Omega; \Gamma \vdash e_2 : \tau_2}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \langle e_1, e_2 \rangle : (\tau_1 \& \tau_2)} \text{T-withI}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l + p_m, T, \langle e_1, e_2 \rangle \delta \gamma) \in \llbracket (\tau_1 \& \tau_2) \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T. \langle e_1, e_2 \rangle \delta \gamma \Downarrow_t \langle v_{f_1}, v_{f_2} \rangle \implies (p_l + p_m, T - t, \langle v_{f_1}, v_{f_2} \rangle) \in \llbracket (\tau_1 \& \tau_2) \sigma \iota \rrbracket$$

This means given $\langle e_1, e_2 \rangle \delta \gamma \Downarrow_t \langle v_{f_1}, v_{f_2} \rangle$ it suffices to prove that

$$(p_l + p_m, T - t, \langle v_{f_1}, v_{f_2} \rangle) \in \llbracket (\tau_1 \& \tau_2) \sigma \iota \rrbracket \quad (\text{F-WI0})$$

IH1:

$$(p_l + p_m, T, e_1 \delta \gamma) \in \llbracket \tau_1 \sigma \iota \rrbracket_{\mathcal{E}}$$

Therefore from Definition 66 we have

$$\forall t_1 < T. e_1 \delta \gamma \Downarrow_{t_1} v_{f_1} \implies (p_l + p_m, T - t_1, v_{f_1}) \in \llbracket \tau_1 \sigma \iota \rrbracket$$

Since we are given that $\langle e_1, e_2 \rangle \delta \gamma \Downarrow_t \langle v_{f_1}, v_{f_2} \rangle$ therefore from E-WI we know that $\exists t_1 < t. e_1 \delta \gamma \Downarrow_{t_1} v_{f_1}$

Since $t_1 < t < T$, therefore we have

$$(p_l + p_m, T - t_1, v_{f_1}) \in \llbracket \tau_1 \sigma \iota \rrbracket \quad (\text{F-WI1})$$

IH2:

$$(p_l + p_m, T, e_2 \delta \gamma) \in \llbracket \tau_2 \sigma \iota \rrbracket_{\mathcal{E}}$$

Therefore from Definition 66 we have

$$\forall t_2 < T . e_2 \delta\gamma \Downarrow_{t_2} v_{f2} \implies (p_l + p_m, T - t_2, v_{f2}) \in \llbracket \tau_2 \sigma \iota \rrbracket$$

Since we are given that $\langle e_1, e_2 \rangle \delta\gamma \Downarrow_t \langle v_{f1}, v_{f2} \rangle$ therefore fom E-WI we also know that $\exists t_2 < t . e_2 \delta\gamma \Downarrow_{t_2} v_{f2}$

Since $t_2 < t < T$, therefore we have

$$(p_l + p_m, T - t_2, v_{f2}) \in \llbracket \tau_2 \sigma \iota \rrbracket \quad (\text{F-WI2})$$

Applying Lemma 69 on (F-W1) and (F-W2) we get the desired.

17. T-fst:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\tau_1 \& \tau_2)}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{fst}(e) : \tau_1} \text{T-fst}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma) \sigma \iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l + p_m, T, (\text{fst}(e)) \delta\gamma) \in \llbracket \tau_1 \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T, v_f . (\text{fst}(e)) \delta\gamma \Downarrow_t v_f \implies (p_l + p_m, T - t, v_f) \in \llbracket \tau_1 \sigma \iota \rrbracket$$

This means given some $t < T, v_f$ s.t $(\text{fst}(e)) \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_l + p_m, T - t, v_f) \in \llbracket \tau_1 \sigma \iota \rrbracket \quad (\text{F-F0})$$

IH

$$(p_l + p_m, T, e \delta\gamma) \in \llbracket (\tau_1 \& \tau_2) \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 66 we have

$$\forall t_1 < T . e \delta\gamma \Downarrow_{t_1} \langle v_1, v_2 \rangle \delta\gamma \implies (p_l + p_m, T - t_1, \langle v_1, v_2 \rangle) \in \llbracket (\tau_1 \& \tau_2) \sigma \iota \rrbracket$$

Since we know that $(\text{fst}(e)) \delta\gamma \Downarrow_t v_f$ therefore from E-fst we know that $\exists t_1 < t . v_1, v_2 . e \delta\gamma \Downarrow_{t_1} \langle v_1, v_2 \rangle$.

Since $t_1 < t < T$, therefore we have

$$(p_l + p_m, T - t_1, \langle v_1, v_2 \rangle) \in \llbracket (\tau_1 \& \tau_2) \sigma \iota \rrbracket$$

From Definition 66 we know that

$$(p_l + p_m, T - t_1, v_1) \in \llbracket \tau_1 \sigma \iota \rrbracket$$

Finally using Lemma 69 we also have

$$(p_l + p_m, T - t, v_1) \in \llbracket \tau_1 \sigma \iota \rrbracket$$

Since from E-fst we know that $v_f = v_1$, therefore we are done.

18. T-snd:

Similar reasoning as in T-fst case above.

19. T-inl:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau_1}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{inl}(e) : \tau_1 \oplus \tau_2} \text{ T-inl}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l + p_m, T, \text{inl}(e) \delta \gamma) \in \llbracket (\tau_1 \oplus \tau_2) \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T . \text{inl}(e) \delta \gamma \Downarrow_t \text{inl}(v) \implies (p_l + p_m, T - t, \text{inl}(v)) \in \llbracket (\tau_1 \oplus \tau_2) \sigma \iota \rrbracket$$

This means given some $t < T$ s.t $\text{inl}(e) \delta \gamma \Downarrow_t \text{inl}(v)$ it suffices to prove that

$$(p_l + p_m, T - t, \text{inl}(v)) \in \llbracket (\tau_1 \oplus \tau_2) \sigma \iota \rrbracket \quad (\text{F-IL0})$$

IH:

$$(p_l + p_m, T, e_1 \delta \gamma) \in \llbracket \tau_1 \sigma \iota \rrbracket_{\mathcal{E}}$$

Therefore from Definition 66 we have

$$\forall t_1 < T . e_1 \delta \gamma \Downarrow_{t_1} v_{f1} \implies (p_l + p_m, T - t_1, v_{f1}) \in \llbracket \tau_1 \sigma \iota \rrbracket$$

Since we are given that $\text{inl}(e) \delta \gamma \Downarrow_t \text{inl}(v)$ therefore fom E-inl we know that $\exists t_1 < t. e \delta \gamma \Downarrow_{t_1} v$

$$\text{Hence we have } (p_l + p_m, T - t_1, v) \in \llbracket \tau_1 \sigma \iota \rrbracket$$

$$\text{From Lemma 69 we get } (p_l + p_m, T - t, v) \in \llbracket \tau_1 \sigma \iota \rrbracket$$

And finally from Definition 66 we get (F-IL0)

20. T-inr:

Similar reasoning as in T-inr case above.

21. T-case:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : (\tau_1 \oplus \tau_2) \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : \tau_1 \vdash e_1 : \tau \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2, y : \tau_2 \vdash e_2 : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \text{case } e \text{ of } e_1; e_2 : \tau} \text{ T-case}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma_1 \oplus \Gamma_2) \sigma \iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l + p_m, T, (\text{case } e \text{ of } e_1; e_2) \delta \gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T, v_f. (\text{case } e \text{ of } e_1; e_2) \delta \gamma \Downarrow_t v_f \implies (p_l + p_m, T - t, v_f) \in \llbracket \tau \sigma \iota \rrbracket$$

This means given some $t < T, v_f$ s.t $(\text{case } e \text{ of } e_1; e_2) \delta \gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_l + p_m, T - t, v_f) \in \llbracket \tau \sigma \iota \rrbracket \quad (\text{F-C0})$$

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2}. p_{l1} + p_{l2} = p_l$ s.t

$$(p_{l1}, T, \gamma) \in \llbracket (\Gamma_1) \sigma \iota \rrbracket_{\mathcal{E}} \text{ and } (p_{l2}, T, \gamma) \in \llbracket (\Gamma_2) \sigma \iota \rrbracket_{\mathcal{E}}$$

Similarly from Definition 7 and Definition 4 we also know that $\exists p_{m1}, p_{m2}. p_{m1} + p_{m2} = p_m$
s.t

$$(p_{m1}, T, \delta) \in \llbracket (\Omega_1) \sigma \iota \rrbracket_{\mathcal{E}} \text{ and } (p_{m2}, T, \delta) \in \llbracket (\Omega_2) \sigma \iota \rrbracket_{\mathcal{E}}$$

IH1

$$(p_{l1} + p_{m1}, T, e \delta \gamma) \in \llbracket (\tau_1 \oplus \tau_2) \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 66 we have

$$\forall t' < T . e \delta \gamma \Downarrow_{t'} v_1 \delta \gamma \implies (p_{l1} + p_{m1}, T - t', v_1) \in \llbracket (\tau_1 \oplus \tau_2) \sigma \iota \rrbracket$$

Since we know that (case e of $e_1; e_2$) $\delta \gamma \Downarrow_t v_f$ therefore from E-case we know that $\exists t' < t, v_1.e \delta \gamma \Downarrow_{t'} v_1$.

Since $t' < t < T$, therefore we have

$$(p_{l1} + p_{m1}, T - t', v_1) \in \llbracket (\tau_1 \oplus \tau_2) \sigma \iota \rrbracket$$

2 cases arise:

(a) $v_1 = \text{inl}(v)$:

IH2

$$(p_{l2} + p_{m2} p_{l1} + p_{m1}, T - t', e_1 \delta \gamma') \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

where

$$\gamma' = \gamma \cup \{x \mapsto v\}$$

This means from Definition 66 we have

$$\forall t_1 < T - t'. e_1 \delta \gamma' \Downarrow_{t_1} v_f \implies (p_{l2} + p_{m2} + p_{l1} + p_{m1}, T - t' - t_1, v_f) \in \llbracket \tau \sigma \iota \rrbracket$$

Since we know that (case e of $e_1; e_2$) $\delta \gamma \Downarrow_t v_f$ therefore from E-case we know that $\exists t_1.e_1 \delta \gamma' \Downarrow v_f$ where $t_1 = t - t' - 1$.

Since $t_1 = t - t' - 1 < T - t'$ therefore we have

$$(p_{l2} + p_{m2} + p_{l1} + p_{m1}, T - t' - t_1, v_f) \in \llbracket \tau \sigma \iota \rrbracket$$

From Lemma 69 we get

$$(p_{l2} + p_{m2} + p_{l1} + p_{m1}, T - t, v_f) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

And we are done

(b) $v_1 = \text{inr}(v)$:

Similar reasoning as in the inl case above.

22. T-subExpI:

$$\frac{\Psi; \Theta, a; \Delta, a < I; \Omega; . \vdash e : \tau}{\Psi; \Theta; \Delta; \sum_{a < I} \Omega; . \vdash !e : !_{a < I} \tau} \text{T-subExpI}$$

Given: $(p_l, \gamma) \in \llbracket \cdot \rrbracket_{\mathcal{E}}$, $(p_m, \delta) \in \llbracket (\sum_{a < I} \Omega) \sigma \iota \rrbracket_{\mathcal{E}}$ and $\models \Delta \iota$

To prove: $(p_l + p_m, !e \delta \gamma) \in \llbracket !_{a < I} \tau \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$\forall t < T . (!e) \delta \gamma \Downarrow_t (!e) \delta \gamma \implies (p_m + p_l, T - t, (!e) \delta \gamma) \in \llbracket !_{a < I} \tau \sigma \iota \rrbracket$$

This means given some $t < T$. s.t $(!e) \delta\gamma \Downarrow_t (!e) \delta\gamma$ it suffices to prove that

$$(p_m + p_l, T - t, (!e) \delta\gamma) \in \llbracket !_{a < I} \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

From Definition 6 it suffices to prove that

$$\exists p_0, \dots, p_{I-1}. p_0 + \dots + p_{I-1} \leq (p_m + p_l) \wedge \forall 0 \leq i < I. (p_i, T, e \delta\gamma) \in \llbracket \tau[i/a] \rrbracket_{\mathcal{E}} \quad (\text{F-SI0})$$

Since we know that $(p_m, T, \delta) \in \llbracket (\sum_{a < I} \Omega) \sigma \iota \rrbracket_{\mathcal{E}}$ therefore from Lemma 11 we know that

$$\exists p'_0, \dots, p'_{I-1}. p'_0 + \dots + p'_{I-1} \leq p_m \wedge \forall 0 \leq i < I. (p_i, T, \delta) \in \llbracket \Omega[i/a] \rrbracket_{\mathcal{E}} \quad (\text{F-SI1})$$

Instantiating IH with each $p'_0 \dots p'_{I-1}$ we get

$$(p'_0, T, e \delta\gamma) \in \llbracket \tau[0/a] \sigma \iota \rrbracket_{\mathcal{E}} \text{ and}$$

...

$$(p'_{I-1}, T, e \delta\gamma) \in \llbracket \tau[I-1/a] \sigma \iota \rrbracket_{\mathcal{E}} \quad (\text{F-SI2})$$

Therefore we get (F-SI0) from (F-SI1) and (F-SI2)

23. T-subExpE:

$$\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e : (!_{a < I} \tau) \quad \Psi; \Theta; \Delta; \Omega_2, x :_{a < I} \tau; \Gamma_2 \vdash e' : \tau'}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash \text{let } !x = e \text{ in } e' : \tau'} \text{ T-subExpE}$$

Given: $(p_l, \gamma) \in \llbracket (\Gamma_1 \oplus \Gamma_2) \sigma \iota \rrbracket_{\mathcal{E}}$, $(p_m, \delta) \in \llbracket (\Omega_1 \oplus \Omega_2) \sigma \iota \rrbracket_{\mathcal{E}}$ and $\models \Delta \iota$

To prove: $(p_l + p_m, (\text{let } !x = e \text{ in } e') \delta\gamma) \in \llbracket \tau' \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$\forall t < T, v_f. (\text{let } !x = e \text{ in } e') \delta\gamma \Downarrow_t v_f \implies (p_m + p_l, T - t, v_f) \in \llbracket \tau' \sigma \iota \rrbracket_{\mathcal{E}}$$

This means given some $t < T$ s.t. $(\text{let } !x = e \text{ in } e') \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_m + p_l, T - t, v_f) \in \llbracket \tau' \sigma \iota \rrbracket_{\mathcal{E}} \quad (\text{F-SE0})$$

From Definition 7 and Definition 5 we know that $\exists p_{l1}, p_{l2}. p_{l1} + p_{l2} = p_l$ s.t

$$(p_{l1}, T, \gamma) \in \llbracket (\Gamma_1) \sigma \iota \rrbracket_{\mathcal{E}} \text{ and } (p_{l2}, T, \gamma) \in \llbracket (\Gamma_2) \sigma \iota \rrbracket_{\mathcal{E}}$$

Similarly from Definition 7 and Definition 4 we also know that $\exists p_{m1}, p_{m2}. p_{m1} + p_{m2} = p_m$ s.t

$$(p_{m1}, T, \delta) \in \llbracket (\Omega_1) \sigma \iota \rrbracket_{\mathcal{E}} \text{ and } (p_{m2}, T, \delta) \in \llbracket (\Omega_2) \sigma \iota \rrbracket_{\mathcal{E}}$$

IH1

$$(p_{l1} + p_{m1}, T, e \delta\gamma) \in \llbracket !_{a < I} \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 6 we have

$$\forall t_1 < T. e \delta\gamma \Downarrow_{t_1} !e_1 \delta\gamma \implies (p_{l1} + p_{m1}, T - t_1, !e_1 \delta\gamma) \in \llbracket !_{a < I} \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

Since we know that $(\text{let } !x = e \text{ in } e') \delta\gamma \Downarrow_t v_f$ therefore from E-subExpE we know that $\exists t_1 < t, e_1. e \delta\gamma \Downarrow_{t_1} !e_1 \delta\gamma$. Therefore we have

$$(p_{l1} + p_{m1}, T - t_1, !e_1 \delta\gamma) \in \llbracket !_{a < I} \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

Therefore from Definition 6 we have

$$\exists p_0, \dots, p_{I-1}. p_0 + \dots + p_{I-1} \leq (p_{l_1} + p_{m_1}) \wedge \forall 0 \leq i < I. (p_i, T - t_1, e_1 \delta\gamma) \in \llbracket \tau[i/a] \rrbracket_{\mathcal{E}} \quad (\text{F-SE1})$$

IH2

$$(p_{l_2} + p_{m_2} + p_0 + \dots + p_{I-1}, T - t_1, e' \delta'\gamma) \in \llbracket \tau' \sigma\iota \rrbracket_{\mathcal{E}}$$

where

$$\delta' = \delta \cup \{x \mapsto e_1\}$$

This means from Definition 6 we have

$$\forall t_2 < T - t_1. e' \delta'\gamma \Downarrow_{t_2} v_f \implies (p_{l_2} + p_{m_2} + p_0 + \dots + p_{I-1}, T - t_1 - t_2, v_f) \in \llbracket \tau' \sigma\iota \rrbracket$$

Since we know that $(\text{let } !x = e \text{ in } e') \delta\gamma \Downarrow_t v_f$ therefore from E-subExpE we know that $\exists t_2. e' \delta'\gamma \Downarrow v_f$ s.t. $t_2 = t - t_1 - 1$. Therefore we have

$$(p_{l_2} + p_{m_2} + p_0 + \dots + p_{I-1}, T - t_1 - t_2, v_f) \in \llbracket \tau' \sigma\iota \rrbracket$$

Since from (F-SE1) we know that $p_0 + \dots + p_{I-1} \leq p_{l_1} + p_{m_1}$ therefore from Lemma 70 we get

$$(p_{l_2} + p_{m_2} + p_{l_1} + p_{m_1}, T - t, v_f) \in \llbracket \tau' \sigma\iota \rrbracket$$

And finally since $p_l = p_{l_1} + p_{l_2}$ and $p_m = p_{m_1} + p_{m_2}$ therefore we get

$$(p_l + p_m, T - t, v_f) \in \llbracket \tau' \sigma\iota \rrbracket$$

And we are done

24. T-tabs:

$$\frac{\Psi, \alpha : K; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \Lambda.e : (\forall \alpha : K . \tau)} \quad \text{T-tabs}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma\iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket \Omega \sigma\iota \rrbracket_{\mathcal{E}}$ and $\models \Delta \iota$

To prove: $(p_l + p_m, T, \Lambda.e \delta\gamma) \in \llbracket (\forall \alpha : K . \tau) \sigma\iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$\forall t < T, v. \Lambda.e \delta\gamma \Downarrow_t v \implies (p_m + p_l, T - t, v) \in \llbracket (\forall \alpha : K . \tau) \sigma\iota \rrbracket$$

This means given some v s.t. $\Lambda.e \delta\gamma \Downarrow v$ and from (E-val) we know that $v = \Lambda.e \delta\gamma$ and $t = 0$ therefore it suffices to prove that

$$(p_l + p_m, T, \Lambda.e \delta\gamma) \in \llbracket (\forall \alpha : K . \tau) \sigma\iota \rrbracket$$

From Definition 6 it suffices to prove that

$$\forall \tau', T' < T. (p_l + p_m, T', e \delta\gamma) \in \llbracket \tau[\tau'/\alpha] \sigma\iota \rrbracket_{\mathcal{E}}$$

This means given some $\tau', T' < T$ it suffices to prove that

$$(p_l + p_m, T', e \delta\gamma) \in \llbracket \tau[\tau'/\alpha] \sigma\iota \rrbracket_{\mathcal{E}} \quad (\text{F-TAb0})$$

$$\underline{\text{IH}} \quad (p_l + p_m, T, e \delta\gamma) \in \llbracket \tau \sigma'\iota \rrbracket_{\mathcal{E}}$$

where

$$\sigma' = \sigma \cup \{\alpha \mapsto \tau'\}$$

We get the desired directly from IH

25. T-tapp:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\forall \alpha : K . \tau) \quad \Psi; \Theta; \Delta \vdash \tau' : K}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e \square : (\tau[\tau'/\alpha])} \text{T-tapp}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$ and $\models \Delta \iota$

To prove: $(p_l + p_m, T, e \square \delta \gamma) \in \llbracket (\tau[\tau'/\alpha]) \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$\forall t < T, v_f. (e \square) \delta \gamma \Downarrow_t v_f \implies (p_m + p_l, T - t, v_f) \in \llbracket (\tau[\tau'/\alpha]) \sigma \iota \rrbracket$$

This means given some $t < T, v_f$ s.t $(e \square) \delta \gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_m + p_l, T - t, v_f) \in \llbracket (\tau[\tau'/\alpha]) \sigma \iota \rrbracket \quad (\text{F-Tap0})$$

IH

$$(p_l + p_m, T, e \delta \gamma) \in \llbracket (\forall \alpha. \tau) \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 6 we have

$$\forall t_1 < T, v'. e \delta \gamma \Downarrow_{t_1} v' \implies (p_l + p_m, T - t_1, v') \in \llbracket (\forall \alpha \tau) \sigma \iota \rrbracket$$

Since we know that $(e \square) \delta \gamma \Downarrow_t v_f$ therefore from E-tapp we know that $\exists t_1 < t. e \delta \gamma \Downarrow_{t_1} \Lambda.e$, therefore we have

$$(p_l + p_m, T - t_1, \Lambda.e) \in \llbracket (\forall \alpha. \tau) \sigma \iota \rrbracket$$

Therefore from Definition 6 we have

$$\forall \tau'', T_1 < T - t_1. (p_l + p_m, T - t_1 - T_1, e \delta \gamma) \in \llbracket \tau[\tau''/\alpha] \sigma \iota \rrbracket_{\mathcal{E}}$$

Instantiating it with the given τ' and $T - t_1 - 1$ we get

$$(p_l + p_m, T - t_1 - 1, e \delta \gamma) \in \llbracket \tau[\tau'/\alpha] \sigma \iota \rrbracket_{\mathcal{E}}$$

From Definition 6 we know that

$$\forall t_2 < T - t_1 - 1, v''. e \delta \gamma \Downarrow_{t_2} v'' \implies (p_l + p_m, T - t_1 - 1 - t_2, v'') \in \llbracket \tau[\tau'/\alpha] \sigma \iota \rrbracket$$

Since we know that $(e \square) \delta \gamma \Downarrow_t v_f$ therefore from E-tapp we know that $\exists t_2. e \Downarrow_{t_2} v_f$ where $t_2 = t - t_1 - 1$

Since $t_2 = t - t_1 - 1 < T - t_1 - 1$, therefore we have

$$(p_l + p_m, T - t, v_f) \in \llbracket \tau[\tau'/\alpha] \sigma \iota \rrbracket$$

And we are done.

26. T-iabs:

$$\frac{\Psi; \Theta, i : S; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \Lambda.e : (\forall i : S . \tau)} \text{ T-iabs}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma, \sigma \iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$ and $\models \Delta \iota$

To prove: $(p_l + p_m, T, \Lambda.e \delta \gamma) \in \llbracket (\forall i : S . \tau) \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$\forall t < T, v. \Lambda.e \delta \gamma \Downarrow_t v \implies (p_m + p_l, T - t, v) \in \llbracket (\forall i : S . \tau) \sigma \iota \rrbracket$$

This means given some $t < T, v$ s.t $\Lambda.e \delta \gamma \Downarrow_t v$ and from (E-val) we know that $v = \Lambda.e \delta \gamma$ and $t = 0$ therefore it suffices to prove that

$$(p_l + p_m, T, \Lambda.e \delta \gamma) \in \llbracket (\forall i : S . \tau) \sigma \iota \rrbracket$$

From Definition 6 it suffices to prove that

$$\forall I. (p_l + p_m, T, e) \in \llbracket \tau[I/i] \sigma \iota \rrbracket_{\mathcal{E}}$$

This means given some I it suffices to prove that

$$(p_l + p_m, T, e) \in \llbracket \tau[I/i] \sigma \iota \rrbracket_{\mathcal{E}} \quad (\text{F-TAb0})$$

$$\underline{\text{IH}} (p_l + p_m, T, e \delta \gamma) \in \llbracket \tau \sigma \iota' \rrbracket_{\mathcal{E}}$$

where

$$\iota' = \iota \cup \{i \mapsto I\}$$

We get the desired directly from IH

27. T-iapp:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\forall i : S . \tau) \quad \Theta \vdash I : S}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e [] : (\tau[I/i])} \text{ T-iapp}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$ and $\models \Delta \iota$

To prove: $(p_l + p_m, T, e [] \delta \gamma) \in \llbracket (\tau[I/i] \sigma \iota) \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$\forall t < T, v_f. (e []) \delta \gamma \Downarrow_t v_f \implies (p_m + p_l, T - t, v_f) \in \llbracket (\tau[I/i] \sigma \iota) \rrbracket$$

This means given some $t < T, v_f$ s.t $(e []) \delta \gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_m + p_l, T - t, v_f) \in \llbracket (\tau[I/i] \sigma \iota) \rrbracket \quad (\text{F-Iap0})$$

IH

$$(p_l + p_m, T, e \delta \gamma) \in \llbracket (\forall i : S . \tau) \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 6 we have

$$\forall t_1 < T, v'. e \delta \gamma \Downarrow_{t_1} v' \implies (p_l + p_m, T - t_1, v') \in \llbracket (\forall i : S . \tau) \sigma \iota \rrbracket$$

Since we know that $(e \square) \delta\gamma \Downarrow_t v_f$ therefore from (E-iapp) we know that $\exists t_1 < t.e\delta\gamma \Downarrow_{t_1} \Lambda.e$, therefore we have

$$(p_l + p_m, T - t_1, \Lambda.e) \in \llbracket (\forall i : S . \tau) \sigma \iota \rrbracket$$

Therefore from Definition 6 we have

$$\forall I'', T_1 < T - t_1. (p_l + p_m, T - t_1 - T_1, e \delta\gamma) \in \llbracket \tau[I''/i] \sigma \iota \rrbracket_{\mathcal{E}}$$

Instantiating it with the given I and $T - t_1 - 1$ we get

$$(p_l + p_m, T - t_1 - 1, e \delta\gamma) \in \llbracket \tau[I/i] \sigma \iota \rrbracket_{\mathcal{E}}$$

From Definition 6 we know that

$$\forall v'', t_2 < T - t_1 - 1. e \delta\gamma \Downarrow_{t_2} v'' \implies (p_l + p_m, T - t_1 - 1 - t_2, v'') \in \llbracket \tau[I/i] \sigma \iota \rrbracket$$

Since we know that $(e \square) \delta\gamma \Downarrow_t v_f$ therefore from E-iapp we know that $\exists t_2. e \Downarrow_{t_2} v_f$ where $t_2 = t - t_1 - 1$

Since $t_2 = t - t_1 - 1 < T - t_1 - 1$, therefore we have

$$(p_l + p_m, v_f) \in \llbracket \tau[I/i] \sigma \iota \rrbracket$$

And we are done.

28. T-CI:

$$\frac{\Psi; \Theta; \Delta, c; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \Lambda.e : (c \implies \tau)} \text{ T-CI}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$ and $\models \Delta \iota$

To prove: $(p_l + p_m, T, \Lambda.e \delta\gamma) \in \llbracket (c \implies \tau) \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$\forall v, t < T. \Lambda.e \delta\gamma \Downarrow_t v \implies (p_m + p_l, T - t, v) \in \llbracket (c \implies \tau) \sigma \iota \rrbracket$$

This means given some $v, t < T$ s.t $\Lambda.e \delta\gamma \Downarrow_t v$ and from (E-val) we know that $v = \Lambda.e \delta\gamma$ and $t = 0$ therefore it suffices to prove that

$$(p_l + p_m, T, \Lambda.e \delta\gamma) \in \llbracket (c \implies \tau) \sigma \iota \rrbracket$$

From Definition 6 it suffices to prove that

$$\forall T' < T. \models c \iota \implies (p_l + p_m, T', e \delta\gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

This means given some $T' < T$ s.t. $\models c \iota$ it suffices to prove that

$$(p_l + p_m, T', e \delta\gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

$$\underline{\text{IH}} (p_l + p_m, T', e \delta\gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

We get the desired directly from IH

29. T-CE:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (c \Rightarrow \tau) \quad \Theta; \Delta \models c}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e [] : \tau} \text{ T-CE}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$ and $\models \Delta \iota$

To prove: $(p_l + p_m, T, e [] \delta \gamma) \in \llbracket (\tau) \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$\forall v_f, t < T . (e []) \delta \gamma \Downarrow_t v_f \implies (p_m + p_l, T - t, v_f) \in \llbracket (\tau) \sigma \iota \rrbracket$$

This means given some $v_f, t < T$ s.t. $(e []) \delta \gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_m + p_l, T - t, v_f) \in \llbracket (\tau) \sigma \iota \rrbracket \quad (\text{F-Tap0})$$

IH

$$(p_l + p_m, T, e \delta \gamma) \in \llbracket (c \Rightarrow \tau) \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 6 we have

$$\forall v', t' < T . e \delta \gamma \Downarrow_{t'} v' \implies (p_l + p_m, T - t', v') \in \llbracket (c \Rightarrow \tau) \sigma \iota \rrbracket$$

Since we know that $(e []) \delta \gamma \Downarrow_t v_f$ therefore from E-CE we know that $\exists t' < t . e \delta \gamma \Downarrow_{t'} \Lambda . e'$, therefore we have

$$(p_l + p_m, T - t', \Lambda . e') \in \llbracket (c \Rightarrow \tau) \sigma \iota \rrbracket$$

Therefore from Definition 6 we have

$$\forall t'' < T - t' . \models c \iota \implies (p_l + p_m, T - t' - t'', e' \delta \gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

Since we are given $\Theta; \Delta \models c$ and $\models \Delta \iota$. Therefore instantiating it with $T - t' - 1$ and since we know that $\models c \iota$. Hence we get

$$(p_l + p_m, T - t' - 1, e' \delta \gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 6 we have

$$\forall v'_f, t'' < T - t' - 1 . (e') \delta \gamma \Downarrow_{t''} v'_f \implies (p_m + p_l, v'_f) \in \llbracket (\tau) \sigma \iota \rrbracket$$

Since from E-CE we know that $e' \delta \gamma \Downarrow_t v_f$ therefore we know that $\exists t'' . e' \delta \gamma \Downarrow_{t''} v'_f$ s.t $t = t' + t'' + 1$

Therefore instantiating (F-CE1) with the given v_f and t'' we get

$$(p_m + p_l, T - t, v_f) \in \llbracket (\tau) \sigma \iota \rrbracket$$

and we are done.

30. T-CAndI:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \quad \Theta; \Delta \models c}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (c \& \tau)} \text{ T-CAndI}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l + p_m, T, e \delta \gamma) \in \llbracket c \& \tau \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$\forall v_f, t < T . e \delta\gamma \Downarrow_t v_f \implies (p_l + p_m, T - t, v_f \delta\gamma) \in \llbracket c \& \tau \sigma \iota \rrbracket$$

This means given some $v_f, t < T$ s.t. $e \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_l + p_m, T - t, v_f) \in \llbracket c \& \tau \sigma \iota \rrbracket$$

From Definition 6 it suffices to prove that

$$. \models c \iota \wedge (p_l + p_m, T - t, v_f) \in \llbracket \tau \sigma \iota \rrbracket$$

Since we are given that $. \models \Delta \iota$ and $\Theta; \Delta \models c$ therefore it suffices to prove that

$$(p_l + p_m, T - t, v_f) \in \llbracket \tau \sigma \iota \rrbracket \quad (\text{F-CAI0})$$

$$\underline{\text{IH}}: (p_l + p_m, T, e \delta\gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 6 we have

$$\forall t' < T . e \delta\gamma \Downarrow_{t'} v_f \implies (p_l + p_m, T - t', v_f) \in \llbracket \tau \sigma \iota \rrbracket$$

Since we are given that $e \delta\gamma \Downarrow_t v_f$ therefore we get

$$(p_l + p_m, T - t, v_f) \in \llbracket \tau \sigma \iota \rrbracket \quad (\text{F-CAI1})$$

We get the desired from (F-CAI1)

31. T-CAndE:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : (c \& \tau) \quad \Psi; \Theta; \Delta, c; \Omega; \Gamma_2, x : \tau \vdash e' : \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \text{clet } x = e \text{ in } e' : \tau'} \text{T-CAndE}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma_1 \oplus \Gamma_2) \sigma \iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket (\Omega) \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l + p_m, T, (\text{clet } x = e \text{ in } e') \delta\gamma) \in \llbracket \tau' \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$\forall v_f, t < T . (\text{clet } x = e \text{ in } e') \delta\gamma \Downarrow_t v_f \implies (p_l + p_m, T - t, v_f) \in \llbracket \tau' \sigma \iota \rrbracket$$

This means given some $v_f, t < T$ s.t. $(\text{clet } x = e \text{ in } e') \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_l + p_m, T - t, v_f) \in \llbracket \tau' \sigma \iota \rrbracket \quad (\text{F-CAE0})$$

From Definition 7 and Definition 5 we know that $\exists p_{l1}, p_{l2}. p_{l1} + p_{l2} = p_l$ s.t

$(p_{l1}, T, \gamma) \in \llbracket (\Gamma_1) \sigma \iota \rrbracket_{\mathcal{E}}$ and $(p_{l2}, T, \gamma) \in \llbracket (\Gamma_2) \sigma \iota \rrbracket_{\mathcal{E}}$

Similarly from Definition 7 and Definition 4 we also know that

$\exists p_{m1}, p_{m2}. p_{m1} + p_{m2} = p_m$ s.t

$(p_{m1}, T, \delta) \in \llbracket (\Omega_1) \sigma \iota \rrbracket_{\mathcal{E}}$ and $(p_{m2}, T, \delta) \in \llbracket (\Omega_2) \sigma \iota \rrbracket_{\mathcal{E}}$

IH1

$$(p_{l1} + p_{m1}, T, e \delta\gamma) \in \llbracket c \& \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 6 we have

$$\forall t_1 < T . e \delta\gamma \Downarrow_{t_1} v_1 \implies (p_{l1}, T - t_1, v_1) \in \llbracket c \& \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

Since we know that $(\text{clet } x = e \text{ in } e') \delta\gamma \Downarrow_t v_f$ therefore from E-CAndE we know that $\exists t_1 < t, v_1.e \delta\gamma \Downarrow_{t_1} v_1$. Therefore we have

$$(p_{l_1} + p_{m_1}, T - t_1, v_1) \in \llbracket c \& \tau \sigma \iota \rrbracket$$

Therefore from Definition 6 we have

$$\cdot \models c \wedge (p_{l_1} + p_{m_1}, T - t_1, v_1) \in \llbracket \tau \sigma \iota \rrbracket \quad (\text{F-CAE1})$$

IH2

$$(p_{l_2} + p_{m_2} + p_{l_1} + p_{m_1}, T - t_1, e' \delta\gamma') \in \llbracket \tau' \sigma \iota \rrbracket_{\mathcal{E}}$$

where

$$\gamma' = \gamma \cup \{x \mapsto v_1\}$$

This means from Definition 6 we have

$$\forall t_2 < T . e' \delta\gamma' \Downarrow_{t_2} v_f \implies (p_{l_2} + p_{m_2} + p_{l_1} + p_{m_1}, T - t_1 - t_2, v_f) \in \llbracket \tau' \sigma \iota \rrbracket$$

Since we know that $(\text{clet } x = e \text{ in } e') \delta\gamma \Downarrow_t v_f$ therefore from E-CAndE we know that $\exists t_2.e' \delta\gamma' \Downarrow_{t_2} v_f$ s.t $t_2 = t - t_1 - 1$

Therefore we have

$$(p_{l_2} + p_{m_2} + p_{l_1} + p_{m_1}, T - t_1 - t_2, v_f) \in \llbracket \tau' \sigma \iota' \rrbracket$$

Since $p_l = p_{l_1} + p_{l_2}$ and $p_m = p_{m_1} + p_{m_2}$ therefore we get

$$(p_l + p_m, T - t, v_f) \in \llbracket \tau' \sigma \iota' \rrbracket$$

And we are done.

32. T-fix:

$$\frac{\Psi; \Theta; b; \Delta, b < L; \Omega, x :_{a < I} \tau[(b + 1 + \frac{b+1,a}{b} I)/b]; \cdot \vdash e : \tau \quad L \geq \frac{0,1}{b} I}{\Psi; \Theta; \Delta; \sum_{b < L} \Omega; \cdot \vdash \text{fix } x.e : \tau[0/b]} \text{ T-fix}$$

Given: $(p_l, T, \gamma) \in \llbracket \cdot \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket \sum_{b < L} \Omega \sigma \iota \rrbracket_{\mathcal{E}}$ and $\models \Delta \iota$

To prove: $(p_l + p_m, T, (\text{fix } x.e) \delta\gamma) \in \llbracket \tau[0/b] \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$\forall T' < T, v_f. (\text{fix } x.e) \delta\gamma \Downarrow_{T'} v_f \implies (p_m + p_l, T - T', v_f) \in \llbracket \tau[0/b] \sigma \iota \rrbracket$$

This means given some $t < T, v_f$ s.t. $\text{fix } x.e \delta\gamma \Downarrow_{T'} v_f$ therefore it suffices to prove that

$$(p_l + p_m, T - T', v_f) \in \llbracket \tau[0/b] \sigma \iota \rrbracket \quad (\text{F-FX0})$$

Also from Lemma 11 we know that

$$\exists p'_0, \dots, p'_{(I-1)}. p'_0 + \dots + p'_{(L-1)} \leq p_m \wedge \forall 0 \leq i < L. (p_i, \delta) \in \llbracket \Omega[i/a] \rrbracket_{\mathcal{E}}$$

We define

$$\begin{aligned}
p_N(\text{leaf}) &\triangleq p'_{\text{leaf}} \\
p_N(t) &\triangleq p'_t + (\sum_{a < I(t)} p_N((t+1 + \frac{\bigtriangleup^{t+1,a}}{b} I(b))))
\end{aligned}$$

Claim

$\forall 0 \leq t < L. (p_N(t), T, e \delta' \gamma) \in \llbracket \tau[t/b] \sigma \iota \rrbracket_{\mathcal{E}}$

where

$$\delta' = \delta \cup \{x \mapsto \text{fix } x.e\delta\}$$

This means given some t it suffices to prove

$$(p_N(t), T, e \delta' \gamma) \in \llbracket \tau[t/b] \sigma \iota \rrbracket_{\mathcal{E}}$$

We prove this by induction on t

Base case: when t is a leaf node (say l)

It suffices to prove that $(p'_l, T, e \delta' \gamma) \in \llbracket \tau[l/b] \sigma \iota \rrbracket_{\mathcal{E}}$

We know that $I(l) = 0$ therefore from IH (of the outer induction) we get the desired

Inductive case: when t is some arbitrary non-leaf node

From IH we know that

$$\forall a < I(t). (p_N(t'), T, e \delta' \gamma) \in \llbracket \tau[t'/b] \sigma \iota \rrbracket_{\mathcal{E}} \text{ where } t' = (t+1 + \frac{\bigtriangleup^{t+1,a}}{b} I(b))$$

Claim

$\forall \tau'. (p_N(t'), T, e \delta' \gamma) \in \llbracket \tau' \sigma \iota \rrbracket_{\mathcal{E}}$ where $\delta' = \delta \cup \{x \mapsto \text{fix } x.e\delta\} \implies$

$$(p_N(t'), T, \text{fix } x.e \delta \gamma) \in \llbracket \tau' \sigma \iota \rrbracket_{\mathcal{E}}$$

Proof is trivial □

Therefore we have

$$\forall a < I(t). (p_N(t'), T, \text{fix } x.e \delta \gamma) \in \llbracket \tau[t'/b] \sigma \iota \rrbracket_{\mathcal{E}} \text{ where } t' = (t+1 + \frac{\bigtriangleup^{t+1,a}}{b} I(b))$$

Now from the IH of the outer induction we get

$$(p'_t + \sum_{a < I} p_N(t'), T, e \delta' \gamma) \in \llbracket \tau[t/b] \sigma \iota \rrbracket_{\mathcal{E}}$$

Which means we get the desired i.e

$$(p_N(t), T, e \delta' \gamma) \in \llbracket \tau[t/b] \sigma \iota \rrbracket_{\mathcal{E}} \quad \square$$

Since we have proved

$$\forall 0 \leq t < L. (p_N(t), T, e \delta' \gamma) \in \llbracket \tau[t/b] \sigma \iota \rrbracket_{\mathcal{E}}$$

where

$$\delta' = \delta \cup \{x \mapsto \text{fix}x.e\}$$

Therefore from Definition 6 we have

$$\forall 0 \leq t < L. \forall T'' < T. e \delta' \gamma \Downarrow_{T''} v_f \implies (p_N(t), T - T'', v_f) \in \llbracket \tau[t/b] \sigma \iota \rrbracket_{\mathcal{E}}$$

Instantiating with t with 0 and since we know that $\text{fix}x.e \delta \gamma \Downarrow_{T'} v_f$ therefore know that $\exists T'' < T' . e \delta' \gamma \Downarrow_{T''} v_f$ where $T'' = T' - 1$

$$(p_N(0), T - T'', v_f) \in \llbracket \tau[0/b] \sigma \iota \rrbracket_{\mathcal{E}}$$

Since $p_N(0) \leq p_m$ therefore $p_N(0) \leq p_l + p_m$

And we get the (F-FX0) from Lemma 69

33. T-ret:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{ret } e : \mathbb{M}0 \tau} \text{T-ret}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$ and $\models \Delta \iota$

To prove: $(p_l + p_m, T, \text{ret } e \delta \gamma) \in \llbracket \mathbb{M}0 \tau \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$\forall t < T. (\text{ret } e) \delta \gamma \Downarrow (\text{ret } e) \delta \gamma \implies (p_m + p_l, T - t, (\text{ret } e) \delta \gamma) \in \llbracket \mathbb{M}0 \tau \sigma \iota \rrbracket$$

Since from E-val we know that $t = 0$ therefore it suffices to prove that

$$(p_m + p_l, T, (\text{ret } e) \delta \gamma) \in \llbracket \mathbb{M}0 \tau \sigma \iota \rrbracket$$

From Definition 6 it suffices to prove that

$$\forall n', t' < T, v_f. (\text{ret } e) \delta \gamma \Downarrow_{t'}^{n'} v_f \implies \exists p'. n' + p' \leq p_l + p_m \wedge (p', T - t', v_f) \in \llbracket \tau \rrbracket$$

This means given some $n', t' < T, v_f$ s.t. $(\text{ret } e) \delta \gamma \Downarrow_{t'}^{n'} v_f$ it suffices to prove that

$$\exists p'. n' + p' \leq p_l + p_m \wedge (p', T - t', v_f) \in \llbracket \tau \rrbracket$$

From (E-ret) we know that $n' = 0$ therefore we choose p' as $p_l + p_m$ and it suffices to prove that

$$(p_l + p_m, T - t', v_f) \in \llbracket \tau \sigma \iota \rrbracket \quad (\text{F-R0})$$

IIH

$$(p_l + p_m, T, e \delta \gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 6 we have

$$\forall t_1 < T. (e) \delta \gamma \Downarrow_{t_1} v_f \implies (p_m + p_l, T - t_1, v_f) \in \llbracket \tau \sigma \iota \rrbracket$$

Since we know that $(\text{ret } e) \delta \gamma \Downarrow_{t'}^0 v_f$ therefore from (E-ret) we know that $\exists t_1 < t. e \delta \gamma \Downarrow_{t'} v_f$ s.t $t_1 + 1 = t'$

Therefore we have $(p_m + p_l, T - t_1, v_f) \in \llbracket \tau \sigma \iota \rrbracket$ and from Lemma 69 we are done

34. T-bind:

$$\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e_1 : \mathbb{M} n_1 \tau_1 \quad \Psi; \Theta; \Delta; \Omega_2; \Gamma_2, x : \tau_1 \vdash e_2 : \mathbb{M} n_2 \tau_2 \quad \Theta \vdash n_1 : \mathbb{R}^+ \quad \Theta \vdash n_2 : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash \text{bind } x = e_1 \text{ in } e_2 : \mathbb{M}(n_1 + n_2) \tau_2} \text{T-bind}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma_1 \oplus \Gamma_2) \sigma \iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket (\Omega_1 \oplus \Omega_2) \sigma \iota \rrbracket_{\mathcal{E}}$ and $\models \Delta \iota$

To prove: $(p_l + p_m, T, \text{bind } x = e_1 \text{ in } e_2 \delta \gamma) \in \llbracket \mathbb{M}(n_1 + n_2) \tau_2 \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$\forall t < T, v. (\text{bind } x = e_1 \text{ in } e_2) \delta \gamma \Downarrow_t v \implies (p_m + p_l, T - t, (\text{bind } x = e_1 \text{ in } e_2) \delta \gamma) \in \llbracket \mathbb{M}(n_1 + n_2) \tau_2 \sigma \iota \rrbracket$$

This means given some $t < T, v$ s.t. $(\text{bind } x = e_1 \text{ in } e_2) \delta \gamma \Downarrow_t v$ and from E-val we know that $v = (\text{bind } x = e_1 \text{ in } e_2) \delta \gamma$ and $t = 0$. It suffices to prove that

$$(p_m + p_l, T, (\text{bind } x = e_1 \text{ in } e_2) \delta \gamma) \in \llbracket \mathbb{M}(n_1 + n_2) \tau_2 \sigma \iota \rrbracket$$

This means from Definition 6 it suffices to prove that

$$\forall s', t' < T, v_f. (\text{bind } x = e_1 \text{ in } e_2) \delta \gamma \Downarrow_{t'}^{s'} v_f \implies \exists p'. s' + p' \leq p_l + p_m + n \wedge (p', T - t', v_f) \in \llbracket \tau_2 \sigma \iota \rrbracket$$

This means given some $s', t' < T, v_f$ s.t. $(\text{bind } x = e_1 \text{ in } e_2) \delta \gamma \Downarrow_{t'}^{s'} v_f$ and we need to prove that

$$\exists p'. s' + p' \leq p_l + p_m + n \wedge (p', T - t', v_f) \in \llbracket \tau_2 \sigma \iota \rrbracket \quad (\text{F-B0})$$

From Definition 7 and Definition 5 we know that $\exists p_{l1}, p_{l2}. p_{l1} + p_{l2} = p_l$ s.t

$$(p_{l1}, T, \gamma) \in \llbracket (\Gamma_1) \sigma \iota \rrbracket_{\mathcal{E}} \text{ and } (p_{l2}, T, \gamma) \in \llbracket (\Gamma_2) \sigma \iota \rrbracket_{\mathcal{E}}$$

Similarly from Definition 7 and Definition 4 we also know that $\exists p_{m1}, p_{m2}. p_{m1} + p_{m2} = p_m$ s.t

$$(p_{m1}, T, \delta) \in \llbracket (\Omega_1) \sigma \iota \rrbracket_{\mathcal{E}} \text{ and } (p_{m2}, T, \delta) \in \llbracket (\Omega_2) \sigma \iota \rrbracket_{\mathcal{E}}$$

IH1

$$(p_{l1} + p_{m1}, T, e_1 \delta \gamma) \in \llbracket \mathbb{M}(n_1) \tau_1 \sigma \iota \rrbracket_{\mathcal{E}}$$

From Definition 6 it means we have

$$\forall t_1 < T. (e_1) \delta \gamma \Downarrow_{t_1} (e_1) \delta \gamma \implies (p_{m1} + p_{l1}, T - t_1, (e_1) \delta \gamma) \in \llbracket \mathbb{M}(n_1) \tau_1 \sigma \iota \rrbracket$$

Since we know that $(\text{bind } x = e_1 \text{ in } e_2) \delta \gamma \Downarrow_{t'}^{s'} v_f$ therefore from E-bind we know that $\exists t_1 < t', v_{m1}. (e_1) \delta \gamma \Downarrow_{t_1} (e_1) \delta \gamma$.

Since $t_1 < t' < T$, therefore we have

$$(p_{m1} + p_{l1}, T - t_1, (e_1) \delta \gamma) \in \llbracket \mathbb{M}(n_1) \tau_1 \sigma \iota \rrbracket$$

This means from Definition 6 we are given that

$$\forall t'_1 < T - t_1. (e_1) \delta \gamma \Downarrow_{t'_1}^{s_1} v_1 \implies \exists p'_1. s_1 + p'_1 \leq p_{l1} + p_{m1} + n_1 \wedge (p'_1, T - t_1 - t'_1, v_1) \in \llbracket \tau_1 \sigma \iota \rrbracket$$

Since we know that $(\text{bind } x = e_1 \text{ in } e_2) \delta \gamma \Downarrow_{t'} v_f$ therefore from E-bind we know that $\exists t'_1 < t' - t_1. (e_1) \delta \gamma \Downarrow_{t'_1}^{s_1} v_1$.

This means we have

$$\exists p'_1. s_1 + p'_1 \leq p_{l1} + p_{m1} + n_1 \wedge (p'_1, T - t_1 - t'_1, v_1) \in \llbracket \tau_1 \sigma \iota \rrbracket \quad (\text{F-B1})$$

IH2

$$(p_{l2} + p_{m2} + p'_1, T - t_1 - t'_1, e_2 \delta\gamma \cup \{x \mapsto v_1\}) \in \llbracket \mathbb{M}(n_2) \tau_2 \sigma \iota \rrbracket_{\mathcal{E}}$$

From Definition 6 it means we have

$$\forall t_2 < T - t_1 - t'_1. (e_2) \delta\gamma \cup \{x \mapsto v_1\} \Downarrow_{t_2} (e_2) \delta\gamma \cup \{x \mapsto v_1\} \implies (p_{m2} + p_{l2} + p'_1 + n_2, T - t_1 - t'_1 - t_2, (e_2) \delta\gamma \cup \{x \mapsto v_1\}) \in \llbracket \mathbb{M}(n_2) \tau_2 \sigma \iota \rrbracket$$

Since we know that $(\text{bind } x = e_1 \text{ in } e_2) \delta\gamma \Downarrow - \Downarrow_{t'}^- v_f$ therefore from E-bind we know that

$$\exists t_2 < t' - t_1 - t'_1. (e_2) \delta\gamma \cup \{x \mapsto v_1\} \Downarrow_{t_2} (e_2) \delta\gamma \cup \{x \mapsto v_1\}.$$

Since $t_2 < t' - t_1 - t'_1 < T - t_1 - t'_1$ therefore we have

$$(p_{m2} + p_{l2} + p'_1 + n_2, T - t_1 - t'_1 - t_2, (e_2) \delta\gamma \cup \{x \mapsto v_1\}) \in \llbracket \mathbb{M}(n_2) \tau_2 \sigma \iota \rrbracket$$

This means from Definition 6 we are given that

$$\forall t'_2 < T - t_1 - t'_1 - t_2. (e_2 \delta\gamma \cup \{x \mapsto v_1\}) \Downarrow_{t'_2}^{s_2} v_2 \implies \exists p'_2. s_2 + p'_2 \leq p_{l2} + p_{m2} + p'_1 + n_2 \wedge (p'_2, T - t_1 - t'_1 - t_2 - t'_2, v_2) \in \llbracket \tau_2 \sigma \iota \rrbracket$$

Since we know that $(\text{bind } x = e_1 \text{ in } e_2) \delta\gamma \Downarrow - \Downarrow_{t'}^- v_f$ therefore from E-bind we know that $\exists t'_2 < t' - t_1 - t'_1 - t_2, s_2, v_2. v_{m2} \Downarrow_{t'_2}^{s_2} v_2$.

This means we have

$$\exists p'_2. s_2 + p'_2 \leq p_{l2} + p_{m2} + p'_1 + n_2 \wedge (p'_2, T - t_1 - t'_1 - t_2 - t'_2, v_2) \in \llbracket \tau_2 \sigma \iota \rrbracket \quad (\text{F-B2})$$

In order to prove (F-B0) we choose p' as p'_2 and it suffices to prove

- (a) $s' + p'_2 \leq p_l + p_m + n$:

Since from (F-B2) we know that

$$s_2 + p'_2 \leq p_{l2} + p_{m2} + p'_1 + n_2$$

Adding s_1 on both sides we get

$$s_1 + s_2 + p'_2 \leq p_{l2} + p_{m2} + s_1 + p'_1 + n_2$$

Since from (F-B1) we know that

$$s_1 + p'_1 \leq p_{l1} + p_{m1} + n_1$$

therefore we also have

$$s_1 + s_2 + p'_2 \leq p_{l2} + p_{m2} + p_{l1} + p_{m1} + n_1 + n_2$$

And finally since we know that $n = n_1 + n_2$, $s' = s_1 + s_2$, $p_l = p_{l1} + p_{l2}$ and $p_m = p_{m1} + p_{m2}$ therefore we get the desired

- (b) $(p'_2, T - t_1 - t'_1 - t_2 - t'_2, v_f) \in \llbracket \tau_2 \sigma \iota \rrbracket$:

From E-bind we know that $v_f = v_2$ therefore we get the desired from (F-B2)

35. T-tick:

$$\frac{\Theta \vdash n : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \uparrow^n : \mathbb{M} n \mathbf{1}} \text{ T-tick}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$ and $\models \Delta \iota$

To prove: $(p_l + p_m, T, \uparrow^n \delta \gamma) \in \llbracket \mathbb{M} n \mathbf{1} \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$(\uparrow^n) \delta \gamma \Downarrow_0 (\uparrow^n) \delta \gamma \implies (p_m + p_l, T, (\uparrow^n) \delta \gamma) \in \llbracket \mathbb{M} n \mathbf{1} \sigma \iota \rrbracket$$

It suffices to prove that

$$(p_m + p_l, T, (\uparrow^n) \delta \gamma) \in \llbracket \mathbb{M} n \mathbf{1} \sigma \iota \rrbracket$$

From Definition 6 it suffices to prove that

$$\forall t' < T, n'. (\uparrow^n) \delta \gamma \Downarrow_{t'}^{n'} () \implies \exists p'. n' + p' \leq p_l + p_m + n \wedge (p', T - t', ()) \in \llbracket \mathbf{1} \rrbracket$$

This means given some $t' < T, n'$ s.t. $(\uparrow^n) \delta \gamma \Downarrow_{t'}^{n'} ()$ it suffices to prove that

$$\exists p'. n' + p' \leq p_l + p_m + n \wedge (p', T - t', ()) \in \llbracket \mathbf{1} \rrbracket$$

From (E-tick) we know that $n' = n$ therefore we choose p' as $p_l + p_m$ and it suffices to prove that

$$(p_l + p_m, T - t', ()) \in \llbracket \mathbf{1} \rrbracket$$

We get this directly from Definition 6

36. T-release:

$$\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e_1 : [n_1] \tau_1 \quad \Psi; \Theta; \Delta; \Omega_2; \Gamma_2, x : \tau_1 \vdash e_2 : \mathbb{M}(n_1 + n_2) \tau_2 \quad \Theta \vdash n_1 : \mathbb{R}^+ \quad \Theta \vdash n_2 : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash \text{release } x = e_1 \text{ in } e_2 : \mathbb{M} n_2 \tau_2} \text{ T-release}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma_1 \oplus \Gamma_2) \sigma \iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket (\Omega_1 \oplus \Omega_2) \sigma \iota \rrbracket_{\mathcal{E}}$ and $\models \Delta \iota$

To prove: $(p_l + p_m, T, \text{release } x = e_1 \text{ in } e_2 \delta \gamma) \in \llbracket \mathbb{M}(n_2) \tau_2 \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$(\text{release } x = e_1 \text{ in } e_2) \delta \gamma \Downarrow_0 (\text{release } x = e_1 \text{ in } e_2 \delta \gamma) \implies (p_m + p_l, (\text{release } x = e_1 \text{ in } e_2) \delta \gamma) \in \llbracket \mathbb{M}(n_2) \tau_2 \sigma \iota \rrbracket$$

This means given $(\text{release } x = e_1 \text{ in } e_2) \delta \gamma \Downarrow_0 (\text{release } x = e_1 \text{ in } e_2) \delta \gamma$ it suffices to prove that

$$(p_m + p_l, (\text{release } x = e_1 \text{ in } e_2) \delta \gamma) \in \llbracket \mathbb{M}(n_2) \tau_2 \sigma \iota \rrbracket$$

This means from Definition 6 it suffices to prove that

$$\forall t' < T, v_f, s'. (\text{release } x = e_1 \text{ in } e_2 \delta \gamma) \Downarrow_{t'}^{s'} v_f \implies \exists p'. s' + p' \leq p_l + p_m + n_2 \wedge (p', T - t', v_f) \in \llbracket \tau_2 \sigma \iota \rrbracket$$

This means given some $t' < T, v_f, s'$ s.t. $(\text{release } x = e_1 \text{ in } e_2 \delta\gamma) \Downarrow_{t'}^{s'} v_f$ and we need to prove that

$$\exists p'. s' + p' \leq p_l + p_m + n_2 \wedge (p', T - t', v_f) \in \llbracket \tau_2 \sigma \iota \rrbracket \quad (\text{F-R0})$$

From Definition 7 and Definition 5 we know that $\exists p_{l1}, p_{l2}. p_{l1} + p_{l2} = p_l$ s.t

$$(p_{l1}, T, \gamma) \in \llbracket (\Gamma_1) \sigma \iota \rrbracket_{\mathcal{E}} \text{ and } (p_{l2}, T, \gamma) \in \llbracket (\Gamma_2) \sigma \iota \rrbracket_{\mathcal{E}}$$

Similarly from Definition 7 and Definition 4 we also know that $\exists p_{m1}, p_{m2}. p_{m1} + p_{m2} = p_m$ s.t

$$(p_{m1}, T, \delta) \in \llbracket (\Omega_1) \sigma \iota \rrbracket_{\mathcal{E}} \text{ and } (p_{m2}, T, \delta) \in \llbracket (\Omega_2) \sigma \iota \rrbracket_{\mathcal{E}}$$

IH1

$$(p_{l1} + p_{m1}, T, e_1 \delta\gamma) \in \llbracket [n_1] \tau_1 \sigma \iota \rrbracket_{\mathcal{E}}$$

From Definition 6 it means we have

$$\forall t_1 < T. (e_1) \delta\gamma \Downarrow_{t_1} v_1 \implies (p_{m1} + p_{l1}, T - t_1, v_1) \in \llbracket [n_1] \tau_1 \sigma \iota \rrbracket$$

Since we know that $(\text{release } x = e_1 \text{ in } e_2) \delta\gamma \Downarrow - \Downarrow_{t'}^- v_f$ therefore from E-rel we know that $\exists t_1 < t'. (e_1) \delta\gamma \Downarrow_{t_1} v_1$. This means we have

$$(p_{m1} + p_{l1}, T - t_1, v_1) \in \llbracket [n_1] \tau_1 \sigma \iota \rrbracket$$

This means from Definition 6 we have

$$\exists p'_1. p'_1 + n_1 \leq p_{l1} + p_{m1} \wedge (p'_1, T - t_1, v_1) \in \llbracket \tau_1 \rrbracket \quad (\text{F-R1})$$

IH2

$$(p_{l2} + p_{m2} + p'_1, T - t_1, e_2 \delta\gamma \cup \{x \mapsto v_1\}) \in \llbracket \mathbb{M}(n_1 + n_2) \tau_2 \sigma \iota \rrbracket_{\mathcal{E}}$$

From Definition 6 it means we have

$$\forall t_2 < T - t_1. (e_2) \delta\gamma \cup \{x \mapsto v_1\} \Downarrow_{t_2} (e_2) \delta\gamma \cup \{x \mapsto v_1\} \implies (p_{m2} + p_{l2} + p'_1 + n_2, T - t_1 - t_2, (e_2) \delta\gamma \cup \{x \mapsto v_1\}) \in \llbracket \mathbb{M}(n_1 + n_2) \tau_2 \sigma \iota \rrbracket$$

Since we know that $(\text{release } x = e_1 \text{ in } e_2) \delta\gamma \Downarrow - \Downarrow_{t'}^- v_f$ therefore from E-rel we know that $\exists t_2 < t - t_1. (e_2) \delta\gamma \cup \{x \mapsto v_1\} \Downarrow_{t_2} (e_2) \delta\gamma \cup \{x \mapsto v_1\}$. This means we have

$$(p_{m2} + p_{l2} + p'_1 + n_2, T - t_1 - t_2, (e_2) \delta\gamma \cup \{x \mapsto v_1\}) \in \llbracket \mathbb{M}(n_1 + n_2) \tau_2 \sigma \iota \rrbracket$$

This means from Definition 6 we are given that

$$\forall t'_2 < T - t_1 - t_2. (e_2) \delta\gamma \cup \{x \mapsto v_1\} \Downarrow_{t'_2}^{s_2} v_2 \implies \exists p'_2. s_2 + p'_2 \leq p_{l2} + p_{m2} + p'_1 + n_1 + n_2 \wedge (p'_2, T - t_1 - t_2 - t'_2, v_2) \in \llbracket \tau_2 \sigma \iota \rrbracket$$

Since we know that $(\text{release } x = e_1 \text{ in } e_2) \delta\gamma \Downarrow - \Downarrow_{t'}^- v_f$ therefore from E-rel we know that $\exists t'_2. (e_2) \delta\gamma \cup \{x \mapsto v_1\} \Downarrow^{s_2} v_2$ s.t. $t'_2 = t' - t_1 - t_2 - 1$

Since $t'_2 = t' - t_1 - t_2 < T - t_1 - t_2 - 1 < T - t_1 - t_2$, therefore we have

$$\exists p'_2. s_2 + p'_2 \leq p_{l2} + p_{m2} + p'_1 + n_1 + n_2 \wedge (p'_2, T - t_1 - t_2 - t'_2, v_2) \in \llbracket \tau_2 \sigma \iota \rrbracket \quad (\text{F-R2})$$

In order to prove (F-R0) we choose p' as p'_2 and it suffices to prove

(a) $s' + p'_2 \leq p_l + p_m + n_2$:

Since from (F-R2) we know that

$$s_2 + p'_2 \leq p_{l2} + p_{m2} + p'_1 + n_1 + n_2$$

Since from (F-R1) we know that

$$p'_1 + n_1 \leq p_{l1} + p_{m1}$$

therefore we also have

$$s_2 + p'_2 \leq p_{l2} + p_{m2} + p_{l1} + p_{m1} + n_2$$

And finally since we know that $s' = s_2$, $p_l = p_{l1} + p_{l2}$ and $p_m = p_{m1} + p_{m2}$ therefore we get the desired

(b) $(p'_2, T - t', v_f) \in \llbracket \tau_2 \sigma \iota \rrbracket$:

From E-rel we know that $v_f = v_2$ therefore we get the desired from (F-R2) and Lemma 69

37. T-store:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \quad \Theta \vdash n : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{store } e : \mathbb{M} n ([n] \tau)} \text{T-store}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}$, $(p_m, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$ and $\models \Delta \iota$

To prove: $(p_l + p_m, T, \text{store } e \delta \gamma) \in \llbracket \mathbb{M} n ([n] \tau) \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 6 it suffices to prove that

$$(\text{store } e) \delta \gamma \Downarrow (\text{store } e) \delta \gamma \implies (p_m + p_l, T, (\text{store } e) \delta \gamma) \in \llbracket \mathbb{M} n ([n] \tau) \sigma \iota \rrbracket$$

It suffices to prove that

$$(p_m + p_l, T, (\text{store } e) \delta \gamma) \in \llbracket \mathbb{M} n ([n] \tau) \sigma \iota \rrbracket$$

From Definition 6 it suffices to prove that

$$\forall t' < T, v_f, n'. (\text{store } e) \delta \gamma \Downarrow_{v_f}^{n'} v_f \implies \exists p'. n' + p' \leq p_l + p_m + n \wedge (p', T - t', v_f) \in \llbracket [n] \tau \sigma \iota \rrbracket$$

This means given some $t' < T, v_f, n'$ s.t. $(\text{store } e) \delta \gamma \Downarrow_{v_f}^{n'}$ it suffices to prove that

$$\exists p'. n' + p' \leq p_l + p_m + n \wedge (p', T - t', v_f) \in \llbracket [n] \tau \sigma \iota \rrbracket$$

From (E-store) we know that $n' = 0$ therefore we choose p' as $p_l + p_m + n$ and it suffices to prove that

$$(p_l + p_m + n, T - t', v_f) \in \llbracket [n] \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

This further means that from Definition 6 we have

$$\exists p''. p'' + n \leq p_l + p_m + n \wedge (p'', T - t', v_f) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

We choose p'' as $p_l + p_m$ and it suffices to prove that

$$(p_l + p_m, T - t', v_f) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}} \quad (\text{F-S0})$$

III

$$(p_l + p_m, T, e \delta\gamma) \in \llbracket \tau \sigma\iota \rrbracket_{\mathcal{E}}$$

This means from Definition 6 we have

$$\forall t_1 < T. (e) \delta\gamma \Downarrow_{t_1} v_f \implies (p_m + p_l, T - t_1, v_f) \in \llbracket \tau \sigma\iota \rrbracket_{\mathcal{E}}$$

Since we know that $(\text{store } e) \delta\gamma \Downarrow - \Downarrow_{t'}^0 v_f$ therefore from (E-store) we know that $\exists t_1 < t'. e \delta\gamma \Downarrow_{t_1} v_f$ where $t_1 + 1 = t'$

Therefore from Lemma 69 we get $(p_m + p_l, T - t_1, v_f) \in \llbracket \tau \sigma\iota \rrbracket_{\mathcal{E}}$ and we are done

□

Lemma 13 (Γ Subtyping: domain containment). $\forall p, \gamma, \Gamma_1, \Gamma_2.$

$$\Psi; \Theta; \Delta \vdash \Gamma_1 <: \Gamma_2 \implies \forall x : \tau \in \Gamma_2. x : \tau' \in \Gamma_1 \wedge \Psi; \Theta; \Delta \vdash \tau' <: \tau$$

Proof. Proof by induction on $\Psi; \Theta; \Delta \vdash \Gamma_1 <: \Gamma_2$

1. sub-lBase:

$$\frac{}{\Psi; \Theta; \Delta \vdash \Gamma_1 <: .} \text{sub-lBase}$$

To prove: $\forall x : \tau' \in (.). x : \tau \in \Gamma_1 \wedge \Psi; \Theta; \Delta \vdash \tau' <: \tau$

Trivial

2. sub-lInd:

$$\frac{x : \tau' \in \Gamma_1 \quad \Psi; \Theta; \Delta \vdash \tau' <: \tau \quad \Psi; \Theta; \Delta \vdash \Gamma_1/x <: \Gamma_2}{\Psi; \Theta; \Delta \vdash \Gamma_1 <: \Gamma_2, x : \tau} \text{sub-lBase}$$

To prove: $\forall y : \tau \in \Gamma_2. y : \tau \in \Gamma_1 \wedge \Psi; \Theta; \Delta \vdash \tau' <: \tau$

This means given some $y : \tau \in (\Gamma_2, x : \tau)$ it suffices to prove that

$$y : \tau \in \Gamma_1 \wedge \Psi; \Theta; \Delta \vdash \tau' <: \tau$$

The following cases arise:

- $y = x$:

In this case we are given that $x : \tau' \in \Gamma_1 \wedge \Psi; \Theta; \Delta \vdash \tau' <: \tau$

Therefore we are done

- $y \neq x$:

Since we are given that $\Psi; \Theta; \Delta \vdash \Gamma_1/x <: \Gamma_2$ therefore we get the desired from IH

□

Lemma 14 (Ω Subtyping: domain containment). $\forall p, \gamma, \Omega_1, \Omega_2.$

$$\Psi; \Theta; \Delta \vdash \Omega_1 <: \Omega_2 \implies$$

$$\forall x :_{a < I} \tau \in \Omega_2. x :_{a < J} \tau' \in \Omega_1 \wedge \Psi; \Theta; \Delta \vdash I \leq J \wedge \Psi; \Theta, a; \Delta, a < I \vdash \tau' <: \tau$$

Proof. Proof by induction on $\Psi; \Theta; \Delta \vdash \Omega_1 <: \Omega_2$

1. sub-lBase:

$$\frac{}{\Psi; \Theta; \Delta \vdash \Omega <: .} \text{ sub-mBase}$$

To prove: $\forall x :_{a < I} \tau \in (\cdot). x :_{a < J} \tau' \in \Omega_1 \wedge \Psi; \Theta; \Delta \vdash I \leq J \wedge \Psi; \Theta, a; \Delta, a < I \vdash \tau' <: \tau$
Trivial

2. sub-lInd:

$$\frac{\Psi; \Theta, a; \Delta, a < I \vdash \tau' <: \tau \quad \frac{x :_{a < J} \tau' \in \Omega_1 \quad \Theta; \Delta \vdash I \leq J \quad \Psi; \Theta; \Delta \vdash \Omega_1/x <: \Omega_2}{\Psi; \Theta; \Delta \vdash \Omega_1/x <: \Omega_2} \text{ sub-mInd}}{\Psi; \Theta; \Delta \vdash \Omega_1 <: \Omega_2, x :_{a < I} \tau} \text{ sub-lInd}$$

To prove: $\forall y :_{a < I} \tau \in \Omega_2. y :_{a < J} \tau' \in \Omega_1 \wedge \Psi; \Theta; \Delta \vdash I \leq J \wedge \Psi; \Theta, a; \Delta, a < I \vdash \tau' <: \tau$

This means given some $y :_{a < I} \tau \in (\Omega_2, x :_{a < I} \tau)$ it suffices to prove that

$y :_{a < J} \tau' \in \Omega_1 \wedge \Psi; \Theta; \Delta \vdash I \leq J \wedge \Psi; \Theta, a; \Delta, a < I \vdash \tau' <: \tau$

The following cases arise:

- $y = x$:

In this case we are given that

$x :_{a < J} \tau' \in \Omega_1 \wedge \Psi; \Theta; \Delta \vdash I \leq J \wedge \Psi; \Theta, a; \Delta, a < I \vdash \tau' <: \tau$

Therefore we are done

- $y \neq x$:

Since we are given that $\Psi; \Theta; \Delta \vdash \Omega_1/x <: \Omega_2$ therefore we get the desired from IH

□

Lemma 15 (Γ subtyping lemma). $\forall p, \gamma, \Gamma_1, \Gamma_2, \sigma, \iota.$

$$\Psi; \Theta; \Delta \vdash \Gamma_1 <: \Gamma_2 \implies \llbracket \Gamma_1 \sigma \iota \rrbracket \subseteq \llbracket \Gamma_2 \sigma \iota \rrbracket$$

Proof. Proof by induction on $\Psi; \Theta; \Delta \vdash \Gamma_1 <: \Gamma_2$

1. sub-lBase:

$$\frac{}{\Psi; \Theta; \Delta \vdash \Gamma <: .} \text{ sub-lBase}$$

To prove: $\forall (p, T, \gamma) \in \llbracket \Gamma_1 \sigma \iota \rrbracket_{\mathcal{E}}. (p, T, \gamma) \in \llbracket \cdot \rrbracket_{\mathcal{E}}$

This means given some $(p, T, \gamma) \in \llbracket \Gamma_1 \sigma \iota \rrbracket_{\mathcal{E}}$ it suffices to prove that $(p, T, \gamma) \in \llbracket \cdot \rrbracket_{\mathcal{E}}$

From Definition 7 it suffices to prove that

$\exists f : \text{Vars} \rightarrow \text{Pots}. (\forall x \in \text{dom}(\cdot). (f(x), T, \gamma(x)) \in \llbracket \Gamma(x) \rrbracket_{\mathcal{E}}) \wedge (\sum_{x \in \text{dom}(\cdot)} f(x) \leq p)$

We choose f as a constant function $f' - = 0$ and we get the desired

2. sub-lInd:

$$\frac{x : \tau' \in \Gamma_1 \quad \Psi; \Theta; \Delta \vdash \tau' <: \tau \quad \Psi; \Theta; \Delta \vdash \Gamma_1/x <: \Gamma_2}{\Psi; \Theta; \Delta \vdash \Gamma_1 <: \Gamma_2, x : \tau} \text{sub-lBase}$$

To prove: $\forall (p, T, \gamma) \in \llbracket \Gamma_1 \sigma \iota \rrbracket_{\mathcal{E}}. (p, T, \gamma) \in \llbracket \Gamma_2, x : \tau \rrbracket_{\mathcal{E}}$

This means given some $(p, T, \gamma) \in \llbracket \Gamma_1 \sigma \iota \rrbracket_{\mathcal{E}}$ it suffices to prove that $(p, T, \gamma) \in \llbracket \Gamma_2, x : \tau \rrbracket_{\mathcal{E}}$

This means from Definition 7 we are given that

$\exists f : \mathcal{V}ars \rightarrow \mathcal{P}ots.$

$$(\forall x \in \text{dom}(\Gamma_1). (f(x), T, \gamma(x)) \in \llbracket \Gamma(x) \rrbracket_{\mathcal{E}}) \quad (\text{L0})$$

$$(\sum_{x \in \text{dom}(\Gamma_1)} f(x) \leq p) \quad (\text{L1})$$

Similarly from Definition 7 it suffices to prove that

$$\exists f' : \mathcal{V}ars \rightarrow \mathcal{P}ots. (\forall y \in \text{dom}(\Gamma_2, x : \tau). (f'(y), T, \gamma(y)) \in \llbracket \Gamma(y) \rrbracket_{\mathcal{E}}) \wedge (\sum_{y \in \text{dom}(\Gamma_2, x : \tau)} f'(y) \leq p)$$

We choose f' as f and it suffices to prove that

$$(a) \forall y \in \text{dom}(\Gamma_2, x : \tau). (f(y), T, \gamma(y)) \in \llbracket \Gamma(y) \rrbracket_{\mathcal{E}}:$$

This means given some $y \in \text{dom}(\Gamma_2, x : \tau)$ it suffices to prove that

$$(f(y), T, \gamma(y)) \in \llbracket \tau_2 \rrbracket_{\mathcal{E}} \text{ where say } \Gamma(y) = \tau_2$$

From Lemma 13 we know that

$$y : \tau_1 \in \Gamma_1 \wedge \Psi; \Theta; \Delta \vdash \tau_1 <: \tau_2$$

By instantiating (L0) with the given y

$$(f(y), T, \gamma(y)) \in \llbracket \tau_1 \rrbracket_{\mathcal{E}}$$

Finally from Lemma 18 we also get $(f(y), T, \gamma(y)) \in \llbracket \tau_2 \rrbracket_{\mathcal{E}}$

And we are done

$$(b) (\sum_{y \in \text{dom}(\Gamma_2, x : \tau)} f(y) \leq p):$$

From (L1) we know that $(\sum_{x \in \text{dom}(\Gamma_1)} f(x) \leq p)$ and since from Lemma 13 we know that $\text{dom}(\Gamma_2, x : \tau) \subseteq \text{dom}(\Gamma_1)$ therefore we also have

$$(\sum_{y \in \text{dom}(\Gamma_2, x : \tau)} f(y) \leq p)$$

□

Lemma 16 (Ω subtyping lemma). $\forall p, \gamma, \Omega_1, \Omega_2, \sigma, \iota.$

$$\Psi; \Theta; \Delta \vdash \Omega_1 <: \Omega_2 \implies \llbracket \Omega_1 \sigma \iota \rrbracket \subseteq \llbracket \Omega_2 \sigma \iota \rrbracket$$

Proof. Proof by induction on $\Psi; \Theta; \Delta \vdash \Omega_1 <: \Omega_2$

1. sub-lBase:

$$\frac{}{\Psi; \Theta; \Delta \vdash \Omega <: .} \text{sub-mBase}$$

To prove: $\forall (p, T, \gamma) \in \llbracket \Omega_1 \sigma \iota \rrbracket_{\mathcal{E}}. (p, T, \gamma) \in \llbracket . \rrbracket_{\mathcal{E}}$

This means given some $(p, T, \gamma) \in \llbracket \Omega_1 \sigma \iota \rrbracket_{\mathcal{E}}$ it suffices to prove that $(p, T, \gamma) \in \llbracket \cdot \rrbracket_{\mathcal{E}}$

From Definition 7 it suffices to prove that

$$\exists f : \mathcal{Vars} \rightarrow \mathcal{Indices} \rightarrow \mathcal{Pots}. (\forall (x :_{a < I} \tau) \in \dots \forall 0 \leq i < I. (f \ x \ i, T, \delta(x)) \in \llbracket \tau[i/a] \rrbracket_{\mathcal{E}}) \wedge (\sum_{x:_{a < I} \tau \in \dots} \sum_{0 \leq i < I} f \ x \ i) \leq p$$

We choose f as a constant function $f' - = 0$ and we get the desired

2. sub-Ind:

$$\frac{\Psi; \Theta, a; \Delta, a < I \vdash \tau' <: \tau \quad x :_{a < J} \tau' \in \Omega_1 \quad \Theta; \Delta \vdash I \leq J \quad \Psi; \Theta; \Delta \vdash \Omega_1/x <: \Omega_2}{\Psi; \Theta; \Delta \vdash \Omega_1 <: \Omega_2, x :_{a < I} \tau} \text{sub-mInd}$$

To prove: $\forall (p, T, \gamma) \in \llbracket \Omega_1 \sigma \iota \rrbracket_{\mathcal{E}}. (p, T, \gamma) \in \llbracket \Omega_2, x : \tau \rrbracket_{\mathcal{E}}$

This means given some $(p, T, \gamma) \in \llbracket \Omega_1 \sigma \iota \rrbracket_{\mathcal{E}}$ it suffices to prove that $(p, T, \gamma) \in \llbracket \Omega_2, x : \tau \rrbracket_{\mathcal{E}}$

This means from Definition 7 we are given that

$$\exists f : \mathcal{Vars} \rightarrow \mathcal{Pots}.$$

$$(\forall (x :_{a < I} \tau) \in \Omega_1. \forall 0 \leq i < I. (f \ x \ i, T, \delta(x)) \in \llbracket \tau[i/a] \rrbracket_{\mathcal{E}}) \quad (\text{L0})$$

$$(\sum_{x:_{a < I} \tau \in \Omega_1} \sum_{0 \leq i < I} f \ x \ i) \leq p \quad (\text{L1})$$

Similarly from Definition 7 it suffices to prove that

$$\exists f' : \mathcal{Vars} \rightarrow \mathcal{Indices} \rightarrow \mathcal{Pots}. (\forall (y :_{a < I_y} \tau_y) \in \Omega_2, x : \tau. \forall 0 \leq i < I_y. (f' \ x \ i, T, \delta(y)) \in \llbracket \tau_y[i/a] \rrbracket_{\mathcal{E}}) \wedge (\sum_{y:_{a < I_y} \tau_y \in \Omega_2, x : \tau} \sum_{0 \leq i < I_y} f' \ y \ i) \leq p$$

We choose f' as f and it suffices to prove that

$$(a) (\forall (y :_{a < I_y} \tau_y) \in \Omega_2, x : \tau. \forall 0 \leq i < I_y. (f \ x \ i, T, \delta(y)) \in \llbracket \tau_y[i/a] \rrbracket_{\mathcal{E}}):$$

This means given some $(y :_{a < I_y} \tau_y) \in \Omega_2, x : \tau$ and some $0 \leq i < I_y$ it suffices to prove that

$$(f \ x \ i, T, \delta(y)) \in \llbracket \tau_y[i/a] \rrbracket_{\mathcal{E}}$$

From Lemma 13 we know that

$$y :_{a < J_y} \tau_1 \in \Omega_1 \wedge \Psi; \Theta; \Delta \vdash I_y \leq J_y \wedge \Psi; \Theta, a; \Delta, a < I_y \vdash \tau_1 <: \tau_y$$

Instantiating (L0) with the given y and i we get

$$(f \ x \ i, T, \delta(y)) \in \llbracket \tau_1[i/a] \rrbracket_{\mathcal{E}}$$

Finally using Lemma 18 we also get

$$(f \ x \ i, T, \delta(y)) \in \llbracket \tau_y[i/a] \rrbracket_{\mathcal{E}}$$

$$(b) (\sum_{y:_{a < I_y} \tau_y \in \Omega_2, x : \tau} \sum_{0 \leq i < I_y} f' \ y \ i) \leq p:$$

From Lemma 14 we know that

$$\forall y :_{a < I_y} \tau_y \in (\Omega_2, x : \tau). y :_{a < J_y} \tau_1 \in \Omega_1 \wedge \Psi; \Theta; \Delta \vdash I_y \leq J_y \wedge \Psi; \Theta, a; \Delta, a < I_y \vdash \tau_1 <: \tau_y$$

And since from (L1) we know that $(\sum_{x:_{a < I} \tau \in \Omega_1} \sum_{0 \leq i < I} f \ x \ i) \leq p$ therefore we also have

$$(\sum_{y:_{a < I_y} \tau_y \in \Omega_2, x : \tau} \sum_{0 \leq i < I_y} f' \ y \ i) \leq p$$

□

Lemma 17 (Value subtyping lemma). $\forall \Psi, \Theta, \Delta, \tau \in \text{Type}, \tau', \sigma, \iota.$
 $\Psi; \Theta; \Delta \vdash \tau <: \tau' \wedge . \models \Delta \iota \implies \llbracket \tau \sigma \iota \rrbracket \subseteq \llbracket \tau' \sigma \iota \rrbracket$

Proof. Proof by induction on the $\Psi; \Theta; \Delta \vdash \tau <: \tau'$ relation

1. sub-refl:

$$\frac{}{\Psi; \Theta; \Delta \vdash \tau <: \tau} \text{sub-refl}$$

To prove: $\forall (p, T, v) \in \llbracket \tau \sigma \iota \rrbracket \implies (p, T, v) \in \llbracket \tau \sigma \iota \rrbracket$

Trivial

2. sub-arrow:

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1' <: \tau_1 \quad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau_2'}{\Psi; \Theta; \Delta \vdash \tau_1 \multimap \tau_2 <: \tau_1' \multimap \tau_2'} \text{sub-arrow}$$

To prove: $\forall (p, T, \lambda x.e) \in \llbracket (\tau_1 \multimap \tau_2) \sigma \iota \rrbracket \implies (p, T, \lambda x.e) \in \llbracket (\tau_1' \multimap \tau_2') \sigma \iota \rrbracket$

This means given some $(p, T, \lambda x.e) \in \llbracket (\tau_1 \multimap \tau_2) \sigma \iota \rrbracket$ we need to prove

$(p, T, \lambda x.e) \in \llbracket (\tau_1' \multimap \tau_2') \sigma \iota \rrbracket$

From Definition 6 we are given that

$$\forall p', e', T' < T . (p', T', e') \in \llbracket \tau_1 \sigma \iota \rrbracket_{\mathcal{E}} \implies (p + p', T', e[e'/x]) \in \llbracket \tau_2 \sigma \iota \rrbracket_{\mathcal{E}} \quad (\text{F-SL0})$$

Also from Definition 6 it suffices to prove that

$$\forall p', e', T'' < T . (p', T'', e') \in \llbracket \tau_1' \sigma \iota \rrbracket_{\mathcal{E}} \implies (p + p', T'', e[e'/x]) \in \llbracket \tau_2' \sigma \iota \rrbracket_{\mathcal{E}}$$

This means given some p', e', T'' s.t $(p', T'', e') \in \llbracket \tau_1' \sigma \iota \rrbracket_{\mathcal{E}}$ we need to prove

$$(p + p', T'', e[e'/x]) \in \llbracket \tau_2' \sigma \iota \rrbracket_{\mathcal{E}} \quad (\text{F-SL1})$$

Since $\Psi; \Theta; \Delta \vdash \tau_1' <: \tau_1$ therefore from Lemma 18 we know that given some $(p', T'', e'') \in \llbracket \tau_1' \sigma \iota \rrbracket$ we also have $(p', T'', e'') \in \llbracket \tau_1 \sigma \iota \rrbracket$

Therefore instantiating (F-SL0) with p', e'', T'' we get

$$(p + p', T'', e[e''/x]) \in \llbracket \tau_2 \sigma \iota \rrbracket_{\mathcal{E}}$$

From Lemma 18 we get the desired

3. sub-tensor:

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau_1' \quad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau_2'}{\Psi; \Theta; \Delta \vdash \tau_1 \otimes \tau_2 <: \tau_1' \otimes \tau_2'} \text{sub-tensor}$$

To prove: $\forall (p, T, \langle\langle v_1, v_2 \rangle\rangle) \in \llbracket (\tau_1 \otimes \tau_2) \sigma \iota \rrbracket \implies (p, T, \langle\langle v_1, v_2 \rangle\rangle) \in \llbracket (\tau_1' \otimes \tau_2') \sigma \iota \rrbracket$

This means given $(p, T, \langle\langle v_1, v_2 \rangle\rangle) \in \llbracket (\tau_1 \otimes \tau_2) \sigma \iota \rrbracket$

It suffices prove that

$$(p, T, \langle\langle v_1, v_2 \rangle\rangle) \in \llbracket (\tau'_1 \otimes \tau'_2) \sigma \rrbracket$$

This means from Definition 6 we are given that

$$\exists p_1, p_2. p_1 + p_2 \leq p \wedge (p_1, T, v_1) \in \llbracket \tau_1 \sigma \rrbracket \wedge (p_2, T, v_2) \in \llbracket \tau_2 \sigma \rrbracket$$

Also from Definition 6 it suffices to prove that

$$\exists p'_1, p'_2. p'_1 + p'_2 \leq p \wedge (p'_1, T, v_1) \in \llbracket \tau'_1 \sigma \rrbracket \wedge (p'_2, T, v_2) \in \llbracket \tau'_2 \sigma \rrbracket$$

$$\underline{\text{IH1}} \llbracket (\tau_1) \sigma \rrbracket \subseteq \llbracket (\tau'_1) \sigma \rrbracket$$

$$\underline{\text{IH2}} \llbracket (\tau_2) \sigma \rrbracket \subseteq \llbracket (\tau'_2) \sigma \rrbracket$$

Instantiating p'_1, p'_2 with p_1, p_2 we get the desired from IH1 and IH2

4. sub-with:

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau'_1 \quad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau'_2}{\Psi; \Theta; \Delta \vdash \tau_1 \& \tau_2 <: \tau'_1 \& \tau'_2} \text{ sub-with}$$

$$\text{To prove: } \forall (p, T, \langle v_1, v_2 \rangle) \in \llbracket (\tau_1 \& \tau_2) \sigma \rrbracket \implies (p, T, \langle v_1, v_2 \rangle) \in \llbracket (\tau'_1 \& \tau'_2) \sigma \rrbracket$$

$$\text{This means given } (p, T, \langle v_1, v_2 \rangle) \in \llbracket (\tau_1 \& \tau_2) \sigma \rrbracket$$

It suffices prove that

$$(p, T, \langle v_1, v_2 \rangle) \in \llbracket (\tau'_1 \& \tau'_2) \sigma \rrbracket$$

This means from Definition 6 we are given that

$$(p, T, v_1) \in \llbracket \tau_1 \sigma \rrbracket \wedge (p, T, v_2) \in \llbracket \tau_2 \sigma \rrbracket \quad (\text{F-SW0})$$

Also from Definition 6 it suffices to prove that

$$(p, T, v_1) \in \llbracket \tau'_1 \sigma \rrbracket \wedge (p, T, v_2) \in \llbracket \tau'_2 \sigma \rrbracket$$

$$\underline{\text{IH1}} \llbracket (\tau_1) \sigma \rrbracket \subseteq \llbracket (\tau'_1) \sigma \rrbracket$$

$$\underline{\text{IH2}} \llbracket (\tau_2) \sigma \rrbracket \subseteq \llbracket (\tau'_2) \sigma \rrbracket$$

We get the desired from (F-SW0), IH1 and IH2

5. sub-sum:

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau'_1 \quad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau'_2}{\Psi; \Theta; \Delta \vdash \tau_1 \oplus \tau_2 <: \tau'_1 \oplus \tau'_2} \text{ sub-sum}$$

$$\text{To prove: } \forall (p, T, \langle v_1, v_2 \rangle) \in \llbracket (\tau_1 \oplus \tau_2) \sigma \rrbracket \implies (p, T, \langle v_1, v_2 \rangle) \in \llbracket (\tau'_1 \oplus \tau'_2) \sigma \rrbracket$$

$$\text{This means given } (p, T, v) \in \llbracket (\tau_1 \oplus \tau_2) \sigma \rrbracket$$

It suffices prove that

$$(p, T, v) \in \llbracket (\tau'_1 \oplus \tau'_2) \sigma \rrbracket$$

This means from Definition 6 2 cases arise

(a) $v = \text{inl}(v')$:

This means from Definition 6 we have $(p, T, v') \in \llbracket \tau_1 \sigma \iota \rrbracket$ (F-SS0)

Also from Definition 6 it suffices to prove that

$(p, T, v') \in \llbracket \tau'_1 \sigma \iota \rrbracket$

IH $\llbracket (\tau_1) \sigma \iota \rrbracket \subseteq \llbracket (\tau'_1) \sigma \iota \rrbracket$

We get the desired from (F-SS0), IH

(b) $v = \text{inr}(v')$:

Symmetric reasoning as in the inl case

6. sub-potential:

$$\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau' \quad \Psi; \Theta; \Delta \vdash n' \leq n}{\Psi; \Theta; \Delta \vdash [n] \tau <: [n'] \tau'} \text{ sub-potential}$$

To prove: $\forall (p, T, v) \in \llbracket [n] \tau \sigma \iota \rrbracket. (p, T, v) \in \llbracket [n'] \tau' \sigma \iota \rrbracket$

This means given $(p, T, v) \in \llbracket [n] \tau \sigma \iota \rrbracket$ and we need to prove

$(p, T, v) \in \llbracket [n'] \tau' \sigma \iota \rrbracket$

This means from Definition 6 we are given

$\exists p'. p' + n \leq p \wedge (p', T, v) \in \llbracket \tau \sigma \iota \rrbracket$ (F-SP0)

And we need to prove

$\exists p''. p'' + n' \leq p \wedge (p'', T, v) \in \llbracket \tau' \sigma \iota \rrbracket$ (F-SP1)

In order to prove (F-SP1) we choose p'' as p'

Since from (F-SP0) we know that $p' + n \leq p$ and we are given that $n' \leq n$ therefore we also have $p' + n' \leq p$

IH $(p', T, v) \in \llbracket \tau' \sigma \iota \rrbracket$

$(p', T, v) \in \llbracket \tau' \sigma \iota \rrbracket$ we get directly from IH

7. sub-monad:

$$\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau' \quad \Psi; \Theta; \Delta \vdash n \leq n'}{\Psi; \Theta; \Delta \vdash \mathbb{M} n \tau <: \mathbb{M} n' \tau'} \text{ sub-monad}$$

To prove: $\forall (p, T, v) \in \llbracket \mathbb{M} n \tau \sigma \iota \rrbracket. (p, T, v) \in \llbracket \mathbb{M} n' \tau' \sigma \iota \rrbracket$

This means given $(p, T, v) \in \llbracket \mathbb{M} n \tau \sigma \iota \rrbracket$ and we need to prove

$(p, T, v) \in \llbracket \mathbb{M} n' \tau' \sigma \iota \rrbracket$

This means from Definition 6 we are given

$\forall t' < T, n_1, v'. v \Downarrow_{\mu'}^{n_1} v' \implies \exists p'. n_1 + p' \leq p + n \wedge (p', T - t', v') \in \llbracket \tau \sigma \iota \rrbracket$ (F-SM0)

Again from Definition 6 we need to prove that

$$\forall t'' < T, n_2, v''. v \Downarrow_{v''}^{n_2} v'' \implies \exists p''. n_1 + p'' \leq p + n' \wedge (p'', T - t'', v') \in \llbracket \tau' \sigma \iota \rrbracket$$

This means given some $t'' < T, n_2, v''$ s.t. $v \Downarrow_{v''}^{n_2} v''$ it suffices to prove that

$$\exists p''. n_1 + p'' \leq p + n' \wedge (p'', T - t'', v') \in \llbracket \tau' \sigma \iota \rrbracket \quad (\text{F-SM1})$$

Instantiating (F-SM0) with t'', n_2, v'' Since $v \Downarrow_{v''}^{n_2} v''$ therefore from (F-SM0) we know that

$$\exists p'. n_1 + p' \leq p + n \wedge (p', T - t'', v') \in \llbracket \tau \sigma \iota \rrbracket \quad (\text{F-SM2})$$

$$\underline{\text{IH}} \llbracket \tau \sigma \iota \rrbracket \subseteq \llbracket \tau' \sigma \iota \rrbracket$$

In order to prove (F-SM1) we choose p'' as p' and we need to prove

$$(a) \ n_1 + p' \leq p + n':$$

Since we are given that $n \leq n'$ therefore we get the desired from (F-SM2)

$$(b) \ (p', T - t'', v') \in \llbracket \tau' \sigma \iota \rrbracket$$

We get this directly from IH

8. sub-subExp:

$$\frac{\Psi; \Theta, a; \Delta, a < J \vdash \tau <: \tau' \quad \Psi; \Theta, a; \Delta \vdash J \leq I}{\Psi; \Theta; \Delta \vdash !_{a < I} \tau <: !_{a < J} \tau'} \text{ sub-subExp}$$

To prove: $\forall (p, T, v) \in \llbracket !_{a < I} \tau \sigma \iota \rrbracket. (p, T, v) \in \llbracket !_{a < J} \tau' \sigma \iota \rrbracket$

This means given $(p, T, !v) \in \llbracket !_{a < I} \tau \sigma \iota \rrbracket$ and we need to prove

$$(p, T, !v) \in \llbracket !_{a < J} \tau' \sigma \iota \rrbracket$$

This means from Definition 6 we are given

$$\exists p_0, \dots, p_{I-1}. p_0 + \dots + p_{I-1} \leq p \wedge \forall 0 \leq i < I. (p_i, T, v) \in \llbracket \tau[i/a] \rrbracket \quad (\text{F-SE0})$$

Again from Definition 6 we need to prove that

$$\exists p'_0, \dots, p'_{J-1}. p'_0 + \dots + p'_{J-1} \leq p \wedge \forall 0 \leq j < J. (p_j, T, v) \in \llbracket \tau'[j/a] \rrbracket \quad (\text{F-SE1})$$

In order to prove (F-SE1) we choose $p'_0 \dots p'_{J-1}$ as $p_0 \dots p_{J-1}$ and we need to prove

$$(a) \ p_0 + \dots + p_{J-1} \leq p:$$

Since we are given that $J \leq I$ therefore we get the desired from (F-SE0)

$$(b) \ \forall 0 \leq j < J. (p_j, T, v) \in \llbracket \tau'[j/a] \sigma \iota \rrbracket$$

We get this directly from IH and (F-SE0)

9. sub-list:

$$\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta \vdash L^n \tau <: L^n \tau'} \text{ sub-list}$$

To prove: $\forall (p, T, v) \in \llbracket L^n \tau \sigma \iota \rrbracket. (p, T, v) \in \llbracket L^n \tau' \sigma \iota \rrbracket$

This means given $(p, T, v) \in \llbracket L^n \tau \sigma \iota \rrbracket$ and we need to prove
 $(p, T, v) \in \llbracket L^n \tau' \sigma \iota \rrbracket$

We induct on $(p, T, v) \in \llbracket L^n \tau \sigma \iota \rrbracket$

(a) $(p, T, nil) \in \llbracket L^0 \tau \sigma \iota \rrbracket$:

We need to prove $(p, T, nil) \in \llbracket L^0 \tau' \sigma \iota \rrbracket$

We get this directly from Definition 6

(b) $(p, T, v' :: l') \in \llbracket L^{m+1} \tau \sigma \iota \rrbracket$:

In this case we are given $(p, T, v' :: l') \in \llbracket L^{m+1} \tau \sigma \iota \rrbracket$

and we need to prove $(p, T, v' :: l') \in \llbracket L^{m+1} \tau' \sigma \iota \rrbracket$

This means from Definition 6 are given

$\exists p_1, p_2. p_1 + p_2 \leq p \wedge (p_1, T, v') \in \llbracket \tau \sigma \iota \rrbracket \wedge (p_2, T, l') \in \llbracket L^m \tau \sigma \iota \rrbracket$ (Sub-List0)

Similarly from Definition 6 we need to prove that

$\exists p'_1, p'_2. p'_1 + p'_2 \leq p \wedge (p'_1, T, v') \in \llbracket \tau' \sigma \iota \rrbracket \wedge (p_2, T, l') \in \llbracket L^m \tau' \sigma \iota \rrbracket$

We choose p'_1 as p_1 and p'_2 as p_2 and we get the desired from (Sub-List0) IH of outer induction and IH of inner induction

10. sub-exist:

$$\frac{\Psi; \Theta, s; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta \vdash \exists s. \tau <: \exists s. \tau'} \text{ sub-exist}$$

To prove: $\forall (p, T, v) \in \llbracket \exists s. \tau \sigma \iota \rrbracket. (p, T, v) \in \llbracket \exists s. \tau' \sigma \iota \rrbracket$

This means given some $(p, T, v) \in \llbracket \exists s. \tau \sigma \iota \rrbracket$ we need to prove
 $(p, T, v) \in \llbracket \exists s. \tau' \sigma \iota \rrbracket$

From Definition 6 we are given that

$\exists s'. (p, T, v) \in \llbracket \tau \sigma \iota[s'/s] \rrbracket$ (F-exist0)

IH: $\llbracket (\tau) \sigma \iota \cup \{s \mapsto s'\} \rrbracket \subseteq \llbracket (\tau') \sigma \iota \cup \{s \mapsto s'\} \rrbracket$

Also from Definition 6 it suffices to prove that

$\exists s''. (p, T, v) \in \llbracket \tau' \sigma \iota[s''/s] \rrbracket$

We choose s'' as s' and we get the desired from IH

11. sub-typePoly:

$$\frac{\Psi, \alpha; \Psi; \Theta; \Delta \vdash \tau_1 <: \tau_2}{\Psi; \Theta; \Delta \vdash \forall \alpha. \tau_1 <: \forall \alpha. \tau_2} \text{ sub-typePoly}$$

To prove: $\forall (p, T, \Lambda \alpha. e) \in \llbracket (\forall \alpha. \tau_1) \sigma \iota \rrbracket. (p, T, \Lambda \alpha. e) \in \llbracket (\forall \alpha. \tau_2) \sigma \iota \rrbracket$

This means given some $(p, T, \Lambda \alpha. e) \in \llbracket (\forall \alpha. \tau_1) \sigma \iota \rrbracket$ we need to prove

$$(p, T, \Lambda\alpha.e) \in \llbracket (\forall\alpha.\tau_2) \sigma\iota \rrbracket$$

From Definition 6 we are given that

$$\forall\tau', T' < T . (p, T', e) \in \llbracket \tau_1[\tau'/\alpha] \rrbracket_{\mathcal{E}} \quad (\text{F-STF0})$$

Also from Definition 6 it suffices to prove that

$$\forall\tau'', T'' < T . (p, T'', e) \in \llbracket \tau_2[\tau''/\alpha] \rrbracket_{\mathcal{E}}$$

This means given some $\tau'', T'' < T$ and we need to prove

$$(p, T'', e[\tau''/\alpha]) \in \llbracket \tau_2[\tau''/\alpha] \rrbracket_{\mathcal{E}} \quad (\text{F-STF1})$$

$$\underline{\text{IH}}: \llbracket (\tau_1) \sigma \cup \{\alpha \mapsto \tau''\} \iota \rrbracket \subseteq \llbracket (\tau_2) \sigma \cup \{\alpha \mapsto \tau''\} \iota \rrbracket$$

Instantiating (F-STF0) with τ'', T'' we get

$$(p, T'', e) \in \llbracket \tau_1[\tau''/\alpha] \rrbracket_{\mathcal{E}}$$

and finally from IH we get the desired

12. sub-indexPoly:

$$\frac{\Psi; \Theta, i; \Delta \vdash \tau_1 <: \tau_2}{\Psi; \Theta; \Delta \vdash \forall i. \tau_1 <: \forall i. \tau_2} \text{sub-indexPoly}$$

$$\text{To prove: } \forall(p, T, \Lambda i.e) \in \llbracket (\forall i. \tau_1) \sigma\iota \rrbracket . (p, T, \Lambda i.e) \in \llbracket (\forall i. \tau_2) \sigma\iota \rrbracket$$

This means given some $(p, T, \Lambda i.e) \in \llbracket (\forall i. \tau_1) \sigma\iota \rrbracket$ we need to prove

$$(p, T, \Lambda i.e) \in \llbracket (\forall i. \tau_2) \sigma\iota \rrbracket$$

From Definition 6 we are given that

$$\forall I, T' < T . (p, T', e) \in \llbracket \tau_1[I/i] \rrbracket_{\mathcal{E}} \quad (\text{F-SIF0})$$

Also from Definition 6 it suffices to prove that

$$\forall I', T'' < T . (p, T'', e) \in \llbracket \tau_2[I'/i] \rrbracket_{\mathcal{E}}$$

This means given some $I', T'' < T$ and we need to prove

$$(p, T'', e) \in \llbracket \tau_2[I'/i] \rrbracket_{\mathcal{E}} \quad (\text{F-SIF1})$$

$$\underline{\text{IH}}: \llbracket (\tau_1) \sigma\iota \cup \{i \mapsto I'\} \rrbracket \subseteq \llbracket (\tau_2) \sigma\iota \cup \{i \mapsto I'\} \rrbracket$$

Instantiating (F-SIF0) with I', T'' we get

$$(p, T'', e) \in \llbracket \tau_1[I'/i] \rrbracket_{\mathcal{E}}$$

and finally from IH we get the desired

13. sub-constraint:

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau_2 \quad \Theta; \Delta \models c_2 \implies c_1}{\Psi; \Theta; \Delta \vdash c_1 \Rightarrow \tau_1 <: c_2 \Rightarrow \tau_2} \text{ sub-constraint}$$

To prove: $\forall (p, T, \Lambda.e) \in \llbracket (c_1 \Rightarrow \tau_1) \sigma \iota \rrbracket . (p, T, \Lambda.e) \in \llbracket (c_2 \Rightarrow \tau_2) \sigma \iota \rrbracket$

This means given some $(p, T, \Lambda.e) \in \llbracket (c_1 \Rightarrow \tau_1) \sigma \iota \rrbracket$ we need to prove $(p, T, \Lambda.e) \in \llbracket (c_2 \Rightarrow \tau_2) \sigma \iota \rrbracket$

From Definition 6 we are given that

$$\forall T' < T . \models c_1 \iota \implies (p, T', e) \in \llbracket \tau_1 \sigma \iota \rrbracket_{\mathcal{E}} \quad (\text{F-SC0})$$

Also from Definition 6 it suffices to prove that

$$\forall T'' < T . \models c_2 \iota \implies (p, T'', e) \in \llbracket \tau_2 \sigma \iota \rrbracket_{\mathcal{E}}$$

This means given some $T'' < T$ s.t. $\models c_2 \iota$ and we need to prove

$$(p, T'', e) \in \llbracket \tau_2 \sigma \iota \rrbracket_{\mathcal{E}} \quad (\text{F-SC1})$$

Since we are given that $\Theta; \Delta \models c_2 \implies c_1$ therefore we know that $\cdot \models c_1 \iota$

Hence from (F-SC0) we have

$$(p, T'', e) \in \llbracket \tau_1 \sigma \iota \rrbracket_{\mathcal{E}} \quad (\text{F-SC2})$$

$$\underline{\text{IH:}} \llbracket (\tau_1) \sigma \iota \rrbracket \subseteq \llbracket (\tau_2) \sigma \iota \rrbracket$$

Therefore we get the desired from IH and (F-SC2)

14. sub-CAnd:

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau_2 \quad \Theta; \Delta \models c_1 \implies c_2}{\Psi; \Theta; \Delta \vdash c_1 \& \tau_1 <: c_2 \& \tau_2} \text{ sub-CAnd}$$

To prove: $\forall (p, T, v) \in \llbracket (c_1 \& \tau_1) \sigma \iota \rrbracket . (p, T, v) \in \llbracket (c_2 \& \tau_2) \sigma \iota \rrbracket$

This means given some $(p, T, v) \in \llbracket (c_1 \& \tau_1) \sigma \iota \rrbracket$ we need to prove

$$(p, T, v) \in \llbracket (c_2 \& \tau_2) \sigma \iota \rrbracket$$

From Definition 6 we are given that

$$\cdot \models c_1 \wedge (p, T, e) \in \llbracket \tau_1 \sigma \iota \rrbracket_{\mathcal{E}} \quad (\text{F-SCA0})$$

Also from Definition 6 it suffices to prove that

$$\cdot \models c_2 \wedge (p, T, e) \in \llbracket \tau_2 \sigma \iota \rrbracket_{\mathcal{E}}$$

Since we are given that $\Theta; \Delta \models c_2 \implies c_1$ and $\cdot \models c_1 \iota$ therefore we also know that $\cdot \models c_2 \iota$

$$\text{Also from (F-SCA0) we have } (p, T, e) \in \llbracket \tau_1 \sigma \iota \rrbracket_{\mathcal{E}} \quad (\text{F-SCA1})$$

$$\underline{\text{IH:}} \llbracket (\tau_1) \sigma \iota \rrbracket \subseteq \llbracket (\tau_2) \sigma \iota \rrbracket$$

Therefore we get the desired from IH and (F-SCA1)

15. sub-potArrow:

$$\frac{\Psi; \Theta; \Delta \vdash k'}{\Psi; \Theta; \Delta \vdash [k](\tau_1 \multimap \tau_2) <: ([k'] \tau_1 \multimap [k' + k] \tau_2)} \text{sub-potArrow}$$

To prove: $\forall (p, T, \lambda x.e) \in \llbracket ([k](\tau_1 \multimap \tau_2)) \sigma \iota \rrbracket . (p, T, \lambda x.e) \in \llbracket ([k'] \tau_1 \multimap [k' + k] \tau_2) \sigma \iota \rrbracket$

This means given some $(p, T, \lambda x.e) \in \llbracket ([k](\tau_1 \multimap \tau_2)) \sigma \iota \rrbracket$ we need to prove
 $(p, T, \lambda x.e) \in \llbracket ([k'] \tau_1 \multimap [k' + k] \tau_2) \sigma \iota \rrbracket$

From Definition 6 we are given that

$$\exists p'. p' + k \leq p \wedge (p', T, \lambda x.e) \in \llbracket (\tau_1 \multimap \tau_2) \sigma \iota \rrbracket \quad (\text{F-SPA0})$$

Again from Definition 6 we know that

$$\forall p'' , e', T' < T . (p'', T', e') \in \llbracket \tau_1 \sigma \iota \rrbracket_{\mathcal{E}} \implies (p' + p'', T', e[e'/x]) \in \llbracket \tau_2 \sigma \iota \rrbracket_{\mathcal{E}} \quad (\text{F-SPA1})$$

Also from Definition 6 it suffices to prove that

$$\forall p'' , e'', T'' < T . (p'', T'', e'') \in \llbracket [k'] \tau_1 \sigma \iota \rrbracket_{\mathcal{E}} \implies (p + p'', T'', e[e''/x]) \in \llbracket [k + k'] \tau_2 \sigma \iota \rrbracket_{\mathcal{E}}$$

This means given some $p'' , e'', T'' < T$ s.t. $(p'', T'', e'') \in \llbracket [k'] \tau_1 \sigma \iota \rrbracket_{\mathcal{E}}$ we need to prove
 $(p + p'', T'', e[e''/x]) \in \llbracket [k + k'] \tau_2 \sigma \iota \rrbracket_{\mathcal{E}} \quad (\text{F-SSP2})$

Applying Definition 6 on (F-SPA2) we get

$$\forall v_f, t' < T'' . e[e''/x] \Downarrow_{t'} v_f \implies (p + p'', T'' - t', v_f) \in \llbracket [k + k'] \tau_2 \sigma \iota \rrbracket$$

This means that given some $v_f, t' < T''$ s.t. $e[e''/x] \Downarrow_{t'} v_f$ and we need to prove that
 $(p + p'', T'' - t', v_f) \in \llbracket [k + k'] \tau_2 \sigma \iota \rrbracket$

This means From Definition 6 it suffices to prove that

$$\exists p_2'' . p_2'' + (k + k') \leq (p + p'') \wedge (p_2'', T'' - t', v_f) \in \llbracket \tau_2 \sigma \iota \rrbracket \quad (\text{F-SPA4})$$

Also since we are given that $(p'', T'', e'') \in \llbracket [k'] \tau_1 \sigma \iota \rrbracket_{\mathcal{E}}$ we apply Definition 6 on it to obtain

$$\forall t < T'' , v' . e'' \Downarrow_t v' \implies (p'', T'' - t, v') \in \llbracket [k'] \tau_1 \sigma \iota \rrbracket$$

Also since we are given that $e[e''/x] \Downarrow_{t'} v_f$ therefore we also know that

$$\exists t'' < t' < T'' . e'' \Downarrow_{t''} v''$$

Instantiating with t'', v'' we get $(p'', T'' - t'', v'') \in \llbracket [k'] \tau_1 \sigma \iota \rrbracket$

Again using Definition 6 we know that we are given

$$\exists p_1'' . p_1'' + k' \leq p'' \wedge (p_1'', T'' - t'', v'') \in \llbracket \tau_1 \sigma \iota \rrbracket \quad (\text{F-SPA3})$$

Since $(p_1'', T'' - t'', v'') \in \llbracket \tau_1 \sigma \iota \rrbracket$ therefore from Definition 6 we also have

$$(p_1'', T'' - t'', v'') \in \llbracket \tau_1 \sigma \iota \rrbracket_{\mathcal{E}}$$

Instantiating (F-SPA1) with $p_1'', v'', T'' - t''$ we get

$$(p' + p_1'', T'' - t'', e[v''/x]) \in \llbracket \tau_2 \sigma \iota \rrbracket \mathcal{E}$$

From Definition 6 this means that

$$\forall t''' < T'' - t'', v_f.e[v''/x] \Downarrow v_f \implies (p' + p_1'', T'' - t'' - t''', v_f) \in \llbracket \tau_2 \sigma \iota \rrbracket \quad (\text{F-SPA4.1})$$

Since we know that $e[v''/x] \Downarrow_{t'} v_f$ therefore we also know that $\exists t''' . e[v''/x] \Downarrow_{t'''} v_f$ s.t. $t''' + t'' \leq t'$

Since we already know that $\exists t'' < t' < T'' . e'' \Downarrow_{t''} v''$ therefore we have $t'' + t''' \leq t' < T''$.

Instantiating (F-SPA4.1) with t''' we get

$$(p' + p_1'', T'' - t'' - t''', v_f) \in \llbracket \tau_2 \sigma \iota \rrbracket \quad (\text{F-SPA5})$$

Since from (F-SPA0) we know that

$$p' + k \leq p$$

And from (F-SPA3) we know that

$$p_1'' + k' \leq p''$$

We add the two to get

$$p' + p_1'' + k + k' \leq p + p'' \quad (\text{F-SPA6})$$

In order to prove (F-SPA4) we choose p_2'' as $p' + p_1''$

and we get the desired from (F-SPA6) and (F-SPA5) and Lemma 69

16. sub-potZero:

$$\frac{}{\Psi; \Theta; \Delta \vdash \tau <: [0] \tau} \text{sub-potZero}$$

To prove: $\forall (p, T, v) \in \llbracket \tau \sigma \iota \rrbracket . (p, T, v) \in \llbracket [0] \tau \sigma \iota \rrbracket$

This means that given $(p, T, v) \in \llbracket \tau \sigma \iota \rrbracket$

And we need to prove $(p, T, v) \in \llbracket [0] \tau \sigma \iota \rrbracket$

From Definition 6 it suffices to prove that

$$\exists p' . p' + 0 \leq p \wedge (p', T, v) \in \llbracket \tau \sigma \iota \rrbracket$$

We choose p' as p and we get the desired

17. sub-familyAbs:

$$\frac{\Psi; \Theta, i : S \vdash \tau <: \tau'}{\Psi; \Theta \vdash \lambda_t i : S . \tau <: \lambda_t i : S . \tau'} \text{sub-familyAbs}$$

To prove:

$$\forall f \in \llbracket \lambda_t i : S . \tau \sigma \iota \rrbracket . f \in \llbracket \lambda_t i : S . \tau' \sigma \iota \rrbracket$$

This means given $f \in \llbracket \lambda_t i : S . \tau \sigma \iota \rrbracket$ and we need to prove

$$f \in \llbracket \lambda_t i : S . \tau' \sigma \iota \rrbracket$$

This means from Definition 6 we are given

$$\forall I. f I \in \llbracket \tau[I/i] \sigma \iota \rrbracket \quad (\text{F-SFAbs0})$$

This means from Definition 6 we need to prove

$$\forall I'. f I' \in \llbracket \tau'[I'/i] \sigma \iota \rrbracket$$

This further means that given some I' we need to prove

$$f I' \in \llbracket \tau'[I'/i] \sigma \iota \rrbracket \quad (\text{F-SFAbs1})$$

Instantiating (F-SFAbs0) with I' we get

$$f I' \in \llbracket \tau[I'/i] \sigma \iota \rrbracket$$

From IH we know that $\llbracket \tau \sigma \iota \cup \{i \mapsto I' \iota\} \rrbracket \subseteq \llbracket \tau' \sigma \iota \cup \{i \mapsto I' \iota\} \rrbracket$

And this completes the proof.

18. Sub-tfamilyApp1:

$$\frac{}{\Psi; \Theta; \Delta \vdash \lambda_t i : S . \tau I <: \tau[I/i]} \text{sub-familyApp1}$$

To prove: $\forall (p, T, v) \in \llbracket \lambda_t i : S . \tau I \sigma \iota \rrbracket. (p, T, v) \in \llbracket \tau[I/i] \sigma \iota \rrbracket$

This means given $(p, T, v) \in \llbracket \lambda_t i : S . \tau I \sigma \iota \rrbracket$ and we need to prove

$$(p, T, v) \in \llbracket \tau[I/i] \sigma \iota \rrbracket$$

This means from Definition 6 we are given

$$(p, T, v) \in \llbracket \lambda_t i : S . \tau \rrbracket I \sigma \iota$$

This further means that we have

$$(p, T, v) \in f I \iota \text{ where } f I = \llbracket \tau \sigma[I\iota/i] \rrbracket$$

This means we have $(p, T, v) \in \llbracket \tau \sigma[I\iota/i] \rrbracket$

And this completes the proof.

19. Sub-tfamilyApp2:

$$\frac{}{\Psi; \Theta; \Delta \vdash \tau[I/i] <: \lambda_t i : S . \tau I} \text{sub-familyApp2}$$

To prove: $\forall (p, T, v) \in \llbracket \tau[I/i] \sigma \iota \rrbracket. (p, T, v) \in \llbracket \lambda_t i : S . \tau I \sigma \iota \rrbracket$

This means given $(p, T, v) \in \llbracket \tau[I/i] \sigma \iota \rrbracket$ (Sub-tF0)

And we need to prove

$$(p, T, v) \in \llbracket \lambda_t i : S . \tau I \sigma \iota \rrbracket$$

This means from Definition 6 it suffices to prove that

$$(p, T, v) \in \llbracket \lambda_t i : S . \tau \rrbracket I \sigma \iota$$

It further suffices to prove that

$(p, T, v) \in f I \iota$ where $f I \iota = \llbracket \tau \sigma [I \iota / i] \rrbracket$

which means we need to show that

$(p, T, v) \in \llbracket \tau \sigma [I \iota / i] \rrbracket$

We get this directly from (Sub-tF0)

20. sub-bSum:

$$\frac{}{\Psi; \Theta; \Delta \vdash \left[\sum_{a < I} K \right] !_{a < I} \tau < : !_{a < I} [K] \tau} \text{sub-bSum}$$

To prove: $\forall (p, T, v) \in \llbracket \left[\sum_{a < I} K \right] !_{a < I} \tau \sigma \iota \rrbracket \implies (p, T, v) \in \llbracket !_{a < I} [K] \tau \sigma \iota \rrbracket$

This means given some (p, T, v) s.t. $(p, T, v) \in \llbracket \left[\sum_{a < I} K \right] !_{a < I} \tau \sigma \iota \rrbracket$ it suffices to prove that $(p, T, v) \in \llbracket !_{a < I} [K] \tau \sigma \iota \rrbracket$

This means from Definition 6 we are given that

$$\exists p'. p' + \sum_{a < I} K \leq p \wedge (p', T, v) \in \llbracket !_{a < I} \tau \sigma \iota \rrbracket \quad (\text{Sub-BS0})$$

Since $(p', T, v) \in \llbracket !_{a < I} \tau \sigma \iota \rrbracket$ therefore again from Definition 6 it means that $\exists e'. v = !e'$ and

$$\exists p_0, \dots, p_{I-1}. p_0 + \dots + p_{I-1} \leq p' \wedge \forall 0 \leq i < I. (p_i, T, e') \in \llbracket \tau [i/a] \sigma \iota \rrbracket_{\mathcal{E}} \quad (\text{Sub-BS1})$$

Since $\forall 0 \leq i < I. (p_i, T, e') \in \llbracket \tau [i/a] \sigma \iota \rrbracket_{\mathcal{E}}$ therefore from Definition 6 we have

$$\forall 0 \leq i < I. \forall t < T. v''. e' \Downarrow_t v'' \implies (p_i, T - t, v') \in \llbracket \tau [i/a] \sigma \iota \rrbracket \quad (\text{Sub-BS1.1})$$

Since we know that $v = !e'$ therefore it suffices to prove that $(p, T, !e') \in \llbracket !_{a < I} [K] \tau \sigma \iota \rrbracket$

From Definition 6 it further suffices to prove that

$$\exists p'_0, \dots, p'_{I-1}. p'_0 + \dots + p'_{I-1} \leq p \wedge \forall 0 \leq i < I. (p'_i, T, e') \in \llbracket [K] \tau [i/a] \sigma \iota \rrbracket_{\mathcal{E}}$$

We choose p'_0 as $p_0 + K[0/a] \dots p'_{I-1}$ as $p_{I-1} + K[(I-1)/a]$ and it suffices to prove that

- $p'_0 + \dots + p'_{I-1} \leq p$:

We need to prove that

$$(p_0 + K[0/a]) + \dots + (p_{I-1} + K[(I-1)/a]) \leq p$$

We get this from (Sub-BS0) and (Sub-BS1)

- $\forall 0 \leq i < I. (p'_i, T, e') \in \llbracket [K] \tau [i/a] \sigma \iota \rrbracket_{\mathcal{E}}$:

Given some $0 \leq i < I$ it suffices to prove that

$$(p'_i, T, e') \in \llbracket [K] \tau [i/a] \sigma \iota \rrbracket_{\mathcal{E}}$$

Since p'_i is $p_i + K[i/a]$ therefore it suffices to prove that

$$(p_i + K[i/a], T, e') \in \llbracket [K[i/a]] \tau [i/a] \sigma \iota \rrbracket_{\mathcal{E}}$$

From Definition 6 we need to prove that

$$\forall v', t'' < T. e' \Downarrow_{t''} v' \implies (p_i + K[i/a], T - t'', v') \in \llbracket [K[i/a]] \tau [i/a] \sigma \iota \rrbracket$$

This means given some v' s.t. $e' \Downarrow_{t''} v'$ we need to prove that

$$(p_i + K[i/a], T - t'', v') \in \llbracket [K[i/a]] \tau[i/a] \sigma \iota \rrbracket$$

From Definition 6 it suffices to prove that

$$\exists p''. p'' + K[i/a] \leq p_i + K[i/a] \wedge (p'', T - t'', v') \in \llbracket \tau[i/a] \sigma \iota \rrbracket$$

We choose p'' as p_i and we need to prove

$$(p_i, T - t'', v') \in \llbracket \tau[i/a] \sigma \iota \rrbracket$$

Instantiating (Sub-BS1.1) with the given i and v', t'' we get the desired

□

Lemma 18 (Expression subtyping lemma). $\forall \Psi, \Theta, \Delta, \tau \in \text{Type}, \tau'$.

$$\Psi; \Theta; \Delta \vdash \tau <: \tau' \implies \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}} \subseteq \llbracket \tau' \sigma \iota \rrbracket_{\mathcal{E}}$$

Proof. To prove: $\forall (p, T, e) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}} \implies (p, T, e) \in \llbracket \tau' \sigma \iota \rrbracket_{\mathcal{E}}$

This means given some $(p, T, e) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$ it suffices to prove that

$$(p, T, e) \in \llbracket \tau' \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 6 we are given

$$\forall v, t < T . e \Downarrow_t v \implies (p, T - t, v) \in \llbracket \tau \sigma \iota \rrbracket \quad (\text{S-E0})$$

Similarly from Definition 6 it suffices to prove that

$$\forall v', t' < T . e \Downarrow_{t'} v' \implies (p, T - t', v') \in \llbracket \tau' \sigma \iota \rrbracket$$

This means given some $v', t' < T$ s.t $e \Downarrow_{t'} v'$ it suffices to prove that

$$(p, T - t', v') \in \llbracket \tau' \sigma \iota \rrbracket$$

Instantiating (S-E0) with v', t' we get $(p, T - t', v') \in \llbracket \tau \sigma \iota \rrbracket$

And finally from Lemma 17 we get the desired.

□

Theorem 19 (Soundness). $\forall e, n, n', \tau \in \text{Type}, t$.

$$\vdash e : \mathbb{M} n \tau \wedge e \Downarrow_t^{n'} v \implies n' \leq n$$

Proof. From Theorem 12 we know that $(0, t + 1, e) \in \llbracket \mathbb{M} n \tau \rrbracket_{\mathcal{E}}$

From Definition 6 this means we have

$$\forall t' < t + 1 . e \Downarrow_{t'} v' \implies (0, t + 1 - t' v') \in \llbracket \mathbb{M} n \tau \rrbracket$$

From the evaluation relation we know that $e \Downarrow_0 e$ therefore we have

$$(0, t + 1, e) \in \llbracket \mathbb{M} n \tau \rrbracket$$

Again from Definition 6 it means we have

$$\forall t'' < t + 1 . e \Downarrow_{t''}^{n'} v \implies \exists p'. n' + p' \leq 0 + n \wedge (p', t + 1 - t'', v) \in \llbracket \tau \rrbracket$$

Since we are given that $e \Downarrow_t^{n'} v$ therefore we have

$$\exists p'. n' + p' \leq n \wedge (p', 1, v) \in \llbracket \tau \rrbracket$$

Since $p' \geq 0$ therefore we get $n' \leq n$

□

Theorem 20 (Soundness). $\forall e, n, n', \tau \in \text{Type}$.

$$\vdash e : [n] \mathbf{1} \multimap \mathbb{M} 0 \tau \wedge e () \Downarrow_{t_1} - \Downarrow_{t_2}^{n'} v \implies n' \leq n$$

Proof. From Theorem 12 we know that $(0, t_1 + t_2 + 2, e) \in \llbracket [n] \mathbf{1} \multimap \mathbb{M} 0 \tau \rrbracket_{\mathcal{E}}$

Therefore from Definition 6 we know that

$$\forall t' < t_1 + t_2 + 2, v.e \Downarrow_{t'} v \implies (0, t_1 + t_2 + 2 - t', v) \in \llbracket [n] \mathbf{1} \multimap \mathbb{M} 0 \tau \rrbracket \quad (\text{S0})$$

Since we know that $e () \Downarrow_{t_1}$ – therefore from E-app we know that $\exists e'.e \Downarrow_{t_1} \lambda x.e'$

Instantiating (S0) with $t_1, \lambda x.e'$ we get $(0, t_2 + 2, \lambda x.e') \in \llbracket [n] \mathbf{1} \multimap \mathbb{M} 0 \tau \rrbracket$

This means from Definition 6 we have

$$\forall p', e', t'' < t_2 + 2. (p', t'', e') \in \llbracket [n] \mathbf{1} \rrbracket_{\mathcal{E}} \implies (0 + p', t'', e'[e''/x]) \in \llbracket \mathbb{M} 0 \tau \rrbracket_{\mathcal{E}} \quad (\text{S1})$$

Claim: $\forall t.(I, t, ()) \in \llbracket [I] \mathbf{1} \rrbracket_{\mathcal{E}}$

Proof:

From Definition 6 it suffices to prove that

$$() \Downarrow_0 v \implies (I, t, v) \in \llbracket [I] \mathbf{1} \rrbracket$$

Since we know that $v = ()$ therefore it suffices to prove that

$$(I, t, v) \in \llbracket [I] \mathbf{1} \rrbracket$$

From Definition 6 it suffices to prove that

$$\exists p'. p' + I \leq I \wedge (p', t, v) \in \llbracket \mathbf{1} \rrbracket$$

We choose p' as 0 and we get the desired

Instantiating (S1) with $n, (), t_2 + 1$ we get $(n, t_2 + 1, e'[()/x]) \in \llbracket \mathbb{M} 0 \tau \rrbracket_{\mathcal{E}}$

This means again from Definition 6 we have

$$\forall t' < t_2 + 1. e'[()/x] \Downarrow_{t'} v' \implies (n, t_2 + 1 - t', v') \in \llbracket \mathbb{M} 0 \tau \rrbracket$$

From E-val we know that $v' = e'[()/x]$ and $t' = 0$ therefore we have

$$(n, t_2 + 1, e'[()/x]) \in \llbracket \mathbb{M} 0 \tau \rrbracket$$

Again from Definition 6 we have

$$\forall t' < t_2 + 1. e'[()/x] \Downarrow_{t'}^{n'} v'' \implies \exists p'. n' + p' \leq n + 0 \wedge (p', t_2 + 1 - t', v'') \in \llbracket \tau \rrbracket$$

Since we are given that $e \Downarrow_{t_1} - \Downarrow_{t_2}^{n'} v$ therefore we get

$$\exists p'. n' + p' \leq n \wedge (p', 1, v'') \in \llbracket \tau \rrbracket$$

Since $p' \geq 0$ therefore we have $n' \leq n$

□

1.5 Embedding dlPCF

Type translation

$$\begin{aligned} \langle b \rangle &= b \\ \langle [a < I] \tau_1 \multimap \tau_2 \rangle &= (!_{a < I} \mathbb{M} 0 \langle \tau_1 \rangle) \multimap [I] \mathbf{1} \multimap \mathbb{M} 0 \langle \tau_2 \rangle \end{aligned}$$

Judgment translation

$$\boxed{\Theta; \Delta; \Gamma \vdash_K e_d : \tau \rightsquigarrow \cdot; \Theta; \Delta; \langle \Gamma \rangle; \cdot \vdash e_a : [K + \text{count}(\Gamma)] \mathbf{1} \multimap \mathbb{M} 0 \langle \tau \rangle}$$

where

$$\begin{aligned} \text{count}(\cdot) &= 0 \\ \text{count}(\Gamma, x : [a < I] \tau) &= \text{count}(\Gamma) + I \end{aligned}$$

Definition 21 (Context translation).

$$\begin{aligned} (\cdot) &= \cdot \\ (\Gamma, x : [a < I]\tau) &= (\Gamma), x :_{a < I} \mathbb{M}0(\tau) \end{aligned}$$

Expression translation

$$\frac{\Theta; \Delta \models I \geq 1 \quad \Theta; \Delta \vdash \sigma[0/a] <: \tau \quad \Theta; \Delta \models [a < I]\sigma \Downarrow \quad \Theta; \Delta \models \Gamma \Downarrow}{\Theta; \Delta; \Gamma, x : [a < I]\sigma \vdash_J x : \tau \rightsquigarrow \lambda p. \text{release} - = p \text{ in } \text{bind} - = \uparrow^1 \text{ in } x} \text{ var}$$

$$\frac{\Theta; \Delta; \Gamma, x : [a < I]\tau_1 \vdash_J e : \tau_2 \rightsquigarrow e_t}{\Theta; \Delta; \Gamma \vdash_J \lambda x. e : ([a < I].\tau_1) \multimap \tau_2 \rightsquigarrow} \text{ lam}$$

$$\lambda p_1. \text{ret } \lambda y. \lambda p_2. \text{let } !x = y \text{ in } \text{release} - = p_1 \text{ in } \text{release} - = p_2 \text{ in } \text{bind } a = \text{store}() \text{ in } e_t a$$

$$\frac{\Theta; \Delta; \Gamma \vdash_J e_1 : ([a < I].\tau_1) \multimap \tau_2 \rightsquigarrow e_{t1} \quad \Theta, a; \Delta, a < I; \Delta \vdash_K e_2 : \tau_1 \rightsquigarrow e_{t2} \quad \Gamma' \sqsupseteq \Gamma \oplus \sum_{a < I} \Delta \quad H \geq J + I + \sum_{a < I} K}{\Theta; \Delta; \Gamma' \vdash_H e_1 e_2 : \tau_2 \rightsquigarrow \lambda p. E_0} \text{ app}$$

$$\begin{aligned} E_0 &= \text{release} - = p \text{ in } E_1 \\ E_1 &= \text{bind } a = \text{store}() \text{ in } E_2 \\ E_2 &= \text{bind } b = e_{t1} a \text{ in } E_3 \\ E_3 &= \text{bind } c = \text{store}!() \text{ in } E_4 \\ E_4 &= \text{bind } d = \text{store}() \text{ in } E_5 \\ E_5 &= b (\text{coerce } !e_{t2} c) d \end{aligned}$$

$$\frac{\Theta, b; \Delta, b < L; \Gamma, x : [a < I]\sigma \vdash_K e : \tau \rightsquigarrow e_t \quad \tau[0/a] <: \mu \quad \Theta, a, b; \Delta, a < I, b < L; \Gamma \vdash \tau[(b+1 + \binom{b+1, a}{b}) I/b] <: \sigma}{\Gamma' \sqsubseteq \sum_{b < L} \Gamma \quad L, M \geq \binom{0,1}{b} I \quad N \geq M - 1 + \sum_{b < L} K} \text{ T-fix}$$

$$\Theta; \Delta; \Gamma' \vdash_N \text{fix } x. e : \mu \rightsquigarrow E_0$$

$$\begin{aligned} E_0 &= \text{fix } Y. E_1 \\ E_1 &= \lambda p. E_2 \\ E_2 &= \text{release} - = p \text{ in } E_3 \\ E_3 &= \text{bind } A = \text{store}() \text{ in } E_4 \\ E_4 &= \text{let } !x = (E_{4.1} E_{4.2}) \text{ in } E_5 \\ E_{4.1} &= \text{coerce } !Y \\ E_{4.2} &= (\lambda u. !()) A \\ E_5 &= \text{bind } C = \text{store}() \text{ in } E_6 \\ E_6 &= e_t C \end{aligned}$$

1.5.1 Type preservation

Theorem 22 (Type preservation: dlPCF to λ -Amor). If $\Theta; \Delta; \Gamma \vdash_I e : \tau$ in dlPCF then there exists e' such that $\Theta; \Delta; \Gamma \vdash_I e : \tau \rightsquigarrow e'$ such that there is a derivation of $\cdot; \Theta; \Delta; (\Gamma); \cdot \vdash e' : [I + \text{count}(\Gamma)] \mathbf{1} \multimap \mathbb{M}0 \langle \tau \rangle$ in λ -Amor .

Proof. Proof by induction on the $\Theta; \Delta; \Gamma \vdash_I e : \tau$

- var:

$$\frac{\Theta; \Delta \models J \geq 0 \quad \Theta; \Delta \models I \geq 1 \quad \Theta; \Delta \vdash \sigma[0/a] <: \tau \quad \Theta; \Delta \models [a < I] \sigma \Downarrow \quad \Theta; \Delta \models \Gamma \Downarrow}{\Theta; \Delta; \Gamma, x : [a < I] \sigma \vdash_J x : \tau \rightsquigarrow \lambda p. \text{release} - = p \text{ in bind} - = \uparrow^1 \text{ in } x} \text{ var}$$

D2:

$$\frac{\Theta; \Delta \vdash \sigma[0/a] <: \tau}{\Theta; \Delta \vdash \langle \sigma[0/a] \rangle <: \langle \tau \rangle} \text{ Lemma 27}$$

D1:

$$\frac{\frac{\cdot; \Theta; \Delta; (\Gamma), x :_{a < I} \mathbb{M}0 \langle \sigma \rangle, \vdash x : \mathbb{M}0 \langle \sigma \rangle [0/a]}{\cdot; \Theta; \Delta; (\Gamma), x :_{a < I} \mathbb{M}0 \langle \sigma \rangle, \vdash x : \mathbb{M}0 \langle \sigma[0/a] \rangle} \text{T-var2}}{\cdot; \Theta; \Delta; (\Gamma), x :_{a < I} \mathbb{M}0 \langle \sigma \rangle, \vdash x : \mathbb{M}0 \langle \sigma[0/a] \rangle} \text{ Lemma 28}$$

D0:

$$\frac{\frac{\cdot; \Theta; \Delta; (\Gamma), x :_{a < I} \mathbb{M}0 \langle \sigma \rangle, \vdash \uparrow^1 : \mathbb{M}1 \mathbf{1}}{\cdot; \Theta; \Delta; (\Gamma), x :_{a < I} \mathbb{M}0 \langle \sigma \rangle, \vdash \uparrow^1 : \mathbb{M}(I + J + \text{count}(\Gamma)) \mathbf{1}} \text{ D1}}{\cdot; \Theta; \Delta; (\Gamma), x :_{a < I} \mathbb{M}0 \langle \sigma \rangle, \vdash \text{bind} - = \uparrow^1 \text{ in } x : \mathbb{M}(I + J + \text{count}(\Gamma)) \langle \sigma[0/a] \rangle} \text{ bind}$$

Main derivation:

$$\frac{\frac{\cdot; \Theta; \Delta; (\Gamma), x :_{a < I} \mathbb{M}0 \langle \sigma \rangle, p : ([I + J + \text{count}(\Gamma)] \mathbf{1}) \vdash \quad p : ([I + J + \text{count}(\Gamma)] \mathbf{1})}{\cdot; \Theta; \Delta; (\Gamma), x :_{a < I} \mathbb{M}0 \langle \sigma \rangle; p : ([I + J + \text{count}(\Gamma)] \mathbf{1}) \vdash \text{release} - = p \text{ in bind} - = \uparrow^1 \text{ in } x : \mathbb{M}0 \langle \tau \rangle} \text{ T-release}}{\lambda p. \text{release} - = p \text{ in bind} - = \uparrow^1 \text{ in } x : (([I + J + \text{count}(\Gamma)] \mathbf{1}) \multimap \mathbb{M}0 \langle \tau \rangle)} \text{ T-lam}$$

- lam:

$$\frac{\Theta; \Delta; \Gamma, x : [a < I] \tau_1 \vdash_J e : \tau_2 \rightsquigarrow e_t}{\Theta; \Delta; \Gamma \vdash_J \lambda x. e : ([a < I]. \tau_1) \multimap \tau_2 \rightsquigarrow \lambda p_1. \text{ret } \lambda y. \lambda p_2. \text{let } !x = y \text{ in release} - = p_1 \text{ in release} - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_t a}$$

$$E_0 = \lambda p_1. \text{ret } \lambda y. \lambda p_2. \text{let } !x = y \text{ in release} - = p_1 \text{ in release} - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_t a$$

$$E_1 = \text{ret } \lambda y. \lambda p_2. \text{let } !x = y \text{ in release} - = p_1 \text{ in release} - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_t a$$

$$E_2 = \lambda y. \lambda p_2. \text{let } !x = y \text{ in release} - = p_1 \text{ in release} - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_t a$$

$E_3 = \lambda p_2. \text{let } !x = y \text{ in release } - = p_1 \text{ in release } - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_t a$

$E_4 = \text{let } !x = y \text{ in release } - = p_1 \text{ in release } - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_t a$

$E_{4.1} = \text{release } - = p_1 \text{ in release } - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_t a$

$E_{4.2} = \text{release } - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_t a$

$E_{4.3} = \text{bind } a = \text{store}() \text{ in } e_t a$

$T_0 = [J + \text{count}(\Gamma)] \mathbf{1} \multimap \mathbb{M}0 \langle ([a < I] \tau_1) \multimap \tau_2 \rangle$

$T_{0.1} = [J + \text{count}(\Gamma)] \mathbf{1} \multimap \mathbb{M}0 \langle (!_{a < I} \mathbb{M}0 \langle \tau_1 \rangle) \multimap [I] \mathbf{1} \multimap \mathbb{M}0 \langle \tau_2 \rangle \rangle$

$T_{0.2} = [J + \text{count}(\Gamma)] \mathbf{1}$

$T_1 = \mathbb{M}0 \langle (!_{a < I} \mathbb{M}0 \langle \tau_1 \rangle) \multimap [I] \mathbf{1} \multimap \mathbb{M}0 \langle \tau_2 \rangle \rangle$

$T_2 = \langle (!_{a < I} \mathbb{M}0 \langle \tau_1 \rangle) \multimap [I] \mathbf{1} \multimap \mathbb{M}0 \langle \tau_2 \rangle \rangle$

$T_{2.1} = !_{a < I} \mathbb{M}0 \langle \tau_1 \rangle$

$T_3 = [I] \mathbf{1} \multimap \mathbb{M}0 \langle \tau_2 \rangle$

$T_{3.1} = [I] \mathbf{1}$

$T_4 = \mathbb{M}0 \langle \tau_2 \rangle$

$T_{4.1} = \mathbb{M}(J + I + \text{count}(\Gamma)) \mathbf{1}$

$T_{4.2} = \mathbb{M}(J + I + \text{count}(\Gamma)) \langle \tau_2 \rangle$

$T_{4.3} = \mathbb{M}(J + \text{count}(\Gamma)) \langle \tau_2 \rangle$

$T_5 = [(J + I + \text{count}(\Gamma))] \mathbf{1} \multimap \mathbb{M}0 \langle \tau_2 \rangle$

D6:

$$\frac{}{;\Theta; \Delta; ; a : [J + I + \text{count}(\Gamma)] \mathbf{1} \vdash a : [J + I + \text{count}(\Gamma)] \mathbf{1}} \text{var}$$

D5:

$$\frac{}{;\Theta; \Delta; (\Gamma), x :_{a < I} \mathbb{M}0 \langle \tau_1 \rangle; \cdot \vdash e_t : T_5} \text{IH}$$

D4:

$$\frac{D5 \quad D6}{;\Theta; \Delta; (\Gamma), x :_{a < I} \mathbb{M}0 \langle \tau_1 \rangle; a : [J + I + \text{count}(\Gamma)] \mathbf{1} \vdash e_t a : T_4} \text{app}$$

D3:

$$\frac{\frac{}{;\Theta; \Delta; ; \cdot \vdash \text{store}() : T_{4.1}} \text{store} \quad D4}{;\Theta; \Delta; (\Gamma), x :_{a < I} \mathbb{M}0 \langle \tau_1 \rangle; \cdot \vdash E_{4.3} : T_{4.2}} \text{bind}}$$

D2:

$$\frac{\frac{}{;\Theta; \Delta; ; p_2 : T_{3.1} \vdash p_2 : T_{3.1}} D3}{;\Theta; \Delta; (\Gamma), x :_{a < I} \mathbb{M}0 \langle \tau_1 \rangle; p_2 : T_{3.1} \vdash E_{4.2} : T_{4.3}} \text{bind}}$$

D1:

$$\frac{\frac{}{;\Theta; \Delta; ; p_1 : T_{0.2} \vdash p_1 : T_{0.2}} D2}{;\Theta; \Delta; (\Gamma), x :_{a < I} \mathbb{M}0 \langle \tau_1 \rangle; p_1 : T_{0.2}, p_2 : T_{3.1} \vdash E_{4.1} : T_4} \text{release}}$$

D0:

$$\frac{\frac{\frac{}{; \Theta; \Delta; ; y : T_{2.1} \vdash y : T_{2.1}}{D1}}{; \Theta; \Delta; (\Gamma); p_1 : T_{0.2}, y : T_{2.1}, p_2 : T_{3.1} \vdash E_4 : T_4} \text{T-subExpE}}{; \Theta; \Delta; (\Gamma); p_1 : T_{0.2}, y : T_{2.1} \vdash E_3 : T_3} \text{lam}}$$

Main derivation:

$$\frac{\frac{\frac{D0}{; \Theta; \Delta; (\Gamma); p_1 : T_{0.2} \vdash E_2 : T_2} \text{lam}}{; \Theta; \Delta; (\Gamma); p_1 : T_{0.2} \vdash E_1 : T_1} \text{ret}}{; \Theta; \Delta; (\Gamma); \cdot \vdash E_0 : T_{0.1}} \text{lam}}$$

• app:

$$\frac{\Theta; \Delta; \Gamma_1 \vdash_J e_1 : ([a < I]\tau_1) \multimap \tau_2 \rightsquigarrow e_{t1} \quad \Theta, a; \Delta, a < I; \Gamma_2 \vdash_K e_2 : \tau_1 \rightsquigarrow e_{t2} \quad \Gamma' \cong \Gamma_1 \oplus \sum_{a < I} \Gamma_2 \quad H \geq J + I + \sum_{a < I} K}{\Theta; \Delta; \Gamma' \vdash_H e_1 e_2 : \tau_2 \rightsquigarrow E_0} \text{app}$$

$$E_0 = \lambda p. E_1$$

$$E_1 = \text{release } - = p \text{ in } E_2$$

$$E_2 = \text{bind } a = \text{store}() \text{ in } E_3$$

$$E_3 = \text{bind } b = e_{t1} a \text{ in } E_4$$

$$E_4 = \text{bind } c = \text{store}!() \text{ in } E_5$$

$$E_5 = \text{bind } d = \text{store}() \text{ in } b (\text{coerce1 } e_{t2} c) d$$

$$T_0 = [H + \text{count}(\Gamma')] \mathbf{1} \multimap \mathbb{M} 0 (\tau_2)$$

$$T_{0.11} = [J + I + \sum_{a < I} K + \text{count}(\Gamma_1) + \text{count}(\sum_{a < I} \Gamma_2)] \mathbf{1} \multimap \mathbb{M} 0 (\tau_2)$$

$$T_{0.1} = [J + I + \sum_{a < I} K + \text{count}(\Gamma_1) + \text{count}(\sum_{a < I} \Gamma_2)] \mathbf{1}$$

$$T_{0.2} = \mathbb{M} 0 (\tau_2)$$

$$T_{0.3} = \mathbb{M}(J + I + \sum_{a < I} K + \text{count}(\Gamma_1) + \text{count}(\sum_{a < I} \Gamma_2)) (\tau_2)$$

$$T_1 = [(J + \text{count}(\Gamma))] \mathbf{1} \multimap \mathbb{M} 0 ([a < I]\tau_1) \multimap \tau_2$$

$$T_{1.1} = [(J + \text{count}(\Gamma))] \mathbf{1}$$

$$T_{1.11} = \mathbb{M}(J + \text{count}(\Gamma)) [(J + \text{count}(\Gamma))] \mathbf{1}$$

$$T_{1.12} = \mathbb{M}(I + \sum_{a < I} K + \text{count}(\sum_{a < I} \Gamma_2)) (\tau_2)$$

$$T_{1.13} = \mathbb{M}(\sum_{a < I} K + \text{count}(\sum_{a < I} \Gamma_2)) T_{1.14}$$

$$T_{1.131} = \mathbb{M}(\sum_{a < I} K + \text{count}(\sum_{a < I} \Gamma_2)) T_{1.15}$$

$$T_{1.14} = [(\sum_{a < I} K + \text{count}(\sum_{a < I} \Gamma_2))] !_{a < I} \mathbf{1} = [\sum_{a < I} (K + \text{count}(\Gamma_2))] !_{a < I} \mathbf{1}$$

$$T_{1.15} = !_{a < I} [(K + \text{count}(\Gamma_2))] \mathbf{1}$$

$$T_{1.2} = \mathbb{M} 0 ([a < I]\tau_1) \multimap \tau_2$$

$$T_2 = [(J + \text{count}(\Gamma))] \mathbf{1} \multimap \mathbb{M} 0 (!_{a < I} \mathbb{M} 0 (\tau_1)) \multimap \mathbb{M} 0 (\tau_2)$$

$$T_{2.1} = [(J + \text{count}(\Gamma))] \mathbf{1}$$

$$T_{2.2} = \mathbb{M} 0 ((!_{a < I} \mathbb{M} 0 (\tau_1)) \multimap [I] \mathbf{1} \multimap \mathbb{M} 0 (\tau_2))$$

$$T_{2.21} = (!_{a < I} \mathbb{M} 0 (\tau_1)) \multimap [I] \mathbf{1} \multimap \mathbb{M} 0 (\tau_2)$$

$$T_{2.22} = [I] \mathbf{1} \multimap \mathbb{M} 0 (\tau_2)$$

$$T_3 = \mathbb{M} 0 (\tau_2)$$

$$T_{3.1} = \mathbb{M} I (\tau_2)$$

$$T_4 = \mathbb{M} 0 (\tau_1)$$

$$T_{4.1} = !_{a < I} \mathbb{M} 0 (\tau_1)$$

$$T_5 = [(K + \text{count}(\Gamma_2))] \mathbf{1} \multimap \mathbb{M} 0 (\tau_1)$$

$$T_{5.0} = !_{a < I} [(K + \text{count}(\Gamma_2))] \mathbf{1} \multimap \mathbb{M} 0 (\tau_1)$$

$$T_{5.1} = !_{a < I} [(K + \text{count}(\Gamma_2))] \mathbf{1} \multimap !_{a < I} \mathbb{M} 0 (\tau_1)$$

D0.7:

$$\frac{}{;\Theta; \Delta; ; c : T_{1.15} \vdash c : T_{1.15}} \text{T-var}$$

D0.6:

$$\frac{\frac{\frac{}{;\Theta; a; \Delta, a < I; (\Gamma_2); \cdot \vdash e_{t2} : T_5} \text{IH}}{;\Theta; \Delta; \sum_{a < I} (\Gamma_2); \cdot \vdash !e_{t2} : T_{5.0}} \text{subExpI} \quad D0.7}}{;\Theta; \Delta; \sum_{a < I} (\Gamma_2); c : T_{1.15} \vdash \text{coerce1 } !e_{t2} c : T_{4.1}} \text{Lemma 32}}$$

D0.5:

$$\frac{\frac{}{;\Theta; \Delta; \sum_{a < I} (\Gamma_2); b : T_{2.21} \vdash b : T_{2.21}} \quad D0.6}}{;\Theta; \Delta; \sum_{a < I} (\Gamma_2); b : T_{2.21}, c : T_{1.15} \vdash b (\text{coerce1 } !e_{t2} c) : T_{2.22}} \text{T-app}}$$

D0.4:

$$\frac{\frac{}{;\Theta; \Delta; ; d : [I] \mathbf{1} \vdash d : [I] \mathbf{1}} \quad D0.5}}{;\Theta; \Delta; \sum_{a < I} (\Gamma_2); b : T_{2.21}, c : T_{1.15}, d : [I] \mathbf{1} \vdash b (\text{coerce1 } !e_{t2} c) d : T_3}}$$

D0.3:

$$\frac{\frac{}{;\Theta; \Delta; ; \cdot \vdash \text{store}() : \mathbb{M} I [I] \mathbf{1}} \quad D0.4}}{;\Theta; \Delta; \sum_{a < I} (\Gamma_2); b : T_{2.21}, c : T_{1.15} \vdash E_5 : T_{3.1}} \text{bind}}$$

D0.21:

$$\frac{}{;\Theta; \Delta \vdash T_{1.14} <: T_{1.15}} \text{sub-bSum}$$

D0.2:

$$\frac{\frac{\frac{}{;\Theta; \Delta; ; \cdot \vdash !() : !_{a < I} \mathbf{1}} \quad D0.21}}{;\Theta; \Delta; ; \cdot \vdash \text{store}() : T_{1.13}} \text{T-sub} \quad D0.3}}{;\Theta; \Delta; ; \cdot \vdash \text{store}() : T_{1.131}} \text{bind}}{;\Theta; \Delta; \sum_{a < I} (\Gamma_2); b : T_{2.21} \vdash E_4 : T_{1.12}}$$

D0.12:

$$\frac{}{;\Theta; \Delta; \cdot; a : T_{2.1} \vdash a : T_{2.1}} \text{T-var}$$

D0.11:

$$\frac{}{;\Theta; \Delta; (\Gamma_1); \cdot \vdash e_{t1} : T_1} \text{IH1}$$

D0.1:

$$\frac{\frac{D0.11 \quad D0.12}{;\Theta; \Delta; (\Gamma_1); a : T_{2.1} \vdash e_{t1} \ a : T_{2.2}} \text{app} \quad D0.2}{;\Theta; \Delta; (\Gamma_1) \oplus \sum_{a < I} (\Gamma_2); a : T_{2.1} \vdash E_3 : T_{1.12}} \text{bind}$$

D0:

$$\frac{\frac{}{;\Theta; \Delta; \cdot; \cdot \vdash \text{store}() : T_{1.11}} \text{D0.1}}{;\Theta; \Delta; (\Gamma_1) \oplus \sum_{a < I} (\Gamma_2); \cdot \vdash E_2 : T_{0.3}} \text{bind}$$

D0.0:

$$\frac{\frac{}{\Theta; \Delta \vdash \Gamma' \sqsubseteq \Gamma_1 \oplus \sum_{a < I} \Gamma_2} \text{By inversion}}{\Theta; \Delta \vdash (\Gamma') <: (\Gamma_1 \oplus \sum_{a < I} \Gamma_2)} \text{Lemma 25}$$

Main derivation:

$$\frac{\frac{\frac{\frac{}{;\Theta; \Delta; \cdot; p : T_{0.1} \vdash p : T_{0.1}} \text{D0}}{;\Theta; \Delta; (\Gamma_1) \oplus \sum_{a < I} (\Gamma_2); p : T_{0.1} \vdash E_1 : T_{0.2}} \text{release}}{;\Theta; \Delta; (\Gamma_1) \oplus \sum_{a < I} (\Gamma_2); \cdot \vdash E_0 : T_{0.11}}}{;\Theta; \Delta; (\Gamma_1) \oplus (\sum_{a < I} \Gamma_2); \cdot \vdash E_0 : T_{0.11}} \text{Lemma 24}}{;\Theta; \Delta; (\Gamma_1 \oplus \sum_{a < I} \Gamma_2); \cdot \vdash E_0 : T_{0.11}} \text{Lemma 23} \quad \text{D0.0}}{\frac{}{;\Theta; \Delta; (\Gamma'); \cdot \vdash E_0 : T_0} \text{T-sub, T-weaken}}$$

• fix:

$$\frac{\Theta, b, \Delta, b < L; \Gamma, x : [a < I] \sigma \vdash_K e : \tau \rightsquigarrow e_t \quad \tau[0/a] <: \mu \quad \Theta, a, b, \Delta, a < I, b < L \vdash \tau[(b+1 + \bigoplus_b^{b+1,a} I)/b] <: \sigma \quad \Gamma' \sqsubseteq \sum_{b < L} \Gamma \quad L, M \geq \bigoplus_b^{0,1} I \quad N \geq M - 1 + \sum_{b < L} K}{\Theta; \Delta; \Gamma' \vdash_N \text{fix}x.e : \mu \rightsquigarrow E_0} \text{T-fix}$$

$$E_0 = \text{fix}Y.E_1$$

$$E_1 = \lambda p. E_2$$

$$E_2 = \text{release } - = p \text{ in } E_3$$

$$E_3 = \text{bind } A = \text{store}() \text{ in } E_4$$

$$E_4 = \text{let } !x = (E_{4.1} E_{4.2}) \text{ in } E_5$$

$$E_{4.1} = \text{coerce1 } !Y$$

$$E_{4.2} = (\lambda u. !()) A$$

$$E_5 = \text{bind } C = \text{store}() \text{ in } E_6$$

$$E_6 = e_t C$$

$$\text{cost}(b') \triangleq$$

$$\text{if } (0 \leq b' < (\bigoplus_b^{0,1} I(b))) \text{ then}$$

$$K(b') + I(b') + \text{count}(\Gamma(b')) + (\sum_{a < I(b')} \text{cost}((b' + 1 + \bigoplus_b^{b'+1,a} I(b))))$$

$$\text{else}$$

$$0$$

$$\tau'(b') = [\text{cost}(b')] \mathbf{1} \multimap \mathbb{M} 0 \langle \tau(b') \rangle$$

$$T_{0.0} = \tau'[(b' + 1 + \bigoplus_b^{b'+1,a} I)/b']$$

$$T_0 = [(N + \text{count}(\Gamma'))] \mathbf{1} \multimap \mathbb{M} 0 \langle \mu \rangle$$

$$T_{0.1} = [(M - 1 + \sum_{b' < L} K) + \text{count}(\sum_{b' < L} \Gamma)] \mathbf{1} \multimap \mathbb{M} 0 \langle \tau(0) \rangle$$

$$b'' = (b' + 1 + \bigoplus_b^{b'+1,a} I)$$

$$T_{1.0} = !_{a < I(b')} ([\text{cost}(b'')] \mathbf{1} \multimap \mathbb{M} 0 \langle \tau(b'') \rangle)$$

$$T_1 = !_{a < I(b')} [\text{cost}(b'')] \mathbf{1} \multimap !_{a < I(b')} \mathbb{M} 0 \langle \tau(b'') \rangle$$

$$T_{1.1} = !_{a < I(b')} \mathbb{M} 0 \langle \tau(b'') \rangle$$

$$T_{1.11} = \mathbb{M} 0 \langle \tau(b'') \rangle$$

$$T_{1.12} = \mathbb{M} 0 \langle \sigma \rangle$$

$$T_2 = [\sum_{a < I(b')} \text{cost}(b'')] \mathbf{1}$$

$$T_{3.0} = \sum_{a < I} \text{cost}(b'') !_{a < I} \mathbf{1}$$

$$T_3 = !_{a < I} [\text{cost}(b'')] \mathbf{1}$$

$$T_4 = \mathbb{M}(K(b') + I(b') + \text{count}(\Gamma(b'))) \langle \tau(b') \rangle$$

$$T_{4.1} = \mathbb{M}(K(b') + I(b') + \text{count}(\Gamma(b'))) [(K(b') + I(b') + \text{count}(\Gamma(b')))] \mathbf{1}$$

$$T_{4.2} = [(K(b') + I(b') + \text{count}(\Gamma(b')))] \mathbf{1}$$

$$T_5 = [(K(b') + I(b') + \text{count}(\Gamma(b')))] \mathbf{1} \multimap \mathbb{M} 0 \langle \tau(b') \rangle$$

$$T_{c0} = \mathbf{1} \multimap !_{a < I} \mathbf{1}$$

$$T_{c0.1} = [0] (\mathbf{1} \multimap !_{a < I} \mathbf{1})$$

$$T_{c1} = [\sum_{a < I} \text{cost}(b'')] \mathbf{1} \multimap [\sum_{a < I} \text{cost}(b'')] !_{a < I} \mathbf{1}$$

D5.2:

$$\frac{}{;\Theta, b'; \Delta, b' < L; ; C : T_{4.2} \vdash C : T_{4.2}} \text{var}$$

D5.10:

$$\frac{\frac{}{;\Theta, b'; \Delta, b' < L \vdash \tau(b'') <: \sigma} \text{Given}}{;\Theta, b'; \Delta, b' < L \vdash \langle \tau(b'') \rangle <: \langle \sigma \rangle} \text{Lemma 27}$$

D5.1:

$$\frac{\frac{}{;\Theta, b'; \Delta, b' < L; \langle \Gamma \rangle, x :_{a < I(b')} T_{1.12}; \cdot \vdash e_t : T_5} \text{IH} \quad D5.10}{;\Theta, b'; \Delta, b' < L; \langle \Gamma \rangle, x :_{a < I(b')} T_{1.11}; \cdot \vdash e_t : T_5} \text{T-weaken}$$

D5:

$$\frac{D5.1 \quad D5.2}{;\Theta, b'; \Delta, b' < L; \langle \Gamma \rangle, x :_{a < I(b')} T_{1.11}; C : T_{4.2} \vdash e_t \quad C : \mathbb{M}0 \langle \tau(b') \rangle} \text{app}$$

D4:

$$\frac{\frac{}{;\Theta, b'; \Delta, b' < L; ; \cdot \vdash \text{store}() : T_{4.1}} \text{D5}}{;\Theta, b'; \Delta, b' < L; \langle \Gamma \rangle, x :_{a < I(b')} T_{1.1}; C : T_{4.2} \vdash E_5 : T_4}$$

D3.2:

$$\frac{}{;\Theta, b'; \Delta, b' < L; Y :_{a < I} T_{0.0}; \cdot \vdash !Y : T_{1.0}} \text{Lemma 29}$$

D3.11:

$$\frac{D3.2}{;\Theta, b'; \Delta, b' < L; Y :_{a < I} T_{0.0}; \cdot \vdash \text{coerce1} (!Y) : T_1} \text{Lemma 32}$$

D3.12:

$$\frac{}{;\Theta, b'; \Delta, b' < L \vdash T_{3.0} <: T_3} \text{sub-bSum}$$

Dc2:

$$\frac{}{;\Theta, b'; \Delta, b' < L \vdash T_{c0.1} <: T_{c1}} \text{sub-potArrow}$$

Dc1:

$$\frac{\frac{\frac{}{;\Theta, b'; \Delta, b' < L, a < I; ; \cdot \vdash () : \mathbf{1}} \text{T-unit}}{;\Theta, b'; \Delta, b' < L; ; u : \mathbf{1} \vdash !() : !_{a < I} \mathbf{1}} \text{T-subExpI, T-weaken}}{;\Theta, b'; \Delta, b' < L; ; \cdot \vdash \lambda u. !() : T_{c0}} \text{T-lam}$$

Dc:

$$\frac{Dc1 \quad \frac{}{;\Theta, b'; \Delta, b' < L \vdash T_{c0} <: T_{c0.1}} \text{sub-potZero}}{;\Theta, b'; \Delta, b' < L; ; \cdot \vdash \lambda u. !() : T_{c0.1}} \text{T-sub} \quad Dc2}{;\Theta, b'; \Delta, b' < L; ; \cdot \vdash \lambda u. !() : T_{c1}} \text{T-sub}$$

D3.1:

$$\begin{array}{c}
Dc \quad \frac{\overline{\cdot; \Theta, b'; \Delta, b' < L; \cdot; A : T_2 \vdash A : T_2} \text{ var}}{\cdot; \Theta, b'; \Delta, b' < L; \cdot; A : T_2 \vdash (\lambda u.!(\cdot)) A : T_{3.0}} \text{ T-app} \quad D3.12 \\
D3.11 \quad \frac{\overline{\cdot; \Theta, b'; \Delta, b' < L; \cdot; A : T_2 \vdash (\lambda u.!(\cdot)) A : T_3} \text{ T-sub}}{\cdot; \Theta, b'; \Delta, b' < L; Y :_{a < I} T_{0.0}; A : T_2 \vdash E_{4.1} E_{4.2} : T_{1.1}} \text{ app}
\end{array}$$

D3:

$$\frac{D3.1 \quad D4}{\cdot; \Theta, b'; \Delta, b' < L; (\Gamma), Y :_{a < I} T_{0.0}; A : T_2 \vdash E_4 : T_4}$$

D2:

$$\frac{\overline{\cdot; \Theta, b'; \Delta, b' < L; (\Gamma'); \cdot \vdash \text{store}() : \mathbb{M}(\sum_{a < I(b')} \text{cost}(b''))T_2} \quad D3}{\cdot; \Theta, b'; \Delta, b' < L; (\Gamma'), Y :_{a < I} T_{0.0}; \cdot \vdash E_3 : \mathbb{M}(\text{cost}(b'))(\tau(b'))}$$

D1:

$$\frac{\overline{\cdot; \Theta, b'; \Delta, b' < L; (\Gamma); p : [\text{cost}(b')] \mathbf{1} \vdash p : [\text{cost}(b')] \mathbf{1}} \quad D2}{\cdot; \Theta, b'; \Delta, b' < L; (\Gamma), Y :_{a < I} T_{0.0}; p : [\text{cost}(b')] \mathbf{1} \vdash E_2 : \mathbb{M}(0)(\tau(b'))} \text{ release}$$

D0:

$$\frac{\overline{\overline{\cdot; \Theta, b'; \Delta, b' < L; (\Gamma), Y :_{a < I} T_{0.0}; \cdot \vdash E_1 : \tau'(b')} \quad D1}{\cdot; \Theta, \Delta; \sum_{a < L} (\Gamma); \cdot \vdash E_0 : \tau'(0)} \text{ T-fix}}{\cdot; \Theta, \Delta; \sum_{a < L} (\Gamma); \cdot \vdash E_0 : T_{0.1}} \text{ Claim}}{\cdot; \Theta, \Delta; (\sum_{a < L} \Gamma); \cdot \vdash E_0 : T_{0.1}} \text{ Lemma 24}}{\cdot; \Theta, \Delta; (\Gamma'); \cdot \vdash E_0 : T_{0.1}} \text{ Lemma 25, T-weaken}$$

Main derivation:

$$\frac{D0}{\cdot; \Theta, \Delta; (\Gamma'); \cdot \vdash E_0 : T_0} \text{ T-sub}$$

Claim:

$$\tau'(0) = [(M - 1 + \sum_{b' < L} K) + \text{count}(\sum_{b' < L} \Gamma)] \mathbf{1} \multimap \mathbb{M} 0 (\tau(0))$$

Proof.

It suffices to prove that

$$\text{cost}(0) = (M - 1 + \sum_{b' < L} K) + \text{count}(\sum_{b' < L} \Gamma)$$

From Definition of cost we know that

$$\begin{aligned}
\text{cost}(0) &= (\sum_{b' < L} I(b') + \sum_{b' < L} K(b')) + \sum_{b' < L} \text{count}(\Gamma) \\
&= (M - 1 + \sum_{b' < L} K(b')) + \sum_{b' < L} \text{count}(\Gamma) && \text{Definition of } I \text{ and } M \\
&= (M - 1 + \sum_{b' < L} K) + \text{count}(\sum_{b' < L} \Gamma) && \text{Lemma 26}
\end{aligned}$$

□

□

Lemma 23 (Relation b/w dlPCF context and its translation - binary sum). $\forall \Gamma_1, \Gamma_2 \in dlPCF$.

$$\langle \Gamma_1 \oplus \Gamma_2 \rangle = \langle \Gamma_1 \rangle \oplus \langle \Gamma_2 \rangle$$

Proof. Proof by induction on Γ_1

$$\begin{aligned} \frac{\Gamma_1 = .}{\langle \cdot \oplus \Gamma_2 \rangle} &= \langle \Gamma_2 \rangle && \text{Definition 3} \\ &= \langle \cdot \rangle + \langle \Gamma_2 \rangle && \text{Definition 4} \end{aligned}$$

$$\frac{\Gamma_1 = \Gamma'_1, x : [-] -}{\text{When } x : [-] - \notin \Gamma_2}$$

When $x : [-] - \notin \Gamma_2$

$$\begin{aligned} \langle \Gamma'_1, x : [a < I] \tau \oplus \Gamma_2 \rangle &= \langle (\Gamma'_1 \oplus \Gamma_2), x : [a < I] \tau \rangle && \text{Definition 3} \\ &= \langle (\Gamma'_1 \oplus \Gamma_2) \rangle, x :_{a < I} \mathbb{M} 0 \langle \tau \rangle && \text{Definition 21} \\ &= \langle (\Gamma'_1) \oplus \langle \Gamma_2 \rangle \rangle, x :_{a < I} \mathbb{M} 0 \langle \tau \rangle && \text{IH} \\ &= \langle \Gamma'_1 \rangle, x :_{a < I} \mathbb{M} 0 \langle \tau \rangle \oplus \langle \Gamma_2 \rangle && \text{Definition 4} \\ &= \langle \Gamma'_1, x : [a < I] \tau \rangle \oplus \langle \Gamma_2 \rangle && \text{Definition 4} \end{aligned}$$

When $x : [b < J] \tau [I + b/c] \in \Gamma_2$

Let $\langle \Gamma'_1, x : [a < I] \tau [a/c] \oplus \Gamma'_2, x : [b < J] \tau [I + b/c] \rangle = \Gamma_r$

$$\begin{aligned} \Gamma_r &= \langle (\Gamma'_1 \oplus \Gamma'_2), x : [c < (I + J)] \tau \rangle && \text{Definition 3} \\ &= \langle (\Gamma'_1 \oplus \Gamma'_2) \rangle, x :_{c < (I + J)} \mathbb{M} 0 \langle \tau \rangle && \text{Definition 21} \\ &= \langle (\Gamma'_1) \oplus \langle \Gamma'_2 \rangle \rangle, x :_{c < (I + J)} \mathbb{M} 0 \langle \tau \rangle && \text{IH} \\ &= \langle \Gamma'_1 \rangle, x :_{a < I} \mathbb{M} 0 \langle \tau \rangle [a/c] \oplus \langle \Gamma'_2 \rangle, x :_{b < J} \mathbb{M} 0 \langle \tau \rangle [I + b/c] && \text{Definition 4} \\ &= \langle \Gamma'_1 \rangle, x :_{a < I} \mathbb{M} 0 \langle \tau [a/c] \rangle \oplus \langle \Gamma'_2 \rangle, x :_{b < J} \mathbb{M} 0 \langle \tau [I + b/c] \rangle && \text{Lemma 28} \\ &= \langle \Gamma'_1, x : [a < I] \tau [a/c] \rangle \oplus \langle \Gamma'_2, x : [b < J] \tau [I + b/c] \rangle && \text{Definition 4} \end{aligned}$$

□

Lemma 24 (Relation b/w dlPCF context and its translation - bounded sum). $\forall \Gamma \in dlPCF$.

$$\langle \sum_{a < I} \Gamma \rangle = \sum_{a < I} \langle \Gamma \rangle$$

Proof. Proof by induction on Γ

$$\begin{aligned} \frac{\Gamma = .}{\langle \sum_{a < I} \cdot \rangle} &= \langle \cdot \rangle && \text{Definition 1} \\ &= . && \text{Definition 21} \\ &= \sum_{a < I} \langle \cdot \rangle && \text{Definition 2} \end{aligned}$$

$$\frac{\Gamma = \Gamma', x : [-] -}{\text{Let } \langle \sum_{a < I} (\Gamma', x : [b < J] \sigma [\sum_{d < a} J[d/a] + b/c]) \rangle = \Gamma_r}$$

$$\begin{aligned} \Gamma_r &= \langle \sum_{a < I} (\Gamma'), x : [c < \sum_{a < I} J] \sigma \rangle && \text{Definition 1} \\ &= \langle \sum_{a < I} (\Gamma') \rangle, x :_{c < \sum_{a < I} J} \mathbb{M} 0 \langle \sigma \rangle && \text{Definition 21} \\ &= \sum_{a < I} \langle \Gamma' \rangle, x :_{c < \sum_{a < I} J} \mathbb{M} 0 \langle \sigma \rangle && \text{IH} \\ &= \sum_{a < I} \langle \Gamma' \rangle, x :_{b < J} \mathbb{M} 0 \langle \sigma [\sum_{d < a} J[d/a] + b/c] \rangle && \text{Definition 2} \\ &= \sum_{a < I} \langle \Gamma' \rangle, x :_{b < J} \mathbb{M} 0 \langle \sigma [\sum_{d < a} J[d/a] + b/c] \rangle && \text{Lemma 28} \\ &= \sum_{a < I} \langle \Gamma', x : [b < J] \sigma [\sum_{d < a} J[d/a] + b/c] \rangle && \text{Definition 21} \end{aligned}$$

□

Lemma 25 (Relation b/w dlPCF context and its translation - subtyping). $\forall \Gamma, \Gamma' \in dlPCF$.

$$\Theta; \Delta \models \Gamma_1 \sqsubseteq \Gamma_2 \implies \cdot; \Theta; \Delta \models \langle \Gamma_1 \rangle <: \langle \Gamma_2 \rangle$$

Proof. Proof by induction on the $\Theta; \Delta \vdash \Gamma_1 \sqsubseteq \Gamma_2$ relation

1. dlpcf-sub-mBase:

$$\frac{}{.; \Theta; \Delta \vdash \langle \Gamma_1 \rangle <: .} \text{ sub-mBase}$$

2. dlpcf-sub-mInd:

D4:

$$\frac{\frac{}{.; \Theta; \Delta \vdash \Gamma_1/x <: \Gamma_2} \text{ By inversion}}{.; \Theta; \Delta \vdash \langle \Gamma_1 \rangle/x <: \langle \Gamma_2 \rangle} \text{ IH}$$

D3:

$$\frac{}{\Theta; \Delta \vdash I \leq J} \text{ By inversion}$$

D2:

$$\frac{\frac{\frac{}{.; \Theta, a; \Delta, a < I \vdash \tau' <: \tau} \text{ By inversion}}{.; \Theta, a; \Delta, a < I \vdash \langle \tau' \rangle <: \langle \tau \rangle} \text{ Lemma 27}}{.; \Theta, a; \Delta, a < I \vdash \text{M}0 \langle \tau' \rangle <: \text{M}0 \langle \tau \rangle}$$

D1:

$$\frac{\frac{}{x : [a < J] \tau' \in \Gamma_1} \text{ By inversion}}{x :_{a < J} \text{M}0 \langle \tau' \rangle \in \langle \Gamma_1 \rangle} \text{ Definition 21}$$

Main derivation:

$$\frac{\frac{D1 \quad D2 \quad D3 \quad D4}{.; \Theta; \Delta \vdash \langle \Gamma_1 \rangle <: \langle \Gamma'_2 \rangle, x :_{a < I} \text{M}0 \langle \tau \rangle}}{.; \Theta; \Delta \vdash \langle \Gamma_1 \rangle <: \langle \Gamma'_2, x : [a < I] \tau \rangle}$$

□

Lemma 26. $\forall L, \Gamma.$

$$\sum_{a < L} \text{count}(\Gamma) = \text{count}(\sum_{a < L} \Gamma)$$

Proof. By induction on Γ

$$\frac{}{\Gamma = .}$$

From Definition of *count* we know that $\text{count}(\cdot) = 0$ therefore

$$\sum_{a < L} \text{count}(\cdot) = 0$$

From Definition 2 we know that $\sum_{a < L} \cdot = \cdot$.

Therefore again from Definition of *count* we know that $\text{count}(\cdot) = 0$

And we are done

$$\begin{aligned} \frac{\Gamma = \Gamma', x :_{b < J} \tau}{\text{count}(\sum_{a < L} \Gamma', x :_{b < J} \tau)} &= \text{count}(\sum_{a < L} \Gamma', x :_{c < \sum_{a < L} J} \sigma) && \text{Definition 2} \\ &\text{where } \tau = \sigma[(\sum_{d < a} J[d/a] + b)/c] \\ &= \text{count}(\sum_{a < L} \Gamma') + \sum_{a < L} J && \text{Definition count}(\cdot) \\ &= \sum_{a < L} \text{count}(\Gamma') + \sum_{a < L} J && \text{IH} \\ &= \sum_{a < L} \text{count}(\Gamma', x :_{b < J} \tau) \end{aligned}$$

□

Lemma 27 (Subtyping is preserved by translation). $\Theta; \Delta \vdash^D \sigma <: \tau \implies \Theta; \Delta \vdash^A (\sigma) <: (\tau)$

Proof. By induction on $\Theta; \Delta \vdash^D \sigma <: \tau$

1. $[a < I]\sigma_1 \multimap \sigma_2 <: [a < J]\tau_1 \multimap \tau_2$:

D1:

$$\frac{\frac{\overline{\Theta; \Delta \vdash^A I \leq J} \text{ By inversion} \quad \overline{\Theta; \Delta \vdash^A (\sigma_2) <: (\tau_2)} \text{ IH2}}{\Theta; \Delta \vdash^A [J] \mathbf{1} <: [I] \mathbf{1}} \quad \overline{\Theta; \Delta \vdash^A \mathbb{M} 0 (\sigma_2) <: \mathbb{M} 0 (\tau_2)}}{\Theta; \Delta \vdash^A [I] \mathbf{1} \multimap \mathbb{M} 0 (\sigma_2) <: [J] \mathbf{1} \multimap \mathbb{M} 0 (\tau_2)}$$

Main derivation:

$$\frac{\frac{\overline{\Theta, a; \Delta \vdash^A I \leq J} \text{ By inversion} \quad \frac{\overline{\Theta; \Delta \vdash^A (\tau_1) <: (\sigma_1)} \text{ IH1}}{\Theta; \Delta \vdash^A \mathbb{M} 0 (\tau_1) <: \mathbb{M} 0 (\sigma_1)} \quad D1}{\Theta; \Delta \vdash^A !_a < J \mathbb{M} 0 (\tau_1) <: !_a < I \mathbb{M} 0 (\sigma_1)}}{\Theta; \Delta \vdash^A !_a < I \mathbb{M} 0 (\sigma_1) \multimap [I] \mathbf{1} \multimap \mathbb{M} 0 (\sigma_2) <: !_a < J \mathbb{M} 0 (\tau_1) \multimap [J] \mathbf{1} \multimap \mathbb{M} 0 (\tau_2)}$$

□

Lemma 28 (Index Substitution lemma). $\forall \tau \in dlPCF, J$.

$$(\tau)[J/b] = (\tau[J/b])$$

Proof. By induction on τ

1. $\tau = b$:

$$\begin{aligned} & (b)[J/b] \\ &= b \\ &= (b[J/b]) \end{aligned}$$

2. $\tau = [a < I]\tau_1 \multimap \tau_2$:

$$\begin{aligned} & ([a < I]\tau_1 \multimap \tau_2)[J/b] \\ &= !_a < I \mathbb{M} 0 (\tau_1) \multimap [I] \mathbf{1} \multimap \mathbb{M} 0 (\tau_2)[J/b] \\ &= !_a < I [J/b] \mathbb{M} 0 (\tau_1)[J/b] \multimap [I][J/b] \mathbf{1} \multimap \mathbb{M} 0 (\tau_2)[J/b] \\ &= !_a < I [J/b] \mathbb{M} 0 (\tau_1[J/b]) \multimap [I][J/b] \mathbf{1} \multimap \mathbb{M} 0 (\tau_2[J/b]) \quad (\text{From IH}) \\ &= ([a < I][J/b]\tau_1[J/b] \multimap \tau_2[J/b]) \end{aligned}$$

□

Lemma 29. $\Psi; \Theta; \Delta; x :_{a < I} \tau; \cdot \vdash !x : !_a < I \tau$

Proof.

$$\frac{\frac{\overline{\Psi; \Theta, a; \Delta, a < I; x :_{b < 1} \tau[a + b/a]; \cdot \vdash x : \tau} \text{ T-var2}}{\Psi; \Theta; \Delta; \sum_{a < I} x :_{b < 1} \tau[a + b/a]; \cdot \vdash !x : !_a < I \tau} \text{ T-subExpI}}{\Psi; \Theta; \Delta; x :_{a < I} \tau; \cdot \vdash !x : !_a < I \tau} \text{ Lemma 30}$$

□

Lemma 30. $\sum_{a < I} x :_{b < 1} \tau[a + b/a] = x_{a < I} \tau$

Proof. It suffices to prove that

$$\sum_{a < I} x :_{b < 1} \tau[a + b/a] = x_{c < I} \tau[c/a]$$

From Definition 2 it suffices to prove that

$$\sum_{a < I} x :_{b < 1} \tau[a + b/a] = x_{c < \sum_{a < I} 1} \tau[c/a]$$

Again from Definition 2 it suffices to prove that

$$\tau[c/a][(\sum_{d < a} 1[d/a] + b)/c] = \tau[a + b/a]$$

$$\tau[c/a][(\sum_{d < a} 1[d/a] + b)/c] =$$

$$\tau[c/a][(\sum_{d < a} 1[d/a] + b)/c] =$$

$$\tau[c/a][(a + b)/c] =$$

$$\tau[(a + b)/a]$$

So, we are done □

Definition 31 (Coercion function). $coerce1 F X \triangleq$
 $let! f = F in let! x = X in!(f x)$

Lemma 32 (Coerce is well-typed). $;\cdot; \cdot; \cdot; \cdot \vdash coerce1 :!_{a < I}(\tau_1 \multimap \tau_2) \multimap !_{a < I}\tau_1 \multimap !_{a < I}\tau_2$

Proof. D2.2

$$\frac{}{;\cdot; a < I; x :_{b < 1} \tau_1[a + b/a]; \cdot \vdash x : \tau_1}$$

D2.1:

$$\frac{}{;\cdot; a < I; f :_{b < 1} (\tau_1 \multimap \tau_2)[a + b/a]; \cdot \vdash f : \tau_1 \multimap \tau_2}$$

D2:

$$\frac{\frac{D2.1 \quad D2.2}{;\cdot; a < I; f :_{b < 1} (\tau_1 \multimap \tau_2)[a + b/a], x :_{b < 1} \tau_1[a + b/a]; \cdot \vdash (f x) : \tau_2} T\text{-subExpI}}{;\cdot; \cdot; \sum_{a < I} f :_{b < 1} (\tau_1 \multimap \tau_2)[a + b/a], x :_{b < 1} \tau_1[a + b/a]; \cdot \vdash !(f x) :!_{a < I}\tau_2} \text{Lemma 33}}{;\cdot; \cdot; f :_{a < I} (\tau_1 \multimap \tau_2), x :_{a < I} \tau_1; \cdot \vdash !(f x) :!_{a < I}\tau_2}$$

D1:

$$\frac{\frac{}{;\cdot; \cdot; f :_{a < I} (\tau_1 \multimap \tau_2); X :!_{a < I}\tau_1 \vdash !(f x)} D2}{;\cdot; \cdot; f :_{a < I} (\tau_1 \multimap \tau_2); \cdot \vdash let! x = X in!(f x)}$$

D0:

$$\frac{\frac{}{;\cdot; \cdot; \cdot; F :!_{a < I}(\tau_1 \multimap \tau_2) \vdash F :!_{a < I}(\tau_1 \multimap \tau_2)} T\text{-var1} \quad D1}{;\cdot; \cdot; \cdot; F :!_{a < I}(\tau_1 \multimap \tau_2) \vdash let! f = F in let! x = X in!(f x)}$$

Main derivation:

$$\frac{\frac{}{;\cdot; \cdot; \cdot; F :!_{a < I}(\tau_1 \multimap \tau_2) \vdash \lambda X. let! f = F in let! x = X in!(f x)} D0}{;\cdot; \cdot; \cdot; \cdot \vdash \lambda F. \lambda X. let! f = F in let! x = X in!(f x) :!_{a < I}(\tau_1 \multimap \tau_2) \multimap !_{a < I}\tau_1 \multimap !_{a < I}\tau_2}$$

□

Lemma 33. $\sum_{a < I} f :_{b < 1} (\tau_1 \multimap \tau_2)[a + b/a], x :_{b < 1} \tau_1[a + b/a] = f :_{a < I} \tau_1 \multimap \tau_2, x :_{a < I} \tau_1$

Proof. It suffices to prove that

$$\sum_{a < I} f :_{b < 1} (\tau_1 \multimap \tau_2)[a + b/a], x :_{b < 1} \tau_1[a + b/a] = f :_{c < I} (\tau_1 \multimap \tau_2)[c/a], x :_{c < I} \tau_1[c/a]$$

From Definition 2 it suffices to prove that

$$\sum_{a < I} f :_{b < 1} (\tau_1 \multimap \tau_2)[a + b/a], x :_{b < 1} \tau_1[a + b/a] = f :_{c < \sum_{a < I} 1} (\tau_1 \multimap \tau_2)[c/a], x :_{c < \sum_{a < I} 1} \tau_1[c/a]$$

Again from Definition 2 it suffices to prove that

1. $(\tau_1 \multimap \tau_2)[c/a][(\sum_{d < a} 1[d/a] + b)/c] = (\tau_1 \multimap \tau_2)[a + b/a]$:

$$\begin{aligned} & (\tau_1 \multimap \tau_2)[c/a][(\sum_{d < a} 1[d/a] + b)/c] = \\ & (\tau_1 \multimap \tau_2)[c/a][(\sum_{d < a} 1[d/a] + b)/c] = \\ & (\tau_1 \multimap \tau_2)[c/a][(\sum_{d < a} 1[d/a] + b)/c] = \\ & (\tau_1 \multimap \tau_2)[c/a][(a + b)/c] = \\ & (\tau_1 \multimap \tau_2)[(a + b)/a] \end{aligned}$$
2. $\tau_1[c/a][(\sum_{d < a} 1[d/a] + b)/c] = \tau_1[a + b/a]$:

$$\begin{aligned} & \tau_1[c/a][(\sum_{d < a} 1[d/a] + b)/c] = \\ & \tau_1[c/a][(\sum_{d < a} 1[d/a] + b)/c] = \\ & \tau_1[c/a][(a + b)/c] = \\ & \tau_1[(a + b)/a] \end{aligned}$$

So, we are done □

1.5.2 Cross-language model: dlPCF to λ -amor

Definition 34 (Logical relation for dlPCF to λ -Amor).

$$\begin{aligned} [\mathbf{b}]_V & \triangleq \{(s v, t v) \mid s v \in [\mathbf{b}] \wedge t v \in [\mathbf{b}] \wedge s v = t v\} \\ [[a < I]\tau_1 \multimap \tau_2]_V & \triangleq \{(\lambda x. e_s, \lambda x. \lambda p. \text{let } !x = y \text{ in } e_t) \mid \forall e'_s, e'_t. \\ & (e'_s, e'_t) \in [[a < I]\tau_1]_{NE} \implies (e_s[e'_s/x], e_t[e'_t/y][() / p]) \in [\tau_2]_E\} \\ [\tau]_E & \triangleq \{(e_s, e_t) \mid \forall^s v. e_s \Downarrow^s v \implies \exists^t v_t, {}^t v_f, J. e_t \Downarrow^t v_t \Downarrow^J {}^t v_f \wedge (s v, {}^t v_f) \in [\tau]_V\} \\ [[a < I]\tau]_{NE} & \triangleq \{(e_s, e_t) \mid \exists e'_t. e_t = \text{coerce } 1 !e'_t !() \wedge \forall 0 \leq i < I. (e_s, e'_t()) \in [\tau[i/a]]_E\} \end{aligned}$$

Definition 35 (Interpretation of typing contexts).

$$[\Gamma]_E = \{(\delta_s, \delta_t) \mid (\forall x : [a < J]\tau \in \text{dom}(\Gamma). \forall 0 \leq j < J. (\delta_s(x), \delta_t(x)) \in [\tau[j/a]]_E)\}$$

Theorem 36 (Fundamental theorem). $\forall \Theta, \Delta, \Gamma, \tau, e_s, e_t, I, \delta_s, \delta_t.$

$$\begin{aligned} & \Theta; \Delta; \Gamma \vdash_I e_s : \tau \rightsquigarrow e_t \wedge (\delta_s, \delta_t) \in [\Gamma \iota]_E \wedge . \models \Delta \iota \\ & \implies \\ & (e_s \delta_s, e_t () \delta_t) \in [\tau \iota]_E \end{aligned}$$

Proof. Proof by induction on the translation relation:

1. var:

$$\frac{\Theta; \Delta \models I \geq 1 \quad \Theta; \Delta \vdash \tau'[0/a] <: \tau \quad \Theta; \Delta \models [a < I]\tau' \Downarrow \quad \Theta; \Delta \models \Gamma \Downarrow}{\Theta; \Delta; \Gamma, x : [a < I]\tau' \vdash_J x : \tau \rightsquigarrow \lambda p.\text{release } - = p \text{ in bind } - = \uparrow^1 \text{ in } x} \text{var}$$

$$E_1 = \lambda p.\text{release } - = p \text{ in bind } - = \uparrow^1 \text{ in } x$$

$$\text{Given: } (\delta_s, \delta_t) \in [\Gamma, x]_E$$

$$\text{To prove: } (x\delta_s, E_1())\delta_t \in [\tau]_E$$

This means from Definition 34 we need to prove that

$$\forall^s v. x\delta_s \Downarrow^s v \implies \exists^t v_t, {}^t v_f, J'. E_1() \Downarrow^t v_t \Downarrow^{J'} {}^t v_f \wedge ({}^s v, {}^t v_f) \in [\tau]_V$$

This means that given some ${}^s v$ s.t. $x\delta_s \Downarrow^s v$ it suffices to prove that

$$\exists^t v_t, {}^t v_f, J'. E_1() \Downarrow^t v_t \Downarrow^{J'} {}^t v_f \wedge ({}^s v, {}^t v_f) \in [\tau]_V \quad (\text{F-DA-V0})$$

Since we are given that $(\delta_s, \delta_t) \in [\Gamma, x]_E$ therefore from Definition 35 we know that

$$\forall y : [a < J]\tau'' \in \text{dom}(\Gamma, x). \forall 0 \leq i < J. (\delta_s(y), \delta_t(y)) \in [\tau''[i/a]]_E$$

This means we also have $(\delta_s(x), \delta_t(x)) \in [\tau'[0/a]]_E$. This further means that from Definition 34 we have

$$\forall^s v''. \delta_s(x) \Downarrow^s v'' \implies \exists J'', {}^t v_t'', {}^t v_f''. \delta_t(x) \Downarrow^t v_t'' \Downarrow^{J''} {}^t v_f'' \wedge ({}^s v'', {}^t v_f'') \in [\tau'[0/a]]_V \quad (\text{F-DA-V1})$$

We instantiate (F-DA-V1) with ${}^s v$ and in order to prove (F-DA-V0) we choose J' as J'' , ${}^t v_t$ as ${}^t v_t''$ and ${}^t v_f$ as ${}^t v_f''$ and we get the desired from (F-DA-V1) and Lemma 37.

2. lam:

$$\frac{\Theta; \Delta; \Gamma, x : [a < I]\tau_1 \vdash_J e : \tau_2 \rightsquigarrow e_t}{\Theta; \Delta; \Gamma \vdash_J \lambda x. e : ([a < I].\tau_1) \multimap \tau_2 \rightsquigarrow} \text{lam}$$

$$\lambda p_1. \text{ret } \lambda y. \lambda p_2. \text{let } !x = y \text{ in release } - = p_1 \text{ in release } - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_t a$$

$$E_1 = \lambda p_1. \text{ret } \lambda y. \lambda p_2. \text{let } !x = y \text{ in release } - = p_1 \text{ in release } - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_t a$$

$$E_2 = \lambda y. \lambda p_2. \text{let } !x = y \text{ in } E_3$$

$$E_3 = \text{release } - = p_1 \text{ in release } - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_t a$$

$$\text{Given: } (\delta_s, \delta_t) \in [\Gamma]_E$$

$$\text{To prove: } (\lambda x. e\delta_s, E_1())\delta_t \in [([a < I].\tau_1) \multimap \tau_2]_E$$

This means from Definition 34 we need to prove that

$$\forall^s v. \lambda x. e\delta_s \Downarrow^s v \implies \exists J', {}^t v_t, {}^t v_f. E_1() \Downarrow^t v_t \Downarrow^{J'} {}^t v_f \wedge ({}^s v, {}^t v_f) \in [([a < I].\tau_1) \multimap \tau_2]_V$$

This means that given some ${}^s v$ s.t. $\lambda x. e\delta_s \Downarrow^s v$ it suffices to prove that

$$\exists J', {}^t v_t, {}^t v_f. E_1() \Downarrow {}^t v_t \Downarrow^{J'} {}^t v_f \wedge ({}^s v, {}^t v_f) \in [([a < I].\tau_1) \multimap \tau_2]_V \quad (\text{F-DA-L0})$$

We know that ${}^s v = \lambda x. e \delta_s$. Also from E-app, E-ret we know that ${}^t v_f = E_2$ and $J' = 0$

Therefore it suffices to show $(\lambda x. e \delta_s, E_2) \in [([a < I].\tau_1) \multimap \tau_2]_V$

From Definition 34 it further suffices to prove that

$$\forall e'_s, e'_t. (e'_s, e'_t) \in [[a < I]\tau_1]_{NE} \implies (e_s[e'_s/x], E_3[e'_t/y][() / p_2]) \in [\tau_2]_E \quad (\text{F-DA-L1})$$

This means given some e'_s, e'_t s.t. $(e'_s, e'_t) \in [[a < I]\tau_1]_{NE}$. We need to prove that

$$(e_s[e'_s/x], E_2[e'_t/x][() / p_2]) \in [\tau_2]_E \quad (\text{F-DA-L1.1})$$

Since $(e'_s, e'_t) \in [[a < I]\tau_1]_{NE}$ therefore from Definition 34 we have

$$\exists e''_t. e'_t = \text{coerce1} !e''_t !() \wedge \forall 0 \leq i < I. (e'_s, e''_t()) \in [\tau_1[i/a]]_E$$

Let

$$\delta'_s = \delta_s \cup \{x \mapsto e'_s\} \text{ and}$$

$$\delta'_t = \delta_t \cup \{x \mapsto e''_t()\}$$

From Definition 35 we know that

$$(\delta'_s, \delta'_t) \in [\Gamma, x : [a < I]\tau_1]_E$$

Therefore from IH we have

$$(e_s \delta'_s, e_t() \delta'_t) \in [\tau_2]_E \quad (\text{F-DA-L2})$$

This means from Definition 34 we have

$$\forall {}^s v_b. e_s \delta'_s \Downarrow {}^s v_b \implies \exists J_b, {}^t v_{t1}, {}^t v_b. e_t() \delta'_t \Downarrow {}^t v_{t1} \Downarrow^{J_b} {}^t v_b \wedge ({}^s v_b, {}^t v_b) \in [\tau_2]_V \quad (\text{F-DA-L3})$$

Applying Definition 34 on (F-DA-L1.1) we need to prove

$$\forall {}^s v_f. e_s[e'_s/x]\delta_s \Downarrow {}^s v_f \implies \exists J_1, {}^t v_t, {}^t v_f. E_2[e'_t/x][() / p_2]\delta_t \Downarrow {}^t v_t \Downarrow^{J_1} {}^t v_f \wedge ({}^s v_f, {}^t v_f) \in [\tau_2]_V$$

This means given some ${}^s v_f$ s.t. $e_s[e'_s/x]\delta_s \Downarrow {}^s v_f$ it suffices to prove

$$\exists J_1, {}^t v_t, {}^t v_f. E_2[e'_t/x][() / p_2]\delta_t \Downarrow {}^t v_t \Downarrow^{J_1} {}^t v_f \wedge ({}^s v_f, {}^t v_f) \in [\tau_2]_V \quad (\text{F-DA-L4})$$

Therefore instantiating (F-DA-L3) with ${}^s v_f$ and we get the desired

3. app:

$$\frac{\Theta; \Delta; \Gamma \vdash_J e_1 : ([a < I].\tau_1) \multimap \tau_2 \rightsquigarrow e_{t1} \quad \Theta, a; \Delta, a < I; \Delta \vdash_K e_2 : \tau_1 \rightsquigarrow e_{t2} \quad \Gamma' \sqsubseteq \Gamma \oplus \sum_{a < I} \Delta \quad H \geq J + I + \sum_{a < I} K}{\Theta; \Delta; \Gamma' \vdash_H e_1 e_2 : \tau_2 \rightsquigarrow E_1} \text{ app}$$

$$E_1 = \lambda p. \text{release} - = p \text{ in } \text{bind } a = \text{store}() \text{ in } \text{bind } b = e_{t1} a \text{ in } \text{bind } c = \text{store}() \text{ in } E'_1$$

$$E'_1 = \text{bind } d = \text{store}() \text{ in } b (\text{coerce1} !e_{t2} c) d$$

$$\text{Given: } (\delta_s, \delta_t) \in [\Gamma']_E$$

To prove: $(e_1 e_2 \delta_s, E_1() \delta_t) \in [\tau_2 \iota]_E$

This means from Definition 34 we need to prove that

$$\forall^s v_f. (e_1 e_2) \delta_s \Downarrow^s v_f \implies \exists J', {}^t v_t, {}^t v_f. E_1() \Downarrow {}^t v_t \Downarrow^{J'} {}^t v_f \wedge ({}^s v_f, {}^t v_f) \in [\tau_2 \iota]_V$$

This means that given some ${}^s v_f$ s.t. $(e_1 e_2) \delta_s \Downarrow^s v_f$ it suffices to prove that

$$\exists J', {}^t v_t, {}^t v_f. E_1() \Downarrow {}^t v_t \Downarrow^{J'} {}^t v_f \wedge ({}^s v_f, {}^t v_f) \in [\tau_2 \iota]_V \quad (\text{F-DA-A0})$$

IH1

$$(e_1 \delta_s, e_{t1}() \delta_t) \in [([a < I] \tau_1 \multimap \tau_2) \iota]_E$$

This means from Definition 34 we have

$$\forall^s v_1. e_1 \delta_s \Downarrow^s v_1 \implies \exists J_1, {}^t v'_1, {}^t v_1. e_{t1}() \delta_t \Downarrow {}^t v'_1 \Downarrow^{J_1} {}^t v_1 \wedge ({}^s v_1, {}^t v_1) \in [([a < I] \tau_1 \multimap \tau_2) \iota]_V$$

Since we know that $(e_1 e_2) \delta_s \Downarrow^n {}^s v_f$ therefore we know that $\exists^s v_1$ s.t. $e_1 \delta_s \Downarrow^s v_1$. Therefore we have

$$\exists J_1, {}^t v'_1, {}^t v_1. e_{t1}() \delta_t \Downarrow {}^t v'_1 \Downarrow^{J_1} {}^t v_1 \wedge ({}^s v_1, {}^t v_1) \in [([a < I] \tau_1 \multimap \tau_2) \iota]_V \quad (\text{F-DA-A1})$$

Since we know that $({}^s v_1, {}^t v_1) \in [([a < I] \tau_1 \multimap \tau_2) \iota]_V$

Let ${}^s v_1 = \lambda x. e_{bs}$ and ${}^t v_1 = \lambda x. \lambda p. \text{let } !x = y \text{ in } e_{bt}$

Therefore from Definition 34 we have

$$\forall e'_s, e'_t. (e'_s, e'_t) \in [([a < I] \tau_1 \iota]_{NE} \implies (e_{bs}[e'_s/x], e_{bt}[e'_t/x][()]/p]) \in [\tau_2 \iota]_E \quad (\text{F-DA-A2})$$

IH2

$$(e_2 \delta_s, e_{t2}() \delta_t) \in [\tau_1 \iota \cup \{a \mapsto 0\}]_E$$

$$(e_2 \delta_s, e_{t2}() \delta_t) \in [\tau_1 \iota \cup \{a \mapsto 1\}]_E$$

...

$$(e_2 \delta_s, e_{t2}() \delta_t) \in [\tau_1 \iota \cup \{a \mapsto I - 1\}]_E \quad (\text{F-DA-A3})$$

We claim that

$$(e_2 \delta_s, \text{coerce } !e_{t2} !() \delta_t) \in [([a < I] \tau_1 \iota]_{NE}$$

From Definition 31 we know that

$$\text{coerce } F X \triangleq$$

$$\text{let } !f = F \text{ in let } !x = X \text{ in } !(f x)$$

therefore the desired holds from Definition 34 and (F-DA-A3)

Instantiating (F-DA-A2) with $e_2 \delta_s, \text{coerce } !e_{t2} !() \delta_t$ we get

$$(e_{bs}[e_2 \delta_s/x], e_{bt}[\text{coerce } !e_{t2} !() \delta_t/x]()) \in [\tau_2 \iota]_E \quad (\text{F-DA-A4})$$

This further means that from Definition 34 we have

$$\forall^s v_{bf}. e_{bs}[e_2 \delta_s/x] \Downarrow^s v_{bf} \implies \exists J_2, {}^t v_{tb}, {}^t v_{bf}. e_{bt}[\text{coerce } !e_{t2} !() \delta_t/x]() \Downarrow {}^t v_{tb} \Downarrow^{J_2} {}^t v_{bf} \wedge ({}^s v_{bf}, {}^t v_{bf}) \in [\tau_2 \iota]_V$$

Since we know that $(e_1 e_2)\delta_s \Downarrow^n s v_f$ therefore we know that $\exists^s v_{bf}, n_2$ s.t $e_{bs}[e_2\delta_s/x] \Downarrow^{n_2} s v_{bf}$. Therefore we have

$$\exists J_2, {}^t v_{tb}, {}^t v_{bf}. e_{bt}[coerce !e_{t2} !(\delta_t/x)]() \Downarrow {}^t v_{tb} \Downarrow^{J_2} {}^t v_{bf} \wedge ({}^s v_{bf}, {}^t v_{bf}) \in [\tau_2 \iota]_V \quad (\text{F-DA-A5})$$

In order to prove (F-DA-A0) we choose J' as $J_1 + J_2$, ${}^t v_t$ as ${}^t v_{tb}$ and ${}^t v_f$ as ${}^t v_{bf}$, we get the desired from (F-DA-A1) and (F-DA-A5)

4. fix:

$$\frac{\begin{array}{c} \Theta, b; \Delta, b < L; \Gamma, x : [a < I] \sigma \vdash_K e : \tau \rightsquigarrow e_t \\ \tau[0/a] <: \mu \quad \Theta, a, b; \Delta, a < I, b < L; \Gamma \vdash \tau[(b+1 + \frac{b+1,a}{b} I)/b] <: \sigma \\ \Gamma' \sqsubseteq \sum_{b < L} \Gamma \quad L, M \geq \frac{0,1}{b} I \quad N \geq M - 1 + \sum_{b < L} K \end{array}}{\Theta; \Delta; \Gamma' \vdash_N \text{fix}.e : \mu \rightsquigarrow E_0} \text{ T-fix}$$

$$E_0 = \text{fix}Y.E_1$$

$$E_1 = \lambda p.E_2$$

$$E_2 = \text{release } - = p \text{ in } E_3$$

$$E_3 = \text{bind } A = \text{store}() \text{ in } E_4$$

$$E_4 = \text{let } !x = (E_{4.1} E_{4.2}) \text{ in } E_5$$

$$E_{4.1} = \text{coerce}1 !Y$$

$$E_{4.2} = (\lambda u.!(\)) A$$

$$E_5 = \text{bind } C = \text{store}() \text{ in } E_6$$

$$E_6 = e_t C$$

Given: $(\delta_s, \delta_t) \in [\Gamma]_E$

To prove: $(\text{fix}.e\delta_s, (\text{fix}Y.E_1)(\delta_t)) \in [\mu \iota]_E$

This means from Definition 34 we need to prove that

$$\forall^s v. \text{fix}.e\delta_s \Downarrow^s v \implies \exists J', {}^t v_t, {}^t v_f. E_0() \Downarrow {}^t v_t \Downarrow^{J'} {}^t v_f \wedge ({}^s v, {}^t v_f) \in [\mu \iota]_V$$

This means that given some ${}^s v$ s.t $\text{fix}.e\delta_s \Downarrow^s v$ it suffices to prove that

$$\exists J', {}^t v_t, {}^t v_f. E_0() \Downarrow {}^t v_t \Downarrow^{J'} {}^t v_f \wedge ({}^s v, {}^t v_f) \in [\mu \iota]_V \quad (\text{F-DA-F0})$$

Claim 1

$$\forall 0 \leq t < L. (e \delta'_s, E_1() \delta'_t) \in [\tau[t/b] \iota]_E$$

where $\delta'_s = \delta_s \cup \{x \mapsto (\text{fix}.e)\delta_s\}$ and $\delta'_t = \delta_t \cup \{x \mapsto (\text{fix}.E_1)\delta_t\}$

We prove this by induction on the recursion tree

Base case: when t is a leaf node

Since for a leaf node $I(t) = 0$ and $x \notin \text{free}(e)$ therefore from IH (outer induction) we get

$$(e \delta_s, e_t () \delta_t) \in [\tau[t/b] \iota]_E$$

This means from Definition 34 we have

$$\forall^s v'. e_s \delta_s \Downarrow^s v \implies \exists^t v'_t, t v'_f, J'. e_t () \delta_t \Downarrow^t v'_t \Downarrow^{J'} t v'_f \wedge ({}^s v', t v'_f) \in [\tau[t/b] \iota]_V \quad (\text{BC0})$$

Since we have to prove $(e \delta'_s, E_1 () \delta'_t) \in [\tau[t/b] \iota]_E$

Therefore from Definition 34 it suffices to prove that

$$\forall^s v. e_s \delta'_s \Downarrow^s v \implies \exists^t v_t, t v_f, J. E_1 () \Downarrow^t v_t \Downarrow^J t v_f \wedge ({}^s v, t v_f) \in [\tau[t/b] \iota]_V$$

This means given some ${}^s v$ s.t $e_s \delta'_s \Downarrow^s v$ it suffices to prove that

$$\exists^t v_t, t v_f, J. E_1 () \Downarrow^t v_t \Downarrow^J t v_f \wedge ({}^s v, t v_f) \in [\tau[t/b] \iota]_V \quad (\text{BC1})$$

Instantiating (BC0) with ${}^s v$ we get

$$\exists^t v'_t, t v'_f, J'. e_t () \delta'_t \Downarrow^t v'_t \Downarrow^{J'} t v'_f \wedge ({}^s v', t v'_f) \in [\tau[t/b] \iota]_V \quad (\text{BC2})$$

From E-release, E-bind, E-subExpE we also know that if

$$e_t () \delta_t \Downarrow^t v'_t \Downarrow^{J'} t v'_f \text{ then } E_1 () \delta'_t \Downarrow^t v'_t \Downarrow^{J'} t v'_f$$

Therefore we get we choose ${}^t v_t, t v_f, J$ as ${}^t v'_t, t v'_f, J'$ in (BC1) and we get the desired from (BC2)

Inductive case: when t is a some internal node

From IH we know that

$$\forall 0 \leq a < I(t). (e \delta'_s, E_1 () \delta'_t) \in [\tau[t'/b] \iota]_E \text{ where } t' = (t + 1 + \bigtriangleup_b^{t+1,a} I(t))$$

Since $\Theta, a, b; \Delta, a < I, b < L; . \vdash \tau[(b + 1 + \bigtriangleup_b^{b+1,a} I)/b] <: \sigma$ therefore from Lemma 38 we know that

$$\forall 0 \leq a < I(t). (e \delta'_s, E_1 () \delta'_t) \in [\sigma \iota]_E \quad (\text{F-DA-F0.1})$$

Claim 2

$$(e \delta'_s, E_1 () \delta'_t) \in [\sigma \iota]_E \implies ((\text{fix}x.e) \delta_s, ((\text{fix}x.(\lambda p.E_2)) ()) \delta_t) \in [\sigma \iota]_E$$

Proof is trivial □

Since from (F-DA-F0.1) we know that

$$\forall 0 \leq a < I(t). (e \delta'_s, E_1 () \delta'_t) \in [\sigma \iota]_E$$

Therefore from Claim2 we also get

$$\forall 0 \leq a < I. (\text{fix}x.e \delta_s, \text{fix}x.E_1 () \delta_t) \in [\sigma \iota]_E$$

Let

$$\delta''_s = \delta_s \cup \{x \mapsto \text{fix}x.e \delta_s\}$$

$$\delta''_t = \delta_t \cup \{x \mapsto ((\text{fix}x.E_1) \delta_t ())\}$$

From Definition 35 it can be seen that $(\delta''_s, \delta''_t) \in [\Gamma, x :_{a < I} \sigma]_E$

Therefore from IH (outer induction) we get

$$(e \delta''_s, e_t \delta''_t) \in [\tau[t/b] \iota]_E$$

This means from Definition 34 we have

$$\forall^s v_0. e_s \delta''_s \Downarrow^s v_0 \implies \exists J_0, {}^t v_t, {}^t v_f. e_t \delta''_t \Downarrow^t v_t \Downarrow^{J_0} {}^t v_f \wedge ({}^s v_0, {}^t v_f) \in [\tau[t/b] \iota]_V \quad (\text{F-DA-F1})$$

In order to prove $(e \delta'_s, E_1 \delta'_t) \in [\tau[t/b] \iota]_E$ from Definition 34 it suffices to prove

$$\forall^s v_s. e \delta'_s \Downarrow^s v_s \implies \exists J_1, {}^t v'_t, {}^t v_t. E_2[() / p] \delta'_t \Downarrow^t v'_t \Downarrow^{J_1} {}^t v_t \wedge ({}^s v_s, {}^t v_t) \in [\tau[t/b] \iota]_V$$

This means given some ${}^s v_s$ s.t $e \delta'_s \Downarrow^s v_s$ and we need to prove that

$$\exists J_1, {}^t v'_t, {}^t v_t. E_2[() / p] \delta'_t \Downarrow^t v'_t \Downarrow^{J_1} {}^t v_t \wedge ({}^s v_s, {}^t v_t) \in [\tau[t/b] \iota]_V \quad (\text{F-DA-F2})$$

From E-release, E-bind, E-subExpE we also know that $E_2[() / p] \delta'_t \xrightarrow{*} e_t[(\text{fix} Y. E_1) () / x] ()$ therefore from (F-DA-F1) we get the desired.

This proves Claim1 □

Since from Claim1 we know that $\forall 0 \leq t < L. (e \delta'_s, E_1 \delta'_t) \in [\tau[t/b] \iota]_E$. Therefore instantiating it with 0 we get

$$(e \delta'_s, E_1 \delta'_t) \in [\tau[0/b] \iota]_E$$

This means from Definition 34 we have

$$\forall^s v'. e \delta'_s \Downarrow^s v' \implies \exists {}^t v'_t, {}^t v'_f, J'. E_1 \delta'_t \Downarrow^t v'_t \Downarrow^{J'} {}^t v'_f \wedge ({}^s v', {}^t v'_f) \in [\tau[0/b] \iota]_V$$

Instantiating it with the given ${}^s v$ and since know that $\text{fix} x. e \delta_s \Downarrow^s v$ therefore from E-fix we also know that $e[\text{fix} x. e / x] \delta_s \Downarrow^s v$. Hence we have

$$\exists {}^t v'_t, {}^t v'_f, J'. E_1 \delta'_t \Downarrow^t v'_t \Downarrow^{J'} {}^t v'_f \wedge ({}^s v', {}^t v'_f) \in [\tau[0/b] \iota]_V \quad (\text{F-DA-F3})$$

Since $E_1 \delta'_t \Downarrow^t v'_t \Downarrow^{J'} {}^t v'_f$ therefore from E-fix we also know that $\text{fix} x. E_1 \delta'_t \Downarrow^t v'_t \Downarrow^{J'} {}^t v'_f$. Also since $\tau[0/b] <: \mu$ therefore from (F-DA-F3) and Lemma 37 we get the desired. □

Lemma 37. $\forall \Theta, \Delta, \tau, \tau', e_s, e_t, \iota.$

$$(a) \Theta; \Delta \vdash \tau <: \tau' \wedge \models \Delta \iota \implies [\tau \iota]_V \subseteq [\tau' \iota]_V$$

$$(b) \Theta; \Delta \vdash [a < I] \tau <: [a < J] \tau' \wedge \models \Delta \iota \implies [[a < I] \tau \iota]_{NE} \subseteq [[a < J] \tau' \iota]_{NE}$$

Proof. Proof by simultaneous induction on $\Theta; \Delta \vdash \tau <: \tau'$ and $\Theta; \Delta \vdash [a < I] \tau <: [a < J] \tau'$

Proof of statement (a)

We case analyze the different cases:

1. \multimap :

$$\frac{\Theta; \Delta \vdash B <: A \quad \Theta; \Delta \vdash \tau <: \tau'}{\Theta; \Delta \vdash A \multimap \tau <: B \multimap \tau'}$$

To prove: $[(A \multimap \tau) \iota]_V \subseteq [(B \multimap \tau') \iota]_V$

This means we need to prove that

$$\forall (\lambda x.e, \lambda x.\lambda p.e_t) \in [A \multimap \tau \iota]_V. (\lambda x.e, \lambda x.\lambda p.e_t) \in [B \multimap \tau' \iota]_E$$

This means given $(\lambda x.e_s, \lambda y.\lambda p.\text{let } !x = y \text{ in } e_t) \in [A \multimap \tau \iota]_V$ and we need to prove

$$(\lambda x.e_s, \lambda y.\lambda p.\text{let } !x = y \text{ in } e_t) \in [B \multimap \tau' \iota]_V$$

This means from Definition 34 we are given that

$$\forall e'_s, e'_t. (e'_s, e'_t) \in [A \iota]_{NE} \implies (e_s[e'_s/x], e_t[e'_t/y][() / p]) \in [\tau \iota]_E \quad (\text{SV-A0})$$

And we need to prove that

$$\forall e''_s, e''_t. (e''_s, e''_t) \in [B \iota]_{NE} \implies (e_s[e''_s/x], e_t[e''_t/y][() / p]) \in [\tau' \iota]_E$$

This means given $(e''_s, e''_t) \in [B \iota]_{NE}$ we need to prove that

$$(e_s[e''_s/x], e_t[e''_t/y][() / y]) \in [\tau' \iota]_E \quad (\text{SV-A1})$$

Since we are given that $(e''_s, e''_t) \in [B \iota]_{NE}$ therefore from IH (Statement (b)) we have

$$(e''_s, e''_t) \in [A \iota]_{NE}$$

In order to prove (SV-A1) we instantiate (SV-A0) with e''_s, e''_t and we get

$$(e_s[e''_s/x], e_t[e''_t/y][() / p]) \in [\tau \iota]_E$$

Finally from Lemma 38 we get

$$(e_s[e''_s/x], e_t[e''_t/y][() / p]) \in [\tau' \iota]_E$$

Proof of statement (b)

$$\frac{\Theta; \Delta \vdash J \leq I \quad \Theta; \Delta \vdash \tau <: \tau'}{\Theta; \Delta \vdash [a < I] \tau <: [a < J] \tau'}$$

To prove: $[[a < I] \tau \iota]_{NE} \subseteq [[a < J] \tau' \iota]_{NE}$

This means we need to prove that

$$\forall (e_s, e_t) \in [[a < I] \tau \iota]_{NE}. (e_s, e_t) \in [[a < J] \tau' \iota]_{NE}$$

This means given $(e_s, e_t) \in [[a < I] \tau \iota]_{NE}$ and we need to prove

$$(e_s, e_t) \in [[a < J] \tau' \iota]_{NE}$$

This means from Definition 34 we are given

$$\exists e'_t. e_t = \text{coerce1 } !e'_t !() \wedge \forall 0 \leq i < I. (e_s, e'_t) \in [\tau[i/a] \iota]_E \quad (\text{SNE0})$$

and we need to prove

$$\exists e''_t. e_t = \text{coerce1 } !e''_t !() \wedge \forall 0 \leq j < J. (e_s, e''_t) \in [\tau'[j/a] \iota]_E \quad (\text{SNE1})$$

In order to prove (SNE1) we choose e''_t as e'_t from (SNE0) and we need to prove

$$\forall 0 \leq j < J.(e_s, e'_t) \in [\tau'[j/a] \iota]_E$$

This means given some $0 \leq j < J$ and we need to prove that

$$(e_s, e'_t) \in [\tau'[j/a] \iota]_E$$

From (SNE0) we get

$$(e_s, e'_t) \in [\tau[j/a] \iota]_E$$

And finally from Lemma 38 we get

$$(e_s, e''_t) \in [\tau'[j/a] \iota]_E$$

□

Lemma 38. $\forall \Theta, \Delta, \tau, \tau', e_s, e_t, \iota.$

$$\Theta; \Delta \vdash \tau <: \tau' \wedge \models \Delta \iota \implies [\tau \iota]_E \subseteq [\tau' \iota]_E$$

Proof. Given: $\Theta; \Delta \vdash \tau <: \tau'$

To prove: $[\tau \iota]_E \subseteq [\tau' \iota]_E$

It suffices to prove that

$$\forall (e_s, e_t) \in [\tau \iota]_E. (e_s, e_t) \in [\tau' \iota]_E$$

This means given $(e_s, e_t) \in [\tau \iota]_E$ it suffices to prove that

$$(e_s, e_t) \in [\tau' \iota]_E$$

This means from Definition 34 we are given that

$$\forall^s v_0. e_s \Downarrow^s v_0 \implies \exists J_0, {}^t v'_0, {}^t v_0. e_t \Downarrow {}^t v'_0 \Downarrow^{J_0} {}^t v_0 \wedge ({}^s v_0, {}^t v_0) \in [\tau \iota]_V \quad (S0)$$

And it suffices to prove that

$$\forall^s v. e_s \Downarrow^s v \implies \exists J, {}^t v_t, {}^t v_f. e_t \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge ({}^s v, {}^t v_f) \in [\tau' \iota]_V$$

This means given some ${}^s v$ s.t $e_s \Downarrow^s v$ and we need to prove

$$\exists J, {}^t v_t, {}^t v_f. e_t \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge ({}^s v, {}^t v_f) \in [\tau' \iota]_V \quad (S1)$$

We get the desired from (S0) and Lemma 37

□

1.5.3 Re-deriving dlPCF's soundness

Definition 39 (Closure translation).

$$\begin{aligned} \llbracket (e, []) \rrbracket &\triangleq e \\ \llbracket (e, \mathbf{C}_1, \dots, \mathbf{C}_n) \rrbracket &\triangleq \lambda x_1 \dots x_n. e \llbracket \mathbf{C}_1 \rrbracket \dots \llbracket \mathbf{C}_n \rrbracket \end{aligned}$$

Definition 40 (Krivine triple translation).

$$\begin{aligned} \llbracket (e, \rho, \epsilon) \rrbracket &\triangleq \llbracket (e, \rho) \rrbracket \\ \llbracket (e, \rho, c.\theta) \rrbracket &\triangleq (\llbracket (e, \rho) \rrbracket \llbracket c \rrbracket, \cdot, \theta) \end{aligned}$$

Lemma 41 (Type preservation for Closure translation). $\forall \Theta, \Delta, e, \rho, \tau.$

$$\Theta; \Delta \vdash_J (e, \rho) : \sigma \implies \Theta; \Delta; \cdot \vdash_J \llbracket (e, \rho) \rrbracket : \sigma$$

Proof.

$$\frac{\Theta; \Delta; x_1 : [a < I_1] \tau_1 \dots x_n : [a < I_n] \tau_n \vdash_K e : \sigma \quad \Theta, a; \Delta, a < I_i \vdash_{H_i} \mathbf{C}_i : \tau_i \quad J \geq K + I_1 + \dots + I_n + \sum_{a < I_1} H_1 + \dots + \sum_{a < I_n} H_n}{\Theta; \Delta \vdash_J (e, (\mathbf{C}_1 \dots \mathbf{C}_n)) : \sigma}$$

$$J' = K + I_1 + \dots + I_n + \sum_{a < I_1} H_1 + \dots + \sum_{a < I_n} H_n$$

D1:

$$\frac{}{\Theta, a; \Delta; . a < I_i \vdash_{H_i} \langle \mathbf{C}_i \rangle : \tau_i} \text{IH}$$

D0:

$$\frac{\frac{}{\Theta; \Delta; x_1 : [a_1 < I_1] \tau_1, \dots, x_n : [a_n < I_n] \tau_n \multimap \vdash_K e : \sigma} \text{Given}}{\Theta; \Delta; . \vdash_K \lambda x_1 \dots x_n. e : [a_1 < I_1] \tau_1 \multimap [a_2 < I_2] \tau_2 \multimap \dots [a_n < I_n] \tau_n \multimap \sigma} \text{D-lam}}$$

Main derivation:

$$\frac{\frac{\frac{D0 \quad D1}{\Theta; \Delta; . \vdash_{J'} \lambda x_1 \dots x_n. e \langle \mathbf{C}_1 \rangle \dots \langle \mathbf{C}_n \rangle : \sigma} \text{D-app}}{\Theta; \Delta; . \vdash_J \lambda x_1 \dots x_n. e \langle \mathbf{C}_1 \rangle \dots \langle \mathbf{C}_n \rangle : \sigma} \text{Lemma 3.5 of [3]}}{\Theta; \Delta; . \vdash_J \langle (e, \mathbf{C}_1 \dots \mathbf{C}_n) \rangle : \sigma} \text{Definition 39}$$

□

Theorem 42 (Type preservation for Krivine triple translation). $\forall \Theta, \Delta, e, \rho, \theta, \tau.$

$$\Theta; \Delta \vdash_I (e, \rho, \theta) : \tau \implies \Theta; \Delta; . \vdash_I \langle (e, \rho, \theta) \rangle : \tau$$

Proof.

$$\frac{\Theta; \Delta \vdash_K (e, \rho) : \sigma \quad \Theta; \Delta \vdash_J \theta : (\sigma, \tau) \quad I \geq K + J}{\Theta; \Delta \vdash_I (e, \rho, \theta) : \tau}$$

Let $I' = K + J$

Proof by induction on θ

1. Case ϵ :

Given: $\Theta; \Delta \vdash_I (e, \rho, \epsilon) : \tau$

To prove: $\Theta; \Delta; . \vdash_I \langle (e, \rho, \epsilon) \rangle : \tau$

D0:

$$\frac{}{\Theta; \Delta; . \vdash_K \langle (e, \rho) \rangle : \sigma} \text{Lemma 41}$$

Main derivation:

$$\frac{\frac{\frac{D0}{\Theta; \Delta; . \vdash_{I'} \langle (e, \rho) \rangle : \tau} \text{Lemma 3.5 of [3]}}{\Theta; \Delta; . \vdash_{I'} \langle (e, \rho, \epsilon) \rangle : \tau} \text{Definition 40}}{\Theta; \Delta; . \vdash_I \langle (e, \rho, \epsilon) \rangle : \tau} \text{Lemma 3.5 of [3]}$$

2. Case $\mathbf{C}.\theta'$:

Given: $\Theta; \Delta \vdash_I (e, \rho, \mathbf{C}.\theta') : \tau$

To prove: $\Theta; \Delta; . \vdash_I \langle (e, \rho, \mathbf{C}.\theta') \rangle : \tau$

Since $\theta = \mathbf{C}.\theta'$ therefore from dlPCF's type rule for $\mathbf{C}.\theta'$ we know that

$$\sigma = [d < L] \gamma \multimap \mu$$

That is we are given that

$$\frac{\Theta, d; \Delta, d < L_g \vdash_{K_g} \mathbf{C} : \gamma \quad \Theta; \Delta \vdash_{H_g} \theta' : (\mu, \tau) \quad J \geq H_g + \sum_{d < L_g} K_g + L_g}{\Theta; \Delta \vdash_J \mathbf{C}.\theta' : ([d < L_g]\gamma \multimap \mu, \tau)}$$

D2:

$$\frac{\overline{\Theta; \Delta \vdash_J \mathbf{C}.\theta' : ([d < L_g]\gamma \multimap \mu, \tau)} \text{ Given}}{\Theta; \Delta \vdash_{H_g} \theta' : (\mu, \tau)} \text{ By inversion}$$

D1:

$$\overline{\Theta; \Delta; \cdot \vdash_K ((e, \rho)) : [d < L_g]\gamma \multimap \mu} \text{ Lemma 41}$$

D0:

$$\frac{D1 \quad \frac{\overline{\Theta, d; \Delta, d < L_g \vdash_{K_g} \mathbf{C} : \gamma} \text{ Given}}{\Theta, d; \Delta, d < L_g \vdash_{K_g} (\mathbf{C}) : \gamma} \text{ Lemma 41}}{\Theta; \Delta; \cdot \vdash_{K+L_g+\sum_{L_g} K_g} ((e, \rho)) (\mathbf{C}) : \mu} \text{ D-app}$$

D0.1:

$$\frac{D0}{\Theta; \Delta; \cdot \vdash_{K+L_g+\sum_{L_g} K_g} ((e, \rho)) (\mathbf{C}), \cdot : \mu}$$

D0.0:

$$\frac{\frac{D0.1 \quad D2}{\Theta; \Delta \vdash_{K+L_g+\sum_{L_g} K_g+H_g} ((e, \rho)) (\mathbf{C}), \cdot, \theta' : \tau} \quad J \geq L_g + \sum_{L_g} K_g + H_g}{\Theta; \Delta \vdash_{K+J} ((e, \rho)) (\mathbf{C}), \cdot, \theta' : \tau} \text{ Lemma 3.5 of [3]}$$

Main derivation:

$$\frac{\frac{D0.0}{\Theta; \Delta; \cdot \vdash_{I'} ((e, \rho)) (\mathbf{C}), \cdot, \theta) : \tau} \text{ IH}}{\Theta; \Delta; \cdot \vdash_{I'} ((e, \rho, \mathbf{C}.\theta)) : \tau} \text{ Definition 40}}{\Theta; \Delta; \cdot \vdash_I ((e, \rho, \mathbf{C}.\theta)) : \tau} \text{ Lemma 3.5 of [3]}$$

□

Definition 43 (Equivalence for λ -amor).

$$v_1 \overset{s}{\approx}_{aV} v_2 \triangleq \left\{ \begin{array}{l} \text{True} \\ \forall e', e'', s' < s. e' \overset{s'}{\approx}_{aE} e'' \implies \\ e_1[e'/x] \overset{s'}{\approx}_{aE} e_2[e''/x] \\ e_1 \overset{s}{\approx}_{aE} e_2 \\ \forall i < s. v_1 \Downarrow_i^k v_a \implies \\ v_2 \Downarrow_i^k v_b \wedge v_a \overset{s-i}{\approx}_{aE} v_b \\ \\ v_{a1} \overset{s}{\approx}_{aV} v_{b1} \wedge v_{a2} \overset{s}{\approx}_{aV} v_{b2} \\ v_{a1} \overset{s}{\approx}_{aV} v_{b1} \wedge v_{a2} \overset{s}{\approx}_{aV} v_{b2} \\ v_a \overset{s}{\approx}_{aV} v_b \\ v_a \overset{s}{\approx}_{aV} v_b \end{array} \right. \begin{array}{l} v_1 = () \wedge v_2 = () \\ v_1 = \lambda x. e_2 \wedge v_2 = \lambda x. e_2 \\ \\ v_1 = !e_1 \wedge v_2 = !e_2 \\ v_1 = \Lambda. e_1 \wedge v_2 = \Lambda. e_2 \\ v_1 = \text{ret } - \wedge v_2 = \text{ret } - \\ \\ v_1 = \text{bind } - = - \text{ in } - \wedge v_2 = \text{bind } - = - \text{ in } - \\ v_1 = \uparrow^n \wedge v_2 = \uparrow^n \\ v_1 = \text{release } - = - \text{ in } - \wedge v_2 = \text{release } - = - \text{ in } - \\ v_1 = \text{store } - \wedge v_2 = \text{store } - \\ v_1 = \langle\langle v_{a1}, v_{a2} \rangle\rangle \wedge v_2 = \langle\langle v_{b1}, v_{b2} \rangle\rangle \\ v_1 = \langle v_{a1}, v_{a2} \rangle \wedge v_2 = \langle v_{b1}, v_{b2} \rangle \\ v_1 = \text{inl}(v_a) \wedge v_2 = \text{inl}(v_b) \\ v_1 = \text{inr}(v_a) \wedge v_2 = \text{inr}(v_b) \end{array}$$

$$e_1 \overset{s}{\approx}_{aE} e_2 \triangleq \forall i < s. e_1 \Downarrow_i v_a \implies e_2 \Downarrow_i v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$$

$$\delta_1 \overset{s}{\approx}_{aE} \delta_2 \triangleq \text{dom}(\delta_1) = \text{dom}(\delta_2) \wedge \forall x \in \text{dom}(\delta_1). \delta_1(x) \overset{s}{\approx}_{aE} \delta_2(x)$$

Lemma 44 (Monotonicity lemma for value equivalence). $\forall v_1, v_2, s.$

$$v_1 \overset{s}{\approx}_{aV} v_2 \implies \forall s' < s. v_1 \overset{s'}{\approx}_{aV} v_2$$

Proof. Given: $v_1 \overset{s}{\approx}_{aV} v_2$

To prove: $\forall s' < s. v_1 \overset{s'}{\approx}_{aV} v_2$

This means given some $s' < s$ and it suffices to prove that $v_1 \overset{s'}{\approx}_{aV} v_2$

We induct on v_1

1. $v_1 = ()$:

Since we are given that $v_1 \overset{s}{\approx}_{aV} v_2$ therefore we get the desired Directly from Definition 43

2. $v_1 = \lambda x. e_1$:

Since we are given that $v_1 \overset{s}{\approx}_{aV} v_2$ therefore from Definition 43 we are given that

$$\forall e', e'', s'' < s. e' \overset{s''}{\approx}_{aE} e'' \implies e_1[e'/x] \overset{s''}{\approx}_{aE} e_2[e''/x] \quad (\text{M-L0})$$

and we need to prove that $v_1 \overset{s'}{\approx}_{aV} v_2$ therefore again from Definition 43 we need to prove that

$$\forall e'_1, e''_1, s''_1 < s. e'_1 \overset{s''_1}{\approx}_{aE} e''_1 \implies e_1[e'_1/x] \overset{s''_1}{\approx}_{aE} e_2[e''_1/x]$$

This means given some $e'_1, e''_1, s''_1 < s'$ s.t $e'_1 \overset{s''_1}{\approx}_{aE} e''_1$ we need to prove that

$$e_1[e'_1/x] \overset{s''_1}{\approx}_{aE} e_2[e''_1/x]$$

Instantiating (M-L0) with e'_1, e''_1, s''_1 we get $e_1[e'_1/x] \overset{s''_1}{\approx}_{aE} e_2[e''_1/x]$

3. $v_1 = !e_1$:

Since we are given $v_1 \overset{s}{\approx}_{aV} v_2$ therefore from Definition 43 we have $e_1 \overset{s}{\approx}_{aE} e_2$ where $v_2 = !e_2$

Similarly from Definition 43 it suffices to prove that $e_1 \overset{s'}{\approx}_{aE} e_2$

We get this directly from Lemma 45

4. $v_1 = \Lambda e_1$:

Similar reasoning as in the $!e_1$ case

5. $v_1 = \text{ret } e_1$:

Since we are given $v_1 \overset{s}{\approx}_{aV} v_2$ therefore from Definition 43 we have

$$\forall i < s.v_1 \Downarrow_i^k v_a \implies v_2 \Downarrow^k v_b \wedge v_a \overset{s-i}{\approx}_{aE} v_b \text{ where } v_2 = \text{ret } e_2 \quad (\text{MV-R0})$$

Similarly from Definition 43 it suffices to prove that

$$\forall j < s'.v_1 \Downarrow_i^k v_a \implies v_2 \Downarrow^k v_b \wedge v_a \overset{s'-j}{\approx}_{aE} v_b$$

This means given some $j < s'$ and $v_1 \Downarrow_i^k v_a$ and it suffices to prove that

$$v_2 \Downarrow^k v_b \wedge v_a \overset{s'-j}{\approx}_{aE} v_b$$

Instantiating (MV-R0) with j we get $v_2 \Downarrow^k v_b \wedge v_a \overset{s-j}{\approx}_{aE} v_b$

Since we have $v_a \overset{s-j}{\approx}_{aE} v_b$ therefore from Lemma 45 we also get $v_a \overset{s'-j}{\approx}_{aE} v_b$

6. $v_1 = \text{bind } - = - \text{ in } -, \uparrow^n, \text{release } - = - \text{ in } -, \text{store } -$:

Similar reasoning as in the $\text{ret } -$ case

7. $v_1 = \langle\langle v_{a1}, v_{a2} \rangle\rangle$:

From Definition 43 and IH we get the desired

8. $v_1 = \langle v_{a1}, v_{a2} \rangle$:

From Definition 43 and IH we get the desired

9. $v_1 = \text{inl}(v)$:

From Definition 43 and IH we get the desired

10. $v_1 = \text{inr}(v)$:

From Definition 43 and IH we get the desired

□

Lemma 45 (Monotonicity lemma for expression equivalence). $\forall e_1, e_2, s.$

$$e_1 \overset{s}{\approx}_{aE} e_2 \implies \forall s' < s.e_1 \overset{s'}{\approx}_{aE} e_2$$

Proof. Given: $e_1 \overset{s}{\approx}_{aE} e_2$

To prove: $\forall s' < s. e_1 \overset{s'}{\approx}_{aE} e_2$

This means given some $s' < s$ and we need to prove $e_1 \overset{s'}{\approx}_{aE} e_2$

Since we are given $e_1 \overset{s}{\approx}_{aE} e_2$ therefore from Definition 43 we have

$$\forall i < s. e_1 \Downarrow_i v_a \implies e_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b \quad (\text{ME0})$$

Similarly from Definition 43 it suffices to prove that

$$\forall j < s'. e_1 \Downarrow_j v_a \implies e_2 \Downarrow v_b \wedge v_a \overset{s'-j}{\approx}_{aV} v_b$$

This means given some $j < s'$ s.t $e_1 \Downarrow_j v_a$ and we need to prove

$$e_2 \Downarrow v_b \wedge v_a \overset{s'-j}{\approx}_{aV} v_b$$

We get the desired from (ME0) and Lemma 44

□

Lemma 46 (Monotonicity lemma for δ equivalence). $\forall \delta_1, \delta_2, s.$

$$\delta_1 \overset{s}{\approx}_{aE} \delta_2 \implies \forall s' < s. \delta_1 \overset{s'}{\approx}_{aE} \delta_2$$

Proof. From Definition 43 and Lemma 45

□

Theorem 47 (Fundamental theorem for equivalence relation of λ -amor). $\forall \delta_1, \delta_2, e, s.$

$$\delta_1 \overset{s}{\approx}_{aE} \delta_2 \implies e \delta_1 \overset{s}{\approx}_{aE} e \delta_2$$

Proof. We induct on e

1. $e = x$:

We need to prove that $x \delta_1 \overset{s}{\approx}_{aE} x \delta_2$

This means it suffices to prove that $\delta_1(x) \overset{s}{\approx}_{aE} \delta_2(x)$

We get this directly from Definition 43

2. $e = \lambda y. e'$:

We need to prove that $\lambda y. e' \delta_1 \overset{s}{\approx}_{aE} \lambda y. e' \delta_2$

This means from Definition 43 it suffices to prove that

$$\forall i < s. \lambda y. e' \delta_1 \Downarrow_i v_a \implies \lambda y. e' \delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$$

This means that given some $i < s$ s.t $\lambda y. e' \delta_1 \Downarrow_i v_a$ it suffices to prove that

$$\lambda y. e' \delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b \quad (\text{FTE-L0})$$

From E-val we know that $v_a = \lambda y. e' \delta_1$

From (FTE-L0) we need to prove that

(a) $\lambda y. e' \delta_2 \Downarrow v_b$:

From E-val we know that $v_b = \lambda y. e' \delta_2$

(b) $v_a \stackrel{s-i}{\approx}_{aV} v_b$:

We need to prove that

$$\lambda y.e'\delta_1 \stackrel{s}{\approx}_{aV} \lambda y.e'\delta_2$$

This means from Definition 43 it suffices to prove that

$$\forall e'_1, e'_2, s' < s. e'_1 \stackrel{s'}{\approx}_{aE} e'_2 \implies e'\delta_1[e'_1/y] \stackrel{s'}{\approx}_{aE} e'\delta_2[e'_2/y]$$

This further means that given some $e'_1, e'_2, s' < s$ s.t $e'_1 \stackrel{s'}{\approx}_{aE} e'_2$ it suffices to prove that

$$e'\delta_1[e'_1/y] \stackrel{s'}{\approx}_{aE} e'\delta_2[e'_2/y]$$

We get this from IH and Lemma 46

3. $e = \text{fix}y.e'$:

We induct on s

$$\text{IH}i: \forall s'' < s. \delta_1 \stackrel{s''}{\approx}_{aE} \delta_2 \implies \text{fix}y.e'\delta_1 \stackrel{s'}{\approx}_{aE} \text{fix}y.e'\delta_2$$

$$\text{To prove: } \delta_1 \stackrel{s}{\approx}_{aE} \delta_2 \implies \text{fix}y.e'\delta_1 \stackrel{s}{\approx}_{aE} \text{fix}y.e'\delta_2$$

This means we are given $\delta_1 \stackrel{s}{\approx}_{aE} \delta_2$ and we need to prove

$$\text{fix}y.e'\delta_1 \stackrel{s}{\approx}_{aE} \text{fix}y.e'\delta_2$$

From Definition 43 it suffices to prove that

$$\forall i < s. \text{fix}y.e'\delta_1 \Downarrow_i v_a \implies \text{fix}y.e'\delta_2 \Downarrow v_b \wedge v_a \stackrel{s-i}{\approx}_{aV} v_b$$

This means given some $i < s$ s.t $\text{fix}y.e'\delta_1 \Downarrow_i v_a$ and we need to prove $\text{fix}y.e'\delta_2 \Downarrow v_b \wedge v_a \stackrel{s-i}{\approx}_{aV} v_b$

Since we are given that $\text{fix}y.e'\delta_1 \Downarrow_i v_a$ therefore from E-fix we know that

$$e'[\text{fix}x.e'\delta_1/y]\delta_1 \Downarrow_{i-1} v_a$$

Instantiating IH*i* with $s-1$ and using Lemma 46 we get

$$\text{fix}y.e'\delta_1 \stackrel{s-1}{\approx}_{aE} \text{fix}y.e'\delta_2 \quad (\text{F1})$$

Let

$$\delta'_1 = \delta_1 \cup \{y \mapsto \text{fix}y.e'\delta_1\}$$

$$\delta'_2 = \delta_2 \cup \{y \mapsto \text{fix}y.e'\delta_2\}$$

From Lemma 46 and (F1) we know that $\delta'_1 \stackrel{s-1}{\approx}_{aE} \delta'_2$

Therefore from IH of outer induction we know that we have

$$e'\delta'_1 \stackrel{s-1}{\approx}_{aE} e'\delta'_2$$

This means from Definition 43 we know that

$$\forall i' < (s-1). e'\delta'_1 \Downarrow_{i'} v_a \implies e'\delta'_2 \Downarrow v_b \wedge v_a \stackrel{s-1-i'}{\approx}_{aV} v_b$$

Instantiating with $i-1$ and since we know that $e'\delta'_1 \Downarrow_{i-1} v_a$ and therefore we get

$$e'\delta'_2 \Downarrow v_b \wedge v_a \stackrel{s-i}{\approx}_{aV} v_b \text{ which is the desired.}$$

4. $e = e_1 e_2$:

We need to prove that $e_1 e_2 \delta_1 \overset{s}{\approx}_{aE} e_1 e_2 \delta_2$

This means from Definition 43 it suffices to prove that

$$\forall i < s. e_1 e_2 \delta_1 \Downarrow_i v_a \implies e_1 e_2 \delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$$

This means that given some $i < s$ s.t $e_1 e_2 \delta_1 \Downarrow_i v_a$ it suffices to prove that

$$e_1 e_2 \delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b \quad (\text{FTE-A0})$$

$$\underline{\text{IH1}}: e_1 \delta_1 \overset{s}{\approx}_{aE} e_1 \delta_2$$

Therefore from Definition 43 we have

$$\forall j < s. e_1 \delta_1 \Downarrow_j v'_a \implies e_1 \delta_2 \Downarrow v'_b \wedge v'_a \overset{s-j}{\approx}_{aV} v'_b \quad (\text{FTE-A1})$$

Since $(e_1 \delta_1 e_2 \delta_1) \Downarrow_i v_a$ therefore from E-app we know that $\exists i_1 < i. e_1 \delta_1 \Downarrow_{i_1} \lambda y. e'$

$$\text{Therefore instantiating (FTE-A1) with } i_1 \text{ we get } e_1 \delta_2 \Downarrow v'_b \wedge v'_a \overset{s-i_1}{\approx}_{aV} v'_b \quad (\text{FTE-A1.1})$$

Since $v'_a = \lambda y. e'$ and since $v'_a \overset{s-i_1}{\approx}_{aV} v'_b$ therefore from Definition 43 we know that $v'_b = \lambda y. e''$

Again since $\lambda y. e' \overset{s-i_1}{\approx}_{aV} \lambda y. e''$ therefore from Definition 43 we know that

$$\forall e'_1, e'_2, s' < (s - i_1). e'_1 \overset{s'}{\approx}_{aE} e'_2 \implies e'[e'_1/y] \overset{s'}{\approx}_{aE} e''[e'_2/y] \quad (\text{FTE-A2})$$

$$\underline{\text{IH2}}: e_2 \delta_1 \overset{s-i_1-1}{\approx}_{aE} e_2 \delta_2$$

Instantiating (FTE-A2) with $e_2 \delta_1, e_2 \delta_2$ we get

$$e'[e_2 \delta_1/y] \overset{s-i_1-1}{\approx}_{aE} e''[e_2 \delta_1/y]$$

Again from Definition 43 we have

$$\forall j < (s - i_1 - 1). e'[e_2 \delta_1/y] \Downarrow_j v''_a \implies e''[e_2 \delta_1/y] \Downarrow v''_b \wedge v''_a \overset{s-i_1-1-j}{\approx}_{aV} v''_b \quad (\text{FTE-A2.1})$$

Since $(e_1 \delta_1 e_2 \delta_1) \Downarrow_i v_a$ therefore from E-app we know that $\exists i_2 = i - i_1 - 1. e'[e_2 \delta_1/x] \Downarrow_{i_2} v_a$

$$\text{Instantiating (FTE-A2.1) with } i_2 \text{ we get } e''[e_2 \delta_1/y] \Downarrow v''_b \wedge v_a \overset{s-i_1-1-i_2}{\approx}_{aV} v''_b$$

Since $i = i_1 + i_2 + 1$ therefore this proves (FTE-A0) and we are done.

5. $e = \langle\langle e_1, e_2 \rangle\rangle$:

We need to prove that $\langle\langle e_1, e_2 \rangle\rangle \delta_1 \overset{s}{\approx}_{aE} \langle\langle e_1, e_2 \rangle\rangle \delta_2$

This means from Definition 43 it suffices to prove that

$$\forall i < s. \langle\langle e_1, e_2 \rangle\rangle \delta_1 \Downarrow_i v_a \implies \langle\langle e_1, e_2 \rangle\rangle \delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$$

This means that given some $i < s$ s.t $\langle\langle e_1, e_2 \rangle\rangle \delta_1 \Downarrow_i v_a$ it suffices to prove that

$$\langle\langle e_1, e_2 \rangle\rangle \delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b \quad (\text{FTE-TI0})$$

From E-TI we know that $v_a = \langle\langle v_{a1}, v_{a2} \rangle\rangle$ and $e_1\delta_1 \Downarrow_{i_1} v_{a1}$ and $e_2\delta_1 \Downarrow_{i_2} v_{a2}$

IH1: $e_1\delta_1 \overset{s}{\approx}_{aE} e_1\delta_2$

Therefore from Definition 43 we have

$$\forall i < s. e_1\delta_1 \Downarrow_i v_{a1} \implies e_1\delta_2 \Downarrow v_{b1} \wedge v_{a1} \overset{s-i}{\approx}_{aV} v_{b1}$$

Since we know that $e_1\delta_1 \Downarrow_{i_1} v_{a1}$ therefore we get

$$e_1\delta_2 \Downarrow v_{b1} \wedge v_{a1} \overset{s-i_1}{\approx}_{aV} v_{b1} \quad (\text{FTE-TI1})$$

IH2: $e_2\delta_1 \overset{s}{\approx}_{aE} e_2\delta_2$

Similarly from Definition 43 we have

$$\forall i < s. e_2\delta_1 \Downarrow_i v_{a1} \implies e_2\delta_2 \Downarrow v_{b1} \wedge v_{a1} \overset{s-i}{\approx}_{aV} v_{b1}$$

Since we know that $e_2\delta_1 \Downarrow_{i_2} v_{a2}$ therefore we get

$$e_2\delta_2 \Downarrow v_{b2} \wedge v_{a2} \overset{s-i_2}{\approx}_{aV} v_{b2} \quad (\text{FTE-TI2})$$

From (FTE-TI0) we need to prove

(a) $\langle\langle e_1, e_2 \rangle\rangle\delta_2 \Downarrow v_b$:

We get this from (FTE-TI1), (FTE-TI2) and E-TI

(b) $v_a \overset{s-i}{\approx}_{aV} v_b$:

Since $i = i_1 + i_2$, $v_a = \langle\langle v_{a1}, v_{a2} \rangle\rangle$ and $v_b = \langle\langle v_{b1}, v_{b2} \rangle\rangle$ it suffices to prove that

$$\langle\langle v_{a1}, v_{a2} \rangle\rangle \overset{s-i_1-i_2}{\approx}_{aV} \langle\langle v_{b1}, v_{b2} \rangle\rangle$$

From Definition 43 it suffices to prove that

$$v_{a1} \overset{s-i_1-i_2}{\approx}_{aV} v_{b1} \text{ and } v_{a2} \overset{s-i_1-i_2}{\approx}_{aV} v_{b2}$$

We get this from (FTE-TI1), (FTE-TI2) and Lemma 44

6. $e = \text{let}\langle\langle x, y \rangle\rangle = e_1$ in e_2 :

We need to prove that $\text{let}\langle\langle x, y \rangle\rangle = e_1$ in $e_2\delta_1 \overset{s}{\approx}_{aE} \text{let}\langle\langle x, y \rangle\rangle = e_1$ in $e_2\delta_2$

This means from Definition 43 it suffices to prove that

$$\forall i < s. \text{let}\langle\langle x, y \rangle\rangle = e_1 \text{ in } e_2\delta_1 \Downarrow_i v_a \implies \text{let}\langle\langle x, y \rangle\rangle = e_1 \text{ in } e_2\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$$

This means that given some $i < s$ s.t $\text{let}\langle\langle x, y \rangle\rangle = e_1$ in $e_2\delta_1 \Downarrow_i v_a$ it suffices to prove that

$$\text{let}\langle\langle x, y \rangle\rangle = e_1 \text{ in } e_2\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b \quad (\text{FTE-TE0})$$

IH1: $e_1\delta_1 \overset{s}{\approx}_{aE} e_1\delta_2$

Therefore from Definition 43 we have

$$\forall i < s. e_1\delta_1 \Downarrow_i v_{a1} \implies e_1\delta_2 \Downarrow v_{b1} \wedge v_{a1} \overset{s-i}{\approx}_{aV} v_{b1}$$

Since we know that $\text{let}\langle\langle x, y \rangle\rangle = e_1$ in $e_2\delta_1 \Downarrow_i v_a$ therefore from E-TE we know that $\exists i_1 < s. e_1\delta_1 \Downarrow_{i_1} \langle\langle v'_{a1}, v'_{a2} \rangle\rangle$. Therefore we get

$$e_1 \delta_2 \Downarrow v_{b1} \wedge v_{a1} \stackrel{s-i_1}{\approx}_{aV} v_{b1} \quad (\text{FTE-TE1})$$

Since $v_{a1} \stackrel{s-i_1}{\approx}_{aV} v_{b1}$ and $v_{a1} = \langle\langle v'_{a1}, v'_{a2} \rangle\rangle$ therefore from Definition 43 we have
 $v_{b1} = \langle\langle v'_{b1}, v'_{b2} \rangle\rangle \quad (\text{FTE-TE1.1})$

Let

$$\delta'_1 = \delta_1 \cup \{x \mapsto \langle\langle v'_{a1}, v'_{a2} \rangle\rangle\}$$

$$\delta'_2 = \delta_2 \cup \{x \mapsto \langle\langle v'_{b1}, v'_{b2} \rangle\rangle\}$$

$$\underline{\text{IH2}}: e_2 \delta'_1 \stackrel{s-i_1}{\approx}_{aE} e_2 \delta'_2$$

Therefore from Definition 43 we have

$$\forall i < (s - i_1). e_2 \delta'_1 \Downarrow_i v_a \implies e_2 \delta'_2 \Downarrow v_{b2} \wedge v_a \stackrel{s-i_1-i}{\approx}_{aV} v_b$$

Since we know that $\text{let}\langle\langle x, y \rangle\rangle = e_1$ in $e_2 \delta_1 \Downarrow_i v_a$ therefore from E-TE we know that
 $\exists i_2 = i - i_1. e_2 \delta'_1 \Downarrow_{i_2} v_a$. Therefore we get

$$e_2 \delta'_2 \Downarrow v_{b2} \wedge v_a \stackrel{s-i_1-i_2}{\approx}_{aV} v_b \quad (\text{FTE-TE2})$$

This proves the desired

7. $e = \langle e_{a1}, e_{a2} \rangle$:

Similar reasoning as in the $\langle\langle e_{a1}, e_{a2} \rangle\rangle$ case above

8. $e = \text{fst}(e')$:

We need to prove that $\text{fst}(e') \delta_1 \stackrel{s}{\approx}_{aE} \text{fst}(e') \delta_2$

This means from Definition 43 it suffices to prove that

$$\forall i < s. \text{fst}(e') \delta_1 \Downarrow_i v_a \implies \text{fst}(e') \delta_2 \Downarrow v_b \wedge v_a \stackrel{s-i}{\approx}_{aV} v_b$$

This means that given some $i < s$ s.t. $\text{fst}(e') \delta_1 \Downarrow_i v_a$ it suffices to prove that

$$\text{fst}(e') \delta_2 \Downarrow v_b \wedge v_a \stackrel{s-i}{\approx}_{aV} v_b \quad (\text{FTE-F0})$$

Since we know that $\text{fst}(e') \delta_1 \Downarrow_i v_a$ therefore from E-fst we know that $e' \delta_1 \Downarrow_i \langle\langle v_a, - \rangle\rangle$

$$\underline{\text{IH}}: e' \delta_1 \stackrel{s}{\approx}_{aE} e' \delta_2$$

This means from Definition 43 we have

$$\forall j < s. e' \delta_1 \Downarrow_j v_{a1} \implies e' \delta_2 \Downarrow v_{b1} \wedge v_{a1} \stackrel{s-j}{\approx}_{aV} v_{b1}$$

Instantiating with i we get $e' \delta_2 \Downarrow v_{b1} \wedge v_{a1} \stackrel{s-j}{\approx}_{aV} v_{b1}$

Since we know that $v_{a1} = \langle\langle v_a, - \rangle\rangle$ therefore from Definition 43 we also know that

$$v_{b1} = \langle\langle v_b, - \rangle\rangle \text{ s.t. } v_a \stackrel{s}{\approx}_{aV} v_b$$

This proves the desired.

9. $e = \text{snd}(e')$:

Similar reasoning as in the $\text{fst}(e')$ case

10. $e = \text{inl}(e')$:

We need to prove that $\text{inl}(e')\delta_1 \overset{s}{\approx}_{aE} \text{inl}(e')\delta_2$

This means from Definition 43 it suffices to prove that

$$\forall i < s. \text{inl}(e')\delta_1 \Downarrow_i v_a \implies \text{inl}(e')\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$$

This means that given some $i < s$ s.t. $\text{inl}(e')\delta_1 \Downarrow_i v_a$ it suffices to prove that

$$\text{inl}(e')\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b \quad (\text{FTE-IL0})$$

Since we know that $\text{inl}(e')\delta_1 \Downarrow_i v_a$ therefore from E-inl we know that $v_a = \text{inl}((v'_a))$ and $e'\delta_1 \Downarrow_i v'_a$

$$\underline{\text{IH}}: e'\delta_1 \overset{s}{\approx}_{aE} e'\delta_2$$

This means from Definition 43 we have

$$\forall j < s. e'\delta_1 \Downarrow_j v_{a1} \implies e'\delta_2 \Downarrow v_{b1} \wedge v_{a1} \overset{s-j}{\approx}_{aV} v_{b1}$$

Instantiating with i we get $e'\delta_2 \Downarrow v_{b1} \wedge v_{a1} \overset{s-i}{\approx}_{aV} v_{b1}$

Since $e'\delta_2 \Downarrow v_{b1}$ therefore from E-inl we have $\text{inl}(e')\delta_2 \Downarrow \text{inl}(v_{b1})$

And since we know that $v_{a1} \overset{s-i}{\approx}_{aV} v_{b1}$ therefore from Definition 43 we also know that

$$\text{inl}(v_{a1}) \overset{s-i}{\approx}_{aV} \text{inl}(v_{b1})$$

This proves the desired.

11. $e = \text{inr}(e')$:

Similar reasoning as in the $\text{inl}(e')$ case

12. $e = \text{case } e_c \text{ of } e_l; e_r$:

We need to prove that $\text{case } e_c \text{ of } e_l; e_r\delta_1 \overset{s}{\approx}_{aE} \text{case } e_c \text{ of } e_l; e_r\delta_2$

This means from Definition 43 it suffices to prove that

$$\forall i < s. \text{case } e_c \text{ of } e_l; e_r\delta_1 \Downarrow_i v_a \implies \text{case } e_c \text{ of } e_l; e_r\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b$$

This means that given some $i < s$ s.t. $\text{case } e_c \text{ of } e_l; e_r\delta_1 \Downarrow_i v_a$ it suffices to prove that

$$\text{case } e_c \text{ of } e_l; e_r\delta_2 \Downarrow v_b \wedge v_a \overset{s-i}{\approx}_{aV} v_b \quad (\text{FTE-C0})$$

Since we know that $\text{case } e_c \text{ of } e_l; e_r\delta_1 \Downarrow_i v_a$ therefore two cases arise:

2 cases arise:

(a) $e_c\delta_1 \Downarrow \text{inl}(v_{c1})$:

$$\underline{\text{IH1}} \ e_c\delta_1 \overset{s}{\approx}_{aE} e_c\delta_2$$

This means from Definition 43 we have

$$\forall j < s. e_c\delta_1 \Downarrow_j v_{c1} \implies e_c\delta_2 \Downarrow v_{c2} \wedge v_{c1} \overset{s-j}{\approx}_{aV} v_{c2}$$

Since we know that $\text{case } e_c \text{ of } e_l; e_r\delta_1 \Downarrow_i v_a$ therefore from E-case1 we know that $\exists i_1$ s.t. $e_c\delta_1 \Downarrow_{i_1} \text{inl}(v'_{c1})$

Therefore instantiating with i_1 we get $e_c \delta_2 \Downarrow v_{c2} \wedge v_{c1} \stackrel{s-i_1}{\approx}_{aV} v_{c2}$

From Definition 43 we know that $\exists v'_{c2}. v_{c2} = \text{inl}(v'_{c2})$ s.t $v'_{c1} \stackrel{s-i_1}{\approx}_{aV} v'_{c2}$

$$\underline{\text{IH2}} \quad e_l \delta_1[v'_{c1}/x] \stackrel{s-i_1}{\approx}_{aE} e_l \delta_2[v'_{c2}/x]$$

This means from Definition 43 we have

$$\forall j < (s - i_1). e_l \delta_1[v'_{c1}/x] \Downarrow_j v_{l1} \implies e_l \delta_2[v'_{c2}/x] \Downarrow v_{l2} \wedge v_{l1} \stackrel{s-i_1-j}{\approx}_{aV} v_b$$

Since we know that case e_c of $e_l; e_r \delta_1 \Downarrow_i v_a$ therefore from E-case1 we know that $\exists i_2$ s.t $e_l \delta_1 \Downarrow_{i_2} v_a$

Therefore instantiating with i_2 we get $e_l \delta_2[v'_{c2}/x] \Downarrow v_{l2} \wedge v_a \stackrel{s-i_1-j}{\approx}_{aV} v_b$

This proves the desired

(b) $e_c \delta_1 \Downarrow \text{inr}(v_{c1})$:

Similar reasoning as in the previous case

13. $e = !e'$:

We need to prove that $!e' \delta_1 \stackrel{s}{\approx}_{aE} !e' \delta_2$

This means from Definition 43 it suffices to prove that

$$\forall i < s. !e' \delta_1 \Downarrow_i v_a \implies !e' \delta_2 \Downarrow v_b \wedge v_a \stackrel{s-i}{\approx}_{aV} v_b$$

This means that given some $i < s$ s.t $!e' \delta_1 \Downarrow_i v_a$ it suffices to prove that

$$!e' \delta_2 \Downarrow v_b \wedge v_a \stackrel{s-i}{\approx}_{aV} v_b \quad (\text{FTE-B0})$$

From E-val we know that $v_a = !e' \delta_1$ and $i = 0$

$$\underline{\text{IH}}: e' \delta_1 \stackrel{s}{\approx}_{aE} e' \delta_2$$

From (FTE-B0) we need to prove that

(a) $!e' \delta_2 \Downarrow v_b$:

From E-val we know that $v_b = !e' \delta_2$

(b) $v_a \stackrel{s-i}{\approx}_{aV} v_b$:

We need to prove that

$$!e' \delta_1 \stackrel{s}{\approx}_{aV} !e' \delta_2$$

This means from Definition 43 it suffices to prove that

$$e' \delta_1 \stackrel{s}{\approx}_{aE} e' \delta_2$$

We get this directly from IH

14. $e = \text{let } !x = e'_1 \text{ in } e'_2$:

We need to prove that $\text{let } !x = e'_1 \text{ in } e'_2 \delta_1 \stackrel{s}{\approx}_{aE} \text{let } !x = e'_1 \text{ in } e'_2 \delta_2$

This means from Definition 43 it suffices to prove that

$$\forall i < s. \text{let } !x = e'_1 \text{ in } e'_2 \delta_1 \Downarrow_i v_a \implies \text{let } !x = e'_1 \text{ in } e'_2 \delta_2 \Downarrow v_b \wedge v_a \stackrel{s-i}{\approx}_{aV} v_b$$

This means that given some $i < s$ s.t. $\text{let! } x = e'_1 \text{ in } e'_2 \delta_1 \Downarrow_i v_a$ it suffices to prove that

$$\text{let! } x = e'_1 \text{ in } e'_2 \delta_2 \Downarrow v_b \wedge v_a \approx_{aV}^{s-i} v_b \quad (\text{FTE-BE0})$$

IH1: $e'_1 \delta_1 \approx_{aE}^s e'_1 \delta_2$

This means from Definition 43 we have

$$\forall j < s. e'_1 \delta_1 \Downarrow_j v_{a11} \implies e'_1 \delta_2 \Downarrow v_{b1} \wedge v_{a1} \approx_{aV}^{s-j} v_{b11}$$

Since we know that $\text{let! } x = e'_1 \text{ in } e'_2 \delta_1 \Downarrow_i v_a$ therefore from E-subExpE we know that $\exists i_1. e'_1 \delta_1 \Downarrow_{i_1} !e_{b1}$

Instantiating with i_1 we get $e'_1 \delta_2 \Downarrow v_{b11} \wedge v_{a11} \approx_{aV}^{s-i_1} v_{b11}$

Since we know that $v_{a11} = !e_{b1}$ therefore from Definition 43 we also know that

$$v_{b11} = !e_{b2} \text{ s.t. } e_{b1} \approx_{aE}^{s-i_1} e_{b2}$$

IH2: $e'_2[e_{b1}/x] \delta_1 \approx_{aE}^{s-i_1} e'_2[e_{b2}/x] \delta_2$

This means from Definition 43 we have

$$\forall j < s. e'_2[e_{b1}/x] \delta_1 \Downarrow_j v_a \implies e'_2[e_{b2}/x] \delta_2 \Downarrow v_b \wedge v_a \approx_{aV}^{s-i_1-j} v_b$$

Since we know that $\text{let! } x = e'_1 \text{ in } e'_2 \delta_1 \Downarrow_i v_a$ therefore from E-subExpE we know that $\exists i_2. e'_1[e_{b1}/x] \delta_1 \Downarrow_{i_2} v_a$

Instantiating with i_2 we get $e'_2[e_{b2}/x] \delta_2 \Downarrow v_b \wedge v_a \approx_{aV}^{s-i_1-i_2} v_b$

This proves the desired

15. $e = \Lambda.e'$:

Similar reasoning as in the $\lambda y.e'$ case

16. $e = e' []$:

Similar reasoning as in the app case

17. $e = \text{ret } e'$:

We need to prove that $\text{ret } e' \delta_1 \approx_{aE}^s \text{ret } e' \delta_2$

This means from Definition 43 it suffices to prove that

$$\forall i < s. \text{ret } e' \delta_1 \Downarrow_i v_a \implies \text{ret } e' \delta_2 \Downarrow v_b \wedge v_a \approx_{aV}^{s-i} v_b$$

This means that given some $i < s$ s.t. $\text{ret } e' \delta_1 \Downarrow_i v_a$ it suffices to prove that

$$\text{ret } e' \delta_2 \Downarrow v_b \wedge v_a \approx_{aV}^{s-i} v_b \quad (\text{FTE-R0})$$

From E-val we know that $v_a = \text{ret } e' \delta_1$ and $i = 0$

From (FTE-R0) we need to prove that

(a) $\text{ret } e' \delta_2 \Downarrow v_b$:

From E-val we know that $v_b = \text{ret } e' \delta_2$

(b) $v_a \stackrel{s-i}{\approx}_{aV} v_b$:

We need to prove that

$$\text{ret } e'\delta_1 \stackrel{s}{\approx}_{aV} \text{ret } e'\delta_2$$

This means from Definition 43 it suffices to prove that

$$\text{ret } e'\delta_1 \Downarrow_i^k v_a \implies \text{ret } e'\delta_2 \Downarrow^k v_b \wedge v_a \stackrel{s-i}{\approx}_{aV} v_b$$

This further means that given some $\text{ret } e'\delta_1 \Downarrow_i^k v_a$ it suffices to prove that

$$\text{ret } e'\delta_2 \Downarrow^k v_b \wedge v_a \stackrel{s-i}{\approx}_{aV} v_b \quad (\text{FTE-R1})$$

From E-return we know that $k = 0$ and $e'\delta_1 \Downarrow_i v_a$

$$\underline{\text{IH}}: e'\delta_1 \stackrel{s}{\approx}_{aE} e'\delta_2$$

This means from Definition 43 we have

$$\forall j < s. e'\delta_1 \Downarrow_j v_a \implies e'\delta_2 \Downarrow v_b \wedge v_a \stackrel{s-j}{\approx}_{aV} v_b$$

Since we are given that $e'\delta_1 \Downarrow_i v_a$ therefore we get

$$e'\delta_2 \Downarrow v_b \wedge v_a \stackrel{s-j}{\approx}_{aV} v_b$$

Since $e'\delta_2 \Downarrow v_b$ therefore from E-return we also have

$$\text{ret } e'\delta_2 \Downarrow^0 v_b$$

This proves the desired

18. $e = \text{bind } x = e_b \text{ in } e_c$:

We need to prove that $\text{bind } x = e_b \text{ in } e_c \delta_1 \stackrel{s}{\approx}_{aE} \text{bind } x = e_b \text{ in } e_c \delta_2$

This means from Definition 43 it suffices to prove that

$$\forall i < s. \text{bind } x = e_b \text{ in } e_c \delta_1 \Downarrow_i v_a \implies \text{bind } x = e_b \text{ in } e_c \delta_2 \Downarrow v_b \wedge v_a \stackrel{s-i}{\approx}_{aV} v_b$$

This means that given some $i < s$ s.t $\text{bind } x = e_b \text{ in } e_c \delta_1 \Downarrow_i v_a$ it suffices to prove that

$$\text{bind } x = e_b \text{ in } e_c \delta_2 \Downarrow v_b \wedge v_a \stackrel{s-i}{\approx}_{aV} v_b \quad (\text{FTE-BI0})$$

From E-val we know that $v_a = \text{bind } x = e_b \text{ in } e_c \delta_1$ and $i = 0$

We need to prove

(a) $\text{bind } x = e_b \text{ in } e_c \delta_2 \Downarrow v_b$:

From E-val we know that $v_b = \text{bind } x = e_b \text{ in } e_c \delta_2$

(b) $v_a \stackrel{s-i}{\approx}_{aV} v_b$:

We need to prove that $\text{bind } x = e_b \text{ in } e_c \delta_1 \stackrel{s}{\approx}_{aV} \text{bind } x = e_b \text{ in } e_c \delta_2$

From Definition 43 it suffices to prove that

$$\text{bind } x = e_b \text{ in } e_c \delta_1 \Downarrow_i^k v_{t1} \implies \text{bind } x = e_b \text{ in } e_c \delta_2 \Downarrow^k v_{t2} \wedge v_{t1} \stackrel{s-i}{\approx}_{aV} v_{t2}$$

This means that given $\text{bind } x = e_b \text{ in } e_c \delta_1 \Downarrow_i^k v_{t1}$ it suffices to prove that

$$\text{bind } x = e_b \text{ in } e_c \delta_2 \Downarrow^k v_{t2} \wedge v_{t1} \stackrel{s-i}{\approx}_{aV} v_{t2} \quad (\text{F-BI1})$$

$$\underline{\text{IH1}}: e_b \delta_1 \stackrel{s}{\approx}_{aE} e_b \delta_2$$

This means from Definition 43 we have

$$\forall j < s.e_b \delta_1 \Downarrow_j v_{a1} \implies e_b \delta_2 \Downarrow v_{b1} \wedge v_{a1} \overset{s-j}{\approx}_{aV} v_{b1}$$

Since we know that $\text{bind } x = e_b$ in $e_c \delta_1 \Downarrow_i v_a$ therefore from E-bind we know that $\exists i_1.e_b \delta_1 \Downarrow_{i_1} v_{a1}$

$$\text{Instantiating with } i_1 \text{ we get } e_b \delta_2 \Downarrow v_{b1} \wedge v_{a1} \overset{s-i_1}{\approx}_{aV} v_{b1}$$

Since v_{a1} is a mondic value and $v_{a1} \Downarrow_{i_1'}^{k1} v'_{a1}$

Since $v_{a1} \overset{s-i_1}{\approx}_{aV} v_{b1}$ therefore from Definition 43 we know that

$$v_{a1} \Downarrow_{i_1'}^{k1} v'_{a1} \implies v_{b1} \Downarrow^{k1} v'_{b1} \wedge v'_{a1} \overset{s-i_1-i_1'}{\approx}_{aV} v'_{b1}$$

Since we are given that $v_{a1} \Downarrow_{i_1'}^{k1} v'_{a1}$ therefore we have

$$v_{b1} \Downarrow^{k1} v'_{b1} \wedge v'_{a1} \overset{s-i_1-i_1'}{\approx}_{aV} v'_{b1}$$

$$\underline{\text{IH2:}} \quad e_c[e'_{a1}/x] \delta_1 \overset{s-i_1-i_1'}{\approx}_{aE} e_c[e'_{b1}/x] \delta_2$$

This means from Definition 43 we have

$$\forall j < s.e_c[e'_{a1}/x] \delta_1 \Downarrow_j v_{a2} \implies e_c[e'_{b1}/x] \delta_2 \Downarrow v_b \wedge v_a \overset{s-i_1-i_1'-j}{\approx}_{aV} v_{b2}$$

Since we know that $\text{bind } x = e_b$ in $e_c \delta_1 \Downarrow_i v_a$ therefore from E-bind we know that $\exists i_2.e_c[e'_{a1}/x] \delta_1 \Downarrow_{i_2} v_{a2}$

$$\text{Instantiating with } i_2 \text{ we get } e_c[e'_{b1}/x] \delta_2 \Downarrow v_b \wedge v_{a2} \overset{s-i_1-i_1'-i_2}{\approx}_{aV} v_{b2}$$

From E-bind we know that v_{a2} is a mondic value and $v_{a2} \Downarrow_{i_2'}^{k2} v'_{a2}$

Since $v_{a2} \overset{s-i_1-i_1'-i_2}{\approx}_{aV} v_{b2}$ therefore from Definition 43 we know that

$$v_{a2} \Downarrow_{i_2'}^{k2} v'_{a2} \implies v_{b2} \Downarrow^{k2} v'_{b2} \wedge v'_{a2} \overset{s-i_1-i_1'-i_2-i_2'}{\approx}_{aV} v'_{b2}$$

Since we are given that $v_{a2} \Downarrow_{i_2'}^{k2} v'_{a2}$ therefore we have

$$v_{b2} \Downarrow^{k2} v'_{b2} \wedge v'_{a2} \overset{s-i_1-i_1'-i_2-i_2'}{\approx}_{aV} v'_{b2}$$

This proves the desired

19. $e = \uparrow^n$:

Trivial

20. $e = \text{release } e_r = x$ in e_c :

Similar reasoning as in the bind case

21. $e = \text{store } e$:

Similar reasoning as in the return case

□

Lemma 48 (Equivalence relation of λ -amor is reflexive for values). $\forall v, s. v \overset{s}{\approx}_{aV} v$

Proof. Instantiating Theorem 47 with \cdot for δ_1 and δ_2 , v for e and with the given s we get $v \stackrel{s}{\approx}_{aE} v$

From Definition 43 this means we have

$$\forall i < s. v \Downarrow_i v_a \implies v \Downarrow v_b \wedge v_a \stackrel{s-i}{\approx}_{aV} v_b$$

Instantiating it with i as 0 and since we know that $v \Downarrow_0 v$ therefore we get the desired \square

Lemma 49 (Property of app rule in λ -Amor). $\forall e_1, e_2, e, s.$

$$e_1 \stackrel{s}{\approx}_{aE} e_2 \implies e e_1 \stackrel{s}{\approx}_{aE} e e_2$$

Proof. We get the desired from Theorem 47 \square

Lemma 50 (Lemma for app1 : empty stack). $\forall t, u, \rho, \theta, v_a, v_1, j.$

$$\Theta; \Delta; \cdot \vdash_{-} \llbracket (t u, \rho, \epsilon) \rrbracket : - \wedge$$

$$\Theta; \Delta; \cdot \vdash_{-} \llbracket (t, \rho, (u, \rho). \epsilon) \rrbracket : - \wedge$$

$$\frac{\llbracket (t u, \rho, \epsilon) \rrbracket () \Downarrow v_a \Downarrow^j v_1}{\exists v_b, v_2. \llbracket (t, \rho, (u, \rho). \epsilon) \rrbracket () \Downarrow v_b \Downarrow^j v_2 \wedge \forall s. v_1 \stackrel{s}{\approx}_{aE} v_2}$$

Proof. From Definition 40 know that

$$\begin{aligned} \llbracket (t u, \rho, \epsilon) \rrbracket &= \llbracket t u, \rho \rrbracket = \\ &(\lambda x_1 \dots x_n. t u) \llbracket \mathbf{C}_1 \rrbracket \dots \llbracket \mathbf{C}_n \rrbracket \end{aligned} \quad (\text{A1.0})$$

Similarly from Definition 40 we also have

$$\begin{aligned} \llbracket (t, \rho, (u, \rho). \epsilon) \rrbracket &= \llbracket \llbracket (t, \rho) \rrbracket \llbracket (u, \rho) \rrbracket, \cdot, \epsilon \rrbracket = \llbracket \llbracket (t, \rho) \rrbracket \llbracket (u, \rho) \rrbracket, \cdot \rrbracket = \llbracket (t, \rho) \rrbracket \llbracket (u, \rho) \rrbracket = \\ &((\lambda x_1 \dots x_n. t) \llbracket \mathbf{C}_1 \rrbracket \dots \llbracket \mathbf{C}_n \rrbracket) (\lambda x_1 \dots x_n. u) \llbracket \mathbf{C}_1 \rrbracket \dots \llbracket \mathbf{C}_n \rrbracket \end{aligned} \quad (\text{A1.1})$$

Since $\Theta; \Delta; \cdot \vdash_{-} \llbracket (t u, \rho, \epsilon) \rrbracket : -$ therefore from Theorem 22 we know that

$$\frac{\llbracket (t u, \rho, \epsilon) \rrbracket =}{(\lambda x_1 \dots x_n. t u) \llbracket \mathbf{C}_1 \rrbracket \dots \llbracket \mathbf{C}_n \rrbracket} =$$

$\lambda p. \text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t1,n} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c$
where

$$\begin{aligned} E_c &= b (\text{coerce1 } !e_{t2,n} c) d \\ e_{t1,n} &= (\lambda x_1 \dots x_n. t u) \llbracket \mathbf{C}_1 \rrbracket \dots \llbracket \mathbf{C}_{n-1} \rrbracket \\ e_{t2,n} &= \llbracket \mathbf{C}_n \rrbracket \end{aligned}$$

$$\overline{e_{t1,n}} =$$

$$(\lambda x_1 \dots x_n. t u) \llbracket \mathbf{C}_1 \rrbracket \dots \llbracket \mathbf{C}_{n-1} \rrbracket =$$

$\lambda p. \text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t1,n-1} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c$

where

$$\begin{aligned} E_c &= b (\text{coerce1 } !e_{t2,n-1} c) d \\ e_{t1,n-1} &= (\lambda x_1 \dots x_n. t u) \llbracket \mathbf{C}_1 \rrbracket \dots \llbracket \mathbf{C}_{n-2} \rrbracket \\ e_{t2,n-1} &= \llbracket \mathbf{C}_{n-1} \rrbracket \end{aligned}$$

...

$$\overline{e_{t1,2}} =$$

$$(\lambda x_1 \dots x_n. t u) \llbracket \mathbf{C}_1 \rrbracket =$$

$\lambda p. \text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t1,1} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c$

where

$$\begin{aligned} E_c &= b (\text{coerce1 } !e_{t2,1} c) d \\ e_{t1,1} &= (\lambda x_1 \dots x_n. t u) \end{aligned}$$

$$e_{t2,1} = \langle \mathbf{C}_1 \rangle$$

$$\frac{e_{t1,1} = \overline{(\lambda x_1 \dots x_n. t u)}}{\lambda p_1. \text{ret } \lambda y. \lambda p_2. \text{let } !x = y \text{ in release } - = p_1 \text{ in release } - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_{t2} a}$$

where

$$e_{t2} = \overline{(\lambda x_2 \dots x_n. t u)}$$

...

$$\frac{e_{tn-1} = \overline{(\lambda x_{n-1} x_n. t u)}}{\lambda p_1. \text{ret } \lambda y. \lambda p_2. \text{let } !x = y \text{ in release } - = p_1 \text{ in release } - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_{tn} a}$$

where

$$e_{tn} = \overline{(\lambda x_n. t u)}$$

$$\frac{e_{tn} = \overline{(\lambda x_n. t u)}}{\lambda p_1. \text{ret } \lambda y. \lambda p_2. \text{let } !x = y \text{ in release } - = p_1 \text{ in release } - = p_2 \text{ in bind } a = \text{store}() \text{ in } e'_t a}$$

where

$$e'_t = \overline{(t u)}$$

$$\frac{e'_t = \overline{(t u)}}{\lambda p. \text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_t a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c}$$

where

$$E_c = b (\text{coerce1 } !e_u c) d$$

$$e_t = \bar{t}$$

$$e_u = \bar{u}$$

Since we know that $\overline{\langle (t u, \rho, \epsilon) \rangle}() \Downarrow v_a \Downarrow^j v_1$ therefore from the reduction rule we know that $\exists j_l, L. \overline{\langle t \rangle}() \Downarrow - \Downarrow^{j_l} L$ and $\exists j_a. L (\text{coerce1 } !\overline{\langle u \rangle}!())() \Downarrow - \Downarrow^{j_l} v_1$ s.t $j = j_l + j_a$

Similarly from (A1.1) we know that

$$\overline{\langle (t, \rho, (u, \rho), \epsilon) \rangle} = \overline{((\lambda x_1 \dots x_n. t) \langle \mathbf{C}_1 \rangle \dots \langle \mathbf{C}_n \rangle) (\lambda x_1 \dots x_n. u) \langle \mathbf{C}_1 \rangle \dots \langle \mathbf{C}_n \rangle}$$

Since $\Theta; \Delta; \cdot \vdash - \overline{\langle (t, \rho, (u, \rho), \epsilon) \rangle} : -$ therefore from Theorem 22 we know that

$$\frac{\overline{\langle (t, \rho, (u, \rho), \epsilon) \rangle} = \overline{((\lambda x_1 \dots x_n. t) \langle \mathbf{C}_1 \rangle \dots \langle \mathbf{C}_n \rangle) ((\lambda x_1 \dots x_n. u) \langle \mathbf{C}_1 \rangle \dots \langle \mathbf{C}_n \rangle)}}{\lambda p. \text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t,n} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c}$$

where

$$E_c = b (\text{coerce1 } !e_{u,n} c) d$$

$$e_{t,n} = \overline{((\lambda x_1 \dots x_n. t) \langle \mathbf{C}_1 \rangle \dots \langle \mathbf{C}_n \rangle)}$$

$$e_{u,n} = \overline{((\lambda x_1 \dots x_n. u) \langle \mathbf{C}_1 \rangle \dots \langle \mathbf{C}_n \rangle)}$$

$$\frac{e_{t,n} = \overline{((\lambda x_1 \dots x_n. t) \langle \mathbf{C}_1 \rangle \dots \langle \mathbf{C}_n \rangle)}}{\lambda p. \text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t1,n} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c}$$

where

$$E_c = b \overline{(\text{coerce1 } !e_{t2,n} \ c) \ d}$$

$$e_{t1,n} = \overline{((\lambda x_1 \dots x_n.t) \ (\mathbb{C}_1) \ \dots \ (\mathbb{C}_{n-1}))}$$

$$e_{t2,n} = \overline{\mathbb{C}_n}$$

$$e_{t1,n} = \overline{((\lambda x_1 \dots x_n.t) \ (\mathbb{C}_1) \ \dots \ (\mathbb{C}_{n-1}))} =$$

$$\lambda p.\text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t1,n-1} \ a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c$$

where

$$E_c = b \overline{(\text{coerce1 } !e_{t2,n-1} \ c) \ d}$$

$$e_{t1,n-1} = \overline{((\lambda x_1 \dots x_n.t) \ (\mathbb{C}_1) \ \dots \ (\mathbb{C}_{n-2}))}$$

$$e_{t2,n-1} = \overline{\mathbb{C}_{n-1}}$$

...

$$e_{t1,2} = \overline{((\lambda x_1 \dots x_n.t) \ (\mathbb{C}_1))} =$$

$$\lambda p.\text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{l1} \ a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c$$

where

$$E_c = b \overline{(\text{coerce1 } !e_{t2,1} \ c) \ d}$$

$$e_{l1} = \overline{(\lambda x_1 \dots x_n.t)}$$

$$e_{t2,1} = \overline{\mathbb{C}_1}$$

$$e_{l1} = \overline{(\lambda x_1 \dots x_n.t)} =$$

$$\lambda p_1.\text{ret } \lambda y.\lambda p_2.\text{let } !x_1 = y \text{ in release } - = p_1 \text{ in release } - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_{l2} \ a$$

where

$$e_{l2} = \overline{(\lambda x_2 \dots x_n.t)}$$

...

$$e_{ln} = \overline{(\lambda x_n.t)} =$$

$$\lambda p_1.\text{ret } \lambda y.\lambda p_2.\text{let } !x_n = y \text{ in release } - = p_1 \text{ in release } - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_T \ a$$

where

$$e_T = \bar{t} \quad (\text{A1.2})$$

Similarly we also have

$$e_{u,n} = \overline{((\lambda x_1 \dots x_n.u) \ (\mathbb{C}_1) \ \dots \ (\mathbb{C}_n))}$$

$$e_{u,n} = \overline{((\lambda x_1 \dots x_n.u) \ (\mathbb{C}_1) \ \dots \ (\mathbb{C}_n))} =$$

$$\lambda p.\text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{u1,n} \ a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c$$

where

$$E_c = b \overline{(\text{coerce1 } !e_{u2,n} \ c) \ d}$$

$$e_{u1,n} = \overline{((\lambda x_1 \dots x_n.u) \ (\mathbb{C}_1) \ \dots \ (\mathbb{C}_{n-1}))}$$

$$e_{u2,n} = \overline{\mathbb{C}_n}$$

$$e_{u1,n} = \overline{((\lambda x_1 \dots x_n.u) \ (\mathbb{C}_1) \ \dots \ (\mathbb{C}_{n-1}))} =$$

$$\lambda p.\text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{u1,n-1} \ a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c$$

where

$$E_c = b \overline{(\text{coerce1 } !e_{u2,n-1} \ c) \ d}$$

$$e_{u1,n-1} = \overline{((\lambda x_1 \dots x_n.u) \ (\mathbb{C}_1) \ \dots \ (\mathbb{C}_{n-2}))}$$

$$e_{u2,n-1} = \overline{\mathbf{C}_{n-1}}$$

$$e_{u1,n-1} = \overline{((\lambda x_1 \dots x_n. u) (\mathbf{C}_1) \dots (\mathbf{C}_{n-2}))} =$$

$\lambda p. \text{release} - = p$ in bind $a = \text{store}()$ in bind $b = e_{u1,n-2} a$ in bind $c = \text{store}!()$ in bind $d = \text{store}()$ in E_c
 where

$$E_c = b (\text{coerce1} !e_{u2,n-2} c) d$$

$$e_{u1,n-2} = \overline{((\lambda x_1 \dots x_n. u) (\mathbf{C}_1) \dots (\mathbf{C}_{n-3}))}$$

$$e_{u2,n-2} = \overline{\mathbf{C}_{n-2}}$$

...

$$e_{u1,2} = \overline{(\lambda x_1 \dots x_n. u) (\mathbf{C}_1)} =$$

$\lambda p. \text{release} - = p$ in bind $a = \text{store}()$ in bind $b = e_{u1,1} a$ in bind $c = \text{store}!()$ in bind $d = \text{store}()$ in E_c
 where

$$E_c = b (\text{coerce1} !e_{u2,1} c) d$$

$$e_{u1,1} = \overline{(\lambda x_1 \dots x_n. u)}$$

$$e_{u2,1} = \overline{\mathbf{C}_1}$$

$$e_{u1,1} = \overline{(\lambda x_1 \dots x_n. u)} =$$

$\lambda p_1. \text{ret } \lambda y. \lambda p_2. \text{let} ! x_1 = y$ in release $- = p_1$ in release $- = p_2$ in bind $a = \text{store}()$ in $e_{U,1} a$
 where

$$e_{U,1} = \overline{(\lambda x_2 \dots x_n. u)}$$

$$e_{U,1} = \overline{(\lambda x_2 \dots x_n. u)} =$$

$\lambda p_1. \text{ret } \lambda y. \lambda p_2. \text{let} ! x_2 = y$ in release $- = p_1$ in release $- = p_2$ in bind $a = \text{store}()$ in $e_{U,2} a$
 where

$$e_{U,2} = \overline{(\lambda x_3 \dots x_n. u)}$$

...

$$e_{U,n-1} = \overline{(\lambda x_n. u)} =$$

$\lambda p_1. \text{ret } \lambda y. \lambda p_2. \text{let} ! x_n = y$ in release $- = p_1$ in release $- = p_2$ in bind $a = \text{store}()$ in $e_{U,n} a$
 where

$$e_{U,n} = \overline{u} \quad (\text{A1.3})$$

$$E_0 = \lambda p. \text{release} - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t,n} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E'_0$$

$$E'_0 = b (\text{coerce1} !e_{u,n} c) d$$

$$v_b = \text{release} - = () \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t,n} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E'_0$$

$$E_{0,1} = \text{bind } a = \text{store}() \text{ in bind } b = e_{t,n} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in}$$

$$b (\text{coerce1} !e_{u,n} c) d$$

$$E_{0,2} = \text{bind } b = e_{t,n} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } b (\text{coerce1} !e_{u,n} c) d$$

$$E_{0,3} = \text{bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } b (\text{coerce1} !e_{u,n} c) d$$

$$E_{0,4} = \text{bind } d = \text{store}() \text{ in } b (\text{coerce1} !e_{u,n} c) d$$

$$e_{t,n} = \lambda p. \text{release} - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t1,n} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E'_{t,n}$$

$$E'_{t,n} = b (\text{coerce1} !e_{t2,n} c) d$$

$$E_{t,n,1} = \text{release} - = () \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t1,n} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E'_{t,n}$$

$$E_{t,n,1,1} = \text{bind } b = e_{t1,n} () \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E'_{t,n}$$

$e_{t1,n} = \lambda p.$
 release $- = p$ in bind $a = \text{store}()$ in bind $b = e_{t1,n-1} a$ in bind $c = \text{store}!$ in bind $d = \text{store}()$ in $E'_{t1,n}$
 $E'_{t1,n} = b (\text{coerce1 } !e_{t2,n-1} c) d$
 $E_{t1,n,1} = \text{release } - = ()$ in bind $a = \text{store}()$ in bind $b = e_{t1,n-1} a$ in bind $c = \text{store}!$ in bind $d = \text{store}()$ in $b (\text{coerce1 } !e_{t2,n-1} c) d$
 $E_{t1,n,2} = \text{bind } b = e_{t1,n-1} ()$ in bind $c = \text{store}!$ in bind $d = \text{store}()$ in $b (\text{coerce1 } !e_{t2,n-1} c) d$
 $E_{t1,n,3} = \text{bind } c = \text{store}!$ in bind $d = \text{store}()$ in $b (\text{coerce1 } !e_{t2,n-1} c) d$
 $E_{t1,n,4} = \text{bind } d = \text{store}()$ in $b (\text{coerce1 } !e_{t2,n-1} c) d$
 $e_{t1,2} = \lambda p.$
 release $- = p$ in bind $a = \text{store}()$ in bind $b = e_{l1} a$ in bind $c = \text{store}!$ in bind $d = \text{store}()$ in $E'_{t1,2}$
 $E'_{t1,2} = b (\text{coerce1 } !e_{t2,1} c) d$
 $E_{t1,2,1} = \text{release } - = ()$ in bind $a = \text{store}()$ in bind $b = e_{l1} a$ in bind $c = \text{store}!$ in bind $d = \text{store}()$ in $b (\text{coerce1 } !e_{t2,1} c) d$
 $E_{t1,2,2} = \text{bind } b = e_{l1} a$ in bind $c = \text{store}!$ in bind $d = \text{store}()$ in $b (\text{coerce1 } !e_{t2,1} c) d$
 $E_{t1,2,3} = \text{bind } c = \text{store}!$ in bind $d = \text{store}()$ in $b (\text{coerce1 } !e_{t2,1} c) d$
 $e_{l1} = \lambda p_1. \text{ret } \lambda y. \lambda p_2. \text{let } !x_1 = y$ in release $- = p_1$ in release $- = p_2$ in bind $a = \text{store}()$ in $e_{l2} a$
 $E_{l1} = \text{ret } \lambda y. \lambda p_2. \text{let } !x_1 = y$ in release $- = ()$ in release $- = p_2$ in bind $a = \text{store}()$ in $e_{l2} a$
 $E_{l,1,1} = \lambda y. \lambda p_2. \text{let } !x_1 = y$ in release $- = ()$ in release $- = p_2$ in bind $a = \text{store}()$ in $e_{l2} a$
 $E_{l,1,2} = \text{let } !x_1 = y$ in release $- = ()$ in release $- = p_2$ in bind $a = \text{store}()$ in $e_{l2} a$
 $E_{l,1,3} = \text{release } - = ()$ in release $- = ()$ in bind $a = \text{store}()$ in $e_{l2} a [(\overline{\mathbb{C}_1}) ()] / x_1$
 $E_{l2} = \text{ret } \lambda y. \lambda p_2. \text{let } !x_2 = y$ in release $- = ()$ in release $- = p_2$ in bind $a = \text{store}()$ in $e_{l3} a [(\overline{\mathbb{C}_1}) ()] / x_1$
 $E_{l,2,1} = \lambda y. \lambda p_2. \text{let } !x_2 = y$ in release $- = ()$ in release $- = p_2$ in bind $a = \text{store}()$ in $e_{l3} a [(\overline{\mathbb{C}_1}) ()] / x_1$
 $E_{l,2,2} = \text{release } - = ()$ in release $- = ()$ in bind $a = \text{store}()$ in $e_{l3} a [(\overline{\mathbb{C}_1}) ()] / x_1 [(\overline{\mathbb{C}_2}) ()] / x_2$
 $E_{l3} = \text{ret } \lambda y. \lambda p_2. \text{let } !x_3 = y$ in release $- = ()$ in release $- = p_2$ in bind $a = \text{store}()$ in $e_{l4} a [(\overline{\mathbb{C}_1}) ()] / x_1 [(\overline{\mathbb{C}_2}) ()] / x_2$
 $E_{l,3,1} = \lambda y. \lambda p_2. \text{let } !x_3 = y$ in release $- = ()$ in release $- = p_2$ in bind $a = \text{store}()$ in $e_{l4} a [(\overline{\mathbb{C}_1}) ()] / x_1 [(\overline{\mathbb{C}_2}) ()] / x_2$

$D_{n-3,2}:$

\vdots

$$\frac{}{E_{l,n-3,1} (\text{coerce1 } !(\overline{\mathbb{C}_{n-3}}) !()) () \Downarrow^- E_{l,n-2,1}}$$

$D_{1,2}:$

$$\frac{}{E_{l,1,1} (\text{coerce1 } !(\overline{\mathbb{C}_1}) !()) () \Downarrow - \Downarrow E_{l,2,1}}$$

$D_{1,1}:$

$$\frac{}{E_{l1} \Downarrow^0 E_{l,1,1}}$$

$D_{2,2,3}:$

$$\frac{}{E_{l3} \Downarrow^- E_{l,3,1}}$$

$D_{2,2,2}:$

$$\frac{}{e_{l3} () [(\overline{\mathbb{C}_1}) ()] / x_1 [(\overline{\mathbb{C}_2}) ()] / x_2 \Downarrow E_{l3}}$$

$D_{2,2,1}:$

$$\frac{}{(\text{coerce1 } !(\overline{\mathbb{C}_2}) !()) \Downarrow !(\overline{\mathbb{C}_2}) ()}$$

$D_22:$

$$\frac{\frac{D_{22.1}}{E_{l,2,1}[(\text{coerce1 } \overline{!(\mathbb{C}_2)} !()) / y][() / p_2] \Downarrow E_{l1,2,2}}{E_{l,2,1}(\text{coerce1 } !e_{t2,1} !()) () \Downarrow^- E_{l,3,1}} \quad D_{22.2} \quad D_{22.3}}$$

$D_21:$

$$\frac{\frac{e_{l1} () \Downarrow E_{l1}}{E_{t1,2,1} \Downarrow^0 E_{l,2,1}} \quad D_{11} \quad D_{12}}$$

$D_32:$

$$\frac{\vdots}{E_{l,3,1}(\text{coerce1 } \overline{!(\mathbb{C}_3)} !()) () \Downarrow^- E_{l,4,1}}$$

$D_31:$

$$\frac{e_{t1,2} () \Downarrow E_{t1,2,1} \quad D_21 \quad D_22}{E_{t1,3,1} \Downarrow E_{l,3,1}}$$

$D_{n-2}2:$

$$\frac{\vdots}{E_{l,(n-2),1}(\text{coerce1 } \overline{!(\mathbb{C}_{n-2})} !()) () \Downarrow^0 E_{l,(n-1),1}}$$

$D_{n-2}1:$

$$\frac{\frac{e_{t1,n-3} () \Downarrow E_{t1,n-3,1}}{E_{t1,n-2,1} \Downarrow^- E_{l,n-2,1}} \quad \frac{\frac{e_{t1,3} () \Downarrow E_{t1,3,1}}{D_{31} \quad D_{32}}}{D_{n-3}2}}{\vdots}$$

$D_{n-1}2:$

$$\frac{\vdots}{E_{l,n-1,1}(\text{coerce1 } \overline{!(\mathbb{C}_{n-1})} !()) () \Downarrow^0 E_{l,n,1}}$$

$D_{n-1}1:$

$$\frac{\frac{e_{t1,n-2} () \Downarrow E_{t1,n-2,1}}{E_{t1,n-1,1} \Downarrow E_{l,n-1,1}} \quad D_{n-2}1 \quad D_{n-2}2}$$

$D_n2:$

$$\frac{\frac{\overline{\overline{t}[(\overline{!(\mathbb{C}_n)} ()) / x_n] \Downarrow - \Downarrow^{j_i} L}}{E_{l,n,1}[(\text{coerce1 } \overline{!(\mathbb{C}_n)} !()) / x_n][() / p_2] \Downarrow^{j_i} L} \quad \text{By inversion}}{E_{l,n,1}(\text{coerce1 } \overline{!(\mathbb{C}_n)} !()) () \Downarrow^{j_i} L}$$

$D_n1:$

$$\frac{\frac{e_{t1,n-1} () \Downarrow E_{t1,n-1,1}}{E_{t1,n,1} \Downarrow^j E_{l,n,1}} \quad D_{(n-1)}1 \quad D_{(n-1)}2}$$

$D2:$

$$\frac{\frac{v_a \Downarrow^j v_1 \quad \text{Given}}{v_b \Downarrow^j v_2} \quad \frac{v_a \overset{s}{\approx}_{aV} v_b}{v_1 \overset{s}{\approx}_{aV} v_2} \quad \text{Definition 43}}$$

T1:

$$\frac{\frac{\overline{L (coerce1 !e_{u,n} !()) () \Downarrow - \Downarrow^{ja} v_b \quad v_a \overset{s}{\approx}_{aV} v_b}}{E_{0.4}[L/b][!()/c] \Downarrow^{ja} v_b}}{E_{0.3}[L/b] \Downarrow^{ja} v_b} \text{ Claim, Lemma 49, Definition 43}$$

T0:

$$\frac{\frac{e_{t1,n} () \Downarrow E_{t1,n,1}}{E_{t,n,1} \Downarrow^j L} \quad \begin{array}{c} D_{n1} \\ D_{n2} \end{array}}{E_{t,n,1} \Downarrow^j L} \text{ E-bind}$$

D0.0:

$$\frac{\frac{\overline{\text{store}() \Downarrow^0 ()} \quad \frac{\frac{e_{t,n} () \Downarrow E_{t,n,1}}{E_{0.2} \Downarrow^j v_2} \quad \begin{array}{c} T0 \\ T1 \\ D2 \end{array}}{E_{0.1} \Downarrow^j v_2} \text{ E-bind}}{E_{0.1} \Downarrow^j v_2} \text{ E-bind}}{v_b \Downarrow^j v_2} \text{ E-release}$$

Main derivation:

$$\frac{\frac{\overline{E_0() \Downarrow v_b} \quad D0.0}{E_0 () \Downarrow v_b \Downarrow^j v_2}}{\frac{\overline{((\lambda x_1 \dots x_n. t) (\mathbb{C}_1) \dots (\mathbb{C}_n)) (\lambda x_1 \dots x_n. u) (\mathbb{C}_1) \dots (\mathbb{C}_n) () \Downarrow v_b \Downarrow^j v_2}}{((t, \rho, (u, \rho). \epsilon)) () \Downarrow v_b \Downarrow^j v_2}}$$

Claim: $\forall s. coerce1 !\overline{u}[\overline{(\mathbb{C}_1)} ()/x_1] \dots [\overline{(\mathbb{C}_n)} ()/x_n] !() \overset{s}{\approx}_{aE} coerce1 !e_{u,n} !()$

Proof

From Definition 43 it suffices to prove

$$\forall i < s. coerce1 !\overline{u}[\overline{(\mathbb{C}_1)} ()/x_1] \dots [\overline{(\mathbb{C}_n)} ()/x_n] !() \Downarrow_i v_1 \implies coerce1 !e_{u,n} !() \Downarrow v_2 \wedge v_1 \overset{s-i}{\approx}_{aV} v_2$$

This further means that given some $i < s$ s.t $coerce1 !\overline{u}[\overline{(\mathbb{C}_1)} ()/x_1] \dots [\overline{(\mathbb{C}_n)} ()/x_n] !() \Downarrow_i v_1$ and we need to prove

$$coerce1 !e_{u,n} !() \Downarrow v_2 \wedge v_1 \overset{s-i}{\approx}_{aV} v_2 \quad (C0)$$

Since we are given that $coerce1 !\overline{u}[\overline{(\mathbb{C}_1)} ()/x_1] \dots [\overline{(\mathbb{C}_n)} ()/x_n] !() \Downarrow v_1$

This means from Definition 31 we have $v_1 = !(\overline{u}[\overline{(\mathbb{C}_1)} ()/x_1] \dots [\overline{(\mathbb{C}_n)} ()/x_n] ())$

Similarly again from Definition 31 we know that

$$v_2 = !(e_{u,n} ())$$

In order to prove that $!(\overline{u}[\overline{(\mathbb{C}_1)} ()/x_1] \dots [\overline{(\mathbb{C}_n)} ()/x_n] ()) \overset{s-i}{\approx}_{aE} !(e_{u,n} ())$

from Definition 43 it suffices to prove that

$$(\overline{u}[\overline{(\mathbb{C}_1)} ()/x_1] \dots [\overline{(\mathbb{C}_n)} ()/x_n] ()) \overset{s-i}{\approx}_{aE} (e_{u,n} ())$$

Using Definition 43 it suffices to prove

$$\forall j < (s - i). (\overline{u}[\overline{(\mathbb{C}_1)} ()/x_1] \dots [\overline{(\mathbb{C}_n)} ()/x_n] ()) \Downarrow_j v'_1 \implies (e_{u,n} ()) \Downarrow v'_2 \wedge v'_1 \overset{s-i-j}{\approx}_{aV} v'_2$$

This means given some $j < (s - i)$ s.t. $(\overline{u}[\overline{(\mathbf{C}_1)}] ()/x_1] \dots [\overline{(\mathbf{C}_n)}] ()/x_n] ()) \Downarrow_j v'_1$
it suffices to prove that

$$(e_{u,n} ()) \Downarrow v'_2 \wedge v'_1 \overset{s-i-j}{\approx}_{aV} v'_2$$

From the embedding of dlPCF into λ -amor we know that v'_1 is a value of monadic type

Since we know that

$$e_{u,n} = \lambda p.$$

release $- = p$ in bind $a = \text{store}()$ in bind $b = e_{u1,n} a$ in bind $c = \text{store}!$ in bind $d = \text{store}()$ in $E'_{u,n}$
where

$$\begin{aligned} E'_{u,n} &= b (\text{coerce1} !e_{u2,n} c) d \\ e_{u1,n} &= \overline{((\lambda x_1 \dots x_n. u) (\mathbf{C}_1) \dots (\mathbf{C}_{n-1}))} \\ e_{u2,n} &= \overline{\mathbf{C}_n} \end{aligned}$$

$e_{u,n} () \Downarrow v'_2$ from E-app where

$v'_2 = \text{release} - = ()$ in bind $a = \text{store}()$ in bind $b = e_{u1,n} a$ in bind $c = \text{store}!$ in bind $d = \text{store}()$ in $b (\text{coerce1} !e_{u2,n} c) d$

Now we need to prove that $v'_1 \overset{s-i-j}{\approx}_{aV} v'_2$

From Definition 43 it suffices to prove that

$$v'_1 \Downarrow_l^k v'_a \implies v'_2 \Downarrow^k v'_b \wedge v'_a \overset{s-i-j-l}{\approx}_{aV} v'_b$$

This means given $v'_1 \Downarrow_l^k v'_a$ it suffices to prove

$$v'_2 \Downarrow^k v'_b \wedge v'_a \overset{s-i-j-l}{\approx}_{aV} v'_b$$

$v'_2 = \text{release} - = ()$ in bind $a = \text{store}()$ in bind $b = e_{u1,n} a$ in bind $c = \text{store}!$ in bind $d = \text{store}()$ in $b (\text{coerce1} !e_{u2,n} c) d$

$$E_{u,n,1} = \text{bind } a = \text{store}() \text{ in bind } b = e_{u1,n} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } b (\text{coerce1} !e_{u2,n} c) d$$

$$E_{u,n,1.1} = \text{bind } b = e_{u1,n} () \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } b (\text{coerce1} !e_{u2,n} c) d$$

$$E_{u,n,1.2} = \text{bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } b (\text{coerce1} !e_{u2,n} c) d$$

$$e_{u1,n} = \lambda p.$$

release $- = p$ in bind $a = \text{store}()$ in bind $b = e_{u1,n-1} a$ in bind $c = \text{store}!$ in bind $d = \text{store}()$ in $E'_{u1,n}$

$$E'_{u1,n} = b (\text{coerce1} !e_{u2,n-1} c) d$$

$E_{u1,n,1} = \text{release} - = ()$ in bind $a = \text{store}()$ in bind $b = e_{u1,n-1} a$ in bind $c = \text{store}!$ in bind $d = \text{store}()$ in $b (\text{coerce1} !e_{u2,n-1} c) d$

$$E_{u1,n,2} = \text{bind } b = e_{u1,n-1} () \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } b (\text{coerce1} !e_{u2,n-1} c) d$$

$$E_{u1,n,3} = \text{bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } b (\text{coerce1} !e_{u2,n-1} c) d$$

$$E_{u1,n,4} = \text{bind } d = \text{store}() \text{ in } b (\text{coerce1} !e_{u2,n-1} c) d$$

$$e_{u1,2} = \lambda p.$$

release $- = p$ in bind $a = \text{store}()$ in bind $b = e_{l1} a$ in bind $c = \text{store}!$ in bind $d = \text{store}()$ in $E'_{u1,2}$

$$E'_{u1,2} = b (\text{coerce1} !e_{u2,1} c) d$$

$E_{u1,2,1} = \text{release} - = ()$ in bind $a = \text{store}()$ in bind $b = e_{l1} a$ in bind $c = \text{store}!$ in bind $d = \text{store}()$ in $b (\text{coerce1} !e_{u2,1} c) d$

$$E_{u1,2,2} = \text{bind } b = e_{l1} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } b (\text{coerce1} !e_{u2,1} c) d$$

$$E_{u1,2,3} = \text{bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } b (\text{coerce1} !e_{u2,1} c) d$$

$$e_{l1} = \lambda p_1. \text{ret } \lambda y. \lambda p_2. \text{let } !x_1 = y \text{ in release} - = p_1 \text{ in release} - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_{U,2} a$$

$$E_{l1} = \text{ret } \lambda y. \lambda p_2. \text{let } !x_1 = y \text{ in release} - = () \text{ in release} - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_{U,2} a$$

$$E_{l,1,1} = \lambda y. \lambda p_2. \text{let } !x_1 = y \text{ in release} - = () \text{ in release} - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_{U,2} a$$

$$E_{l,1,2} = \text{let } !x_1 = y \text{ in release} - = () \text{ in release} - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_{U,2} a$$

$$\begin{aligned}
E_{l,1,3} &= \text{release } - = () \text{ in release } - = () \text{ in bind } a = \text{store}() \text{ in } e_{U,2} a[(\overline{\mathbb{C}_1}) ()]/x_1] \\
E_{l,2} &= \text{ret } \lambda y. \lambda p_2. \text{let } !x_2 = y \text{ in release } - = () \text{ in release } - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_{U,3} a[(\overline{\mathbb{C}_1}) ()]/x_1] \\
E_{l,2,1} &= \lambda y. \lambda p_2. \text{let } !x_2 = y \text{ in release } - = () \text{ in release } - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_{U,3} a[(\overline{\mathbb{C}_1}) ()]/x_1] \\
E_{l,2,2} &= (\text{release } - = () \text{ in release } - = () \text{ in bind } a = \text{store}() \text{ in } e_{U,3} a) S_2 \\
E_{l,3} &= (\text{ret } \lambda y. \lambda p_2. \text{let } !x_3 = y \text{ in release } - = () \text{ in release } - = p_2 \text{ in bind } a = \text{store}() \text{ in } \\
&e_{U,4} a) S_2 \\
E_{l,3,1} &= (\lambda y. \lambda p_2. \text{let } !x_3 = y \text{ in release } - = () \text{ in release } - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_{U,4} a) S_2 \\
S_2 &= [(\overline{\mathbb{C}_1}) ()]/x_1][(\overline{\mathbb{C}_2}) ()]/x_2] \\
E_{l,n,1} &= (\lambda y. \lambda p_2. \text{let } !x_n = y \text{ in release } - = () \text{ in release } - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_{U,n} a) S_{n-1} \\
S_{n-1} &= [(\overline{\mathbb{C}_1}) ()]/x_1] \dots [(\overline{\mathbb{C}_{n-1}}) ()]/x_{n-1}]
\end{aligned}$$

$D_{n-3,2}$:

$$\frac{\vdots}{E_{l,(n-3),1} (\text{coerce1 } !(\overline{\mathbb{C}_{n-3}}) !()) () \Downarrow^0 E_{l,(n-2),1}}$$

$D_1,2$:

$$\frac{}{E_{l,1,1} (\text{coerce1 } !(\overline{\mathbb{C}_1}) !()) () \Downarrow - \Downarrow E_{l,2,1}}$$

$D_1,1$:

$$\frac{}{E_{l1} \Downarrow^0 E_{l,1,1}}$$

$D_2,2,3$:

$$\frac{}{E_{l3} \Downarrow^0 E_{l,3,1}}$$

$D_2,2,2$:

$$\frac{}{e_{l3} () [(\overline{\mathbb{C}_1}) ()]/x_1][(\overline{\mathbb{C}_2}) ()]/x_2] \Downarrow E_{l3}}$$

$D_2,2,1$:

$$\frac{}{(\text{coerce1 } !(\overline{\mathbb{C}_2}) !()) \Downarrow !(\overline{\mathbb{C}_2}) ()}$$

$D_2,2$:

$$\frac{\frac{D_2,2,1}{E_{l,2,1}[(\text{coerce1 } !(\overline{\mathbb{C}_2}) !()) / y][() / p_2] \Downarrow E_{l1,2,2}} \quad D_2,2,2 \quad D_2,2,3}{E_{l,2,1} (\text{coerce1 } !e_{u,2,1} !()) () \Downarrow^0 E_{l,3,1}}}$$

$D_2,1$:

$$\frac{\frac{}{e_{l1} () \Downarrow E_{l1}} \quad D_1,1 \quad D_1,2}{E_{u1,2,1} \Downarrow^0 E_{l,2,1}}}$$

$D_3,2$:

$$\frac{\vdots}{E_{l,3,1} (\text{coerce1 } !(\overline{\mathbb{C}_3}) !()) () \Downarrow^0 E_{l,4,1}}$$

$D_3,1$:

$$\frac{e_{u1,2} () \Downarrow E_{u1,2,1} \quad D_2,1 \quad D_2,2}{E_{u1,3,1} \Downarrow E_{l,3,1}}$$

$D_{n-2}2:$

$$\frac{\vdots}{\overline{E_{l,(n-2),1}(\text{coerce1}!(\mathbf{C}_{n-2})!())} \Downarrow^0 E_{l,(n-1),1}}$$

$D_{n-2}1:$

$$\frac{\overline{e_{u1,n-3}() \Downarrow E_{u1,n-3,1}} \quad \frac{\overline{e_{u1,3}() \Downarrow E_{u1,3,1}} \quad D_{31} \quad D_{32}}{\vdots} \quad D_{n-3}2}{\overline{E_{u1,n-2,1} \Downarrow^0 E_{l,n-2,1}}}$$

$D_{n-1}2:$

$$\frac{\vdots}{\overline{E_{l,n-1,1}(\text{coerce1}!(\mathbf{C}_{n-1})!())} \Downarrow^0 E_{l,n,1}}$$

$D_{n-1}1:$

$$\frac{\overline{e_{u1,n-2}() \Downarrow E_{u1,n-2,1}} \quad D_{n-2}1 \quad D_{n-2}2}{\overline{E_{u1,n-1,1} \Downarrow^0 E_{l,n-1,1}}}$$

$D_n2:$

$$\frac{\overline{\overline{\overline{\overline{\overline{\bar{u}!(\mathbf{C}_1)())}/x_1} \dots [!(\mathbf{C}_n)()]/x_n} \Downarrow v'_1 \Downarrow^k v'_a} \quad \text{Given}}{\overline{E_{l,n,1}[(\text{coerce1}!(\mathbf{C}_n)!())]/x_n}[(\text{coerce1}!(\mathbf{C}_n)!())]/p_2} \Downarrow v'_1 \Downarrow^k v'_a} \quad \overline{E_{l,n,1}(\text{coerce1}!(\mathbf{C}_n)!())} \Downarrow^k v'_a}$$

$D_n1:$

$$\frac{\overline{e_{u1,n-1}() \Downarrow E_{u1,n-1,1}} \quad D_{(n-1)1} \quad D_{(n-1)2}}{\overline{E_{u1,n,1} \Downarrow^0 E_{l,n,1}}}$$

Main derivation:

$$\frac{\overline{e_{u1,n}() \Downarrow E_{u1,n,1}} \quad D_n1 \quad D_n2}{\overline{E_{u,n,1} \Downarrow^k v'_a} \quad \text{E-bind}} \quad \text{E-release} \quad \frac{}{v'_2 \Downarrow^k v'_a}$$

From Lemma 48 we get $v'_a \stackrel{s-i-j-l}{\approx} v'_a$

□

Lemma 51 (Lemma for appl: non-empty stack). $\forall t, u, \rho, \theta, v'_{\epsilon 1}, v_{\epsilon 1}, v'_{\epsilon 2}, v_{\epsilon 2}, v_{\theta 1}, j, j', j''$.

$(t u, \rho, \epsilon)$ and $(t, \rho, (u, \rho).\epsilon)$ are well-typed

$(t u, \rho, \theta)$ and $(t, \rho, (u, \rho).\theta)$ are well-typed

$(t u, \rho, \epsilon) \rightarrow (t, \rho, (u, \rho).\epsilon) \wedge (t u, \rho, \theta) \rightarrow (t, \rho, (u, \rho).\theta) \wedge$

$\overline{\overline{(t u, \rho, \epsilon)}}() \Downarrow v'_{\epsilon 1} \Downarrow^j v_{\epsilon 1} \wedge \overline{\overline{(t, \rho, (u, \rho).\epsilon)}}() \Downarrow v'_{\epsilon 2} \Downarrow^{j'} v_{\epsilon 2} \wedge \forall s.v_{\epsilon 1} \stackrel{s}{\approx}_{aV} v_{\epsilon 2} \wedge$

$\overline{\overline{(t u, \rho, \theta)}}() \Downarrow v'_{\theta 1} \Downarrow^{j''} v_{\theta 1}$

\implies

$\exists v'_{\theta 2}, v_{\theta 2}, j''' . \overline{\overline{(t, \rho, (u, \rho).\theta)}}() \Downarrow v'_{\theta 2} \Downarrow^{j'''} v_{\theta 2} \wedge (j - j') = (j'' - j''') \wedge \forall s.v_{\theta 1} \stackrel{s}{\approx}_{aV} v_{\theta 2}$

Proof. We prove this by induction on θ

1. Case $\theta = \epsilon$:

Directly from given

2. Case $\theta = \mathbf{C}'.\theta'$:

Let $\theta' = \mathbf{C}'_1 \dots \mathbf{C}'_n$ and $\theta'' = \mathbf{C}'_1 \dots \mathbf{C}'_{n-1}$

Given:

$(t \ u, \rho, \mathbf{C}'.\theta')$ and $(t, \rho, (u, \rho).\mathbf{C}'.\theta')$ are well-typed \wedge

$(t \ u, \rho, \mathbf{C}'.\theta') \rightarrow (t, \rho, (u, \rho).\mathbf{C}'.\theta') \wedge \overline{\llbracket (t \ u, \rho, \mathbf{C}'.\theta') \rrbracket} () \Downarrow v'_{\theta 1} \Downarrow^{j''} v_{\theta 1}$

We need to prove that

$\exists v'_{\theta 2}, v_{\theta 2}, j'''$.

$$\overline{\llbracket (t, \rho, (u, \rho).\mathbf{C}'.\theta') \rrbracket} () \Downarrow v'_{\theta 2} \Downarrow^{j''} v_{\theta 2} \wedge (j - j') = (j'' - j''') \wedge \forall s.v_{\theta 1} \overset{s}{\approx}_{aV} v_{\theta 2} \quad (\text{ET-0})$$

From IH we know

$(t \ u, \rho, \mathbf{C}'.\theta'')$ and $(t, \rho, (u, \rho).\mathbf{C}'.\theta'')$ are well-typed \wedge

$(t \ u, \rho, \mathbf{C}'.\theta'') \rightarrow (t, \rho, (u, \rho).\mathbf{C}'.\theta'') \wedge \overline{\llbracket (t \ u, \rho, \mathbf{C}'.\theta'') \rrbracket} () \Downarrow v'_{\theta 11} \Downarrow^{j''} v_{\theta 11} \implies \exists j'''_1, v'_{\theta 22}, v_{\theta 22}$.

$$\overline{\llbracket (t, \rho, (u, \rho).\mathbf{C}'.\theta'') \rrbracket} () \Downarrow v'_{\theta 22} \Downarrow^{j''} v_{\theta 22} \wedge (j - j') = (j''_1 - j''_1) \wedge \forall s.v_{\theta 11} \overset{s}{\approx}_{aV} v_{\theta 22} \quad (\text{ET-IH})$$

From Definition 39 and Definition 40 we know that

$$\overline{\llbracket (t \ u, \rho, \mathbf{C}'.\theta') \rrbracket} = \overline{\llbracket (t \ u, \rho) \llbracket \mathbf{C}' \rrbracket \dots \llbracket \mathbf{C}_{n-1} \rrbracket \llbracket \mathbf{C}_n \rrbracket \rrbracket} \quad (\text{ET-1})$$

Since $(t \ u, \rho, \mathbf{C}'.\theta')$ is well typed therefore we know that

$$\overline{\llbracket (t \ u, \rho, \mathbf{C}'.\theta') \rrbracket} = \overline{\llbracket (t \ u, \rho) \llbracket \mathbf{C}' \rrbracket \dots \llbracket \mathbf{C}_{n-1} \rrbracket \llbracket \mathbf{C}_n \rrbracket \rrbracket} =$$

$\lambda p.\text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t1} \ a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c$

where

$$E_c = b \ (\text{coerce1 } !e_{t2} \ c) \ d$$

$$e_{t1} = \overline{\llbracket (t \ u, \rho) \llbracket \mathbf{C}' \rrbracket \dots \llbracket \mathbf{C}_{n-1} \rrbracket \rrbracket}$$

$$e_{t2} = \overline{\llbracket \mathbf{C}_n \rrbracket} \quad (\text{ET-1.1})$$

From Krivine reduction (app rule) we also know that $(t \ u, \rho, \mathbf{C}'.\theta'') \rightarrow (t, \rho, (u, \rho).\mathbf{C}'.\theta'')$

Also since we know that $\overline{\llbracket (t \ u, \rho, \mathbf{C}'.\theta'') \rrbracket} () \Downarrow v'_{\theta 11} \Downarrow^{j''} v_{\theta 11}$ therefore we also know that $\exists j'''_1, v'_1, v_1.e_{t1}() \Downarrow v_1 \Downarrow^{j''} v'_1$

Also since we know that

$(t \ u, \rho, \mathbf{C}'.\theta')$ and $(t, \rho, (u, \rho).\mathbf{C}'.\theta')$ are well-typed

therefore from Lemma 56 we also know that

$(t \ u, \rho, \mathbf{C}'.\theta'')$ and $(t, \rho, (u, \rho).\mathbf{C}'.\theta'')$ are well-typed

Therefore from (ET-IH) we have

$$\begin{aligned} & \exists j_1''', v'_{\theta 22}, v_{\theta 22}. \overline{\langle (t, \rho, (u, \rho). \mathbf{C}'. \theta'') \rangle} () \Downarrow v'_{\theta 22} \Downarrow^{j_1'''} v_{\theta 22} \wedge (j - j') = (j_1'' - j_1''') \\ & \wedge \forall s. v_{\theta 11} \overset{s}{\approx}_{aV} v_{\theta 22} \quad (\text{ET-2}) \end{aligned}$$

From (ET-0) and Definition 39, Definition 40 it suffices to prove that

$$\begin{aligned} & \exists j''', v'_{\theta 2}, v_{\theta 2}. \overline{\langle (t, \rho) \langle (u, \rho) \rangle \langle \mathbf{C}' \rangle \dots \langle \mathbf{C}_{n-1} \rangle \langle \mathbf{C}_n \rangle \rangle} () \Downarrow v'_{\theta 2} \Downarrow^{j'''} v_{\theta 2} \wedge (j - j') = (j'' - j''') \wedge \\ & \forall s. v_{\theta 1} \overset{s}{\approx}_{aV} v_{\theta 2} \quad (\text{ET-3}) \end{aligned}$$

Since $(t, \rho, (u, \rho). \mathbf{C}'. \theta')$ is well typed therefore we know that

$$\overline{\langle (t, \rho) \langle (u, \rho) \rangle \langle \mathbf{C}' \rangle \dots \langle \mathbf{C}_{n-1} \rangle \langle \mathbf{C}_n \rangle \rangle} =$$

$\lambda p. \text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b' = e'_{t1} \ a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c$
where

$$\begin{aligned} E_c &= b' (\text{coerce1 } !e'_{t2} \ c) \ d \\ e'_{t1} &= \overline{\langle (t, \rho) \langle (u, \rho) \rangle \langle \mathbf{C}' \rangle \dots \langle \mathbf{C}_{n-1} \rangle \rangle} \\ e'_{t2} &= \overline{\langle \mathbf{C}_n \rangle} \end{aligned}$$

From (ET-2) we know that $e'_{t1} () \Downarrow v'_{\theta 22} \Downarrow^{j_1'''} v_{\theta 22}$

and we need to prove that $v_{\theta 22} (\text{coerce1 } !e'_{t2} \ c) \ d \Downarrow v_t \Downarrow^{j'' - j'''} v_{\theta 2}$ (ET-p)

Since we are given that $\langle (t \ u, \rho, \mathbf{C}'. \theta') \rangle () \Downarrow v'_{\theta 1} \Downarrow^{j''} v_{\theta 1}$ this means from (ET-1.1) we have

$\lambda p. \text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t1} \ a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c$
 $\Downarrow v'_{\theta 1} \Downarrow^{j''} v_{\theta 1}$

where

$$E_c = b (\text{coerce1 } !e_{t2} \ c) \ d$$

Also since we are given that $\overline{\langle (t \ u, \rho, \mathbf{C}'. \theta'') \rangle} () \Downarrow v'_{\theta 11} \Downarrow^{j_1'''} v_{\theta 11}$ this means we have

$$e_{t1} () \Downarrow v'_{\theta 11} \Downarrow^{j_1'''} v_{\theta 11}$$

This means $v_{\theta 11} (\text{coerce1 } !e_{t2} \ c) \ d \Downarrow - \Downarrow^y v_{\theta 1}$ for some y s.t $y + j_1'' = j''$

Since $\forall s. v_{\theta 11} \overset{s}{\approx}_{aV} v_{\theta 22}$ and $e_{t2} = e'_{t2} = \overline{\langle \mathbf{C}_n \rangle}$ therefore from Definition 43 we get $\forall s. v_{\theta 1} \overset{s}{\approx}_{aV} v_{\theta 2}$. Also from Definition 43 we have

$$\begin{aligned} j'' - j_1'' &= j''' - j_1''' = \\ j'' - j''' &= j_1'' - j_1''' = \\ j'' - j''' &= j - j' \quad (\text{From ET-IH}) \end{aligned}$$

□

Lemma 52 (Cost and size lemma). $\forall e_s, D_s, E_s.$

$$\begin{aligned} & (e_s, \epsilon, \epsilon) \overset{*}{\rightarrow} D_s \rightarrow E_s \wedge \\ & D_s \text{ is well-typed } \wedge \\ & E_s \text{ is well-typed } \wedge \\ & e_t = \overline{\langle D_s \rangle} \wedge e_t () \Downarrow v_a \Downarrow^j v_1 \\ & \implies \\ & \exists e'_t. e'_t = \overline{\langle E_s \rangle} \wedge e'_t () \Downarrow v_b \Downarrow^{j'} v_2 \wedge \forall s. v_1 \overset{s}{\approx}_{aE} v_2 \wedge \end{aligned}$$

1. $j' = j \wedge |D_s| > |E_s|$ or
2. $j' = j - 1 \wedge |E_s| < |D_s| + |e_s|$

Proof. We case analyze on the $D_s \rightarrow E_s$ reduction

1. App1:

Given $D_s = (t \ u, \rho, \theta)$ and $E_s = (t, \rho, (u, \rho).\theta)$

Let $D'_s = (t \ u, \rho, \epsilon)$ and $E'_s = (t, \rho, (u, \rho).\epsilon)$

Since we are given that D_s is well-typed and E_s is well-typed therefore from Lemma 53 we also have

D'_s is well-typed and E'_s is well-typed

Also since we know that $e_t () \Downarrow v_a \Downarrow^j v_1$ therefore from Lemma 54 we also know that

$\exists j_e. \overline{\langle D'_s \rangle} () \Downarrow v'_d \Downarrow^{j_e} v_d$

From Lemma 50 we know that $\exists v_e. \overline{\langle E'_s \rangle} () \Downarrow v'_e \Downarrow^{j_e} v_e$ s.t $\forall s.v_d \overset{s}{\approx}_{aV} v_e$

And finally from Lemma 51 we know that $\overline{\langle E_s \rangle} () \Downarrow v_b \Downarrow^j v_2$ s.t $\forall s.v_1 \overset{s}{\approx}_{aV} v_2$

$|D_s| > |E_s|$ holds directly from the Definition of $|-|$

2. App2:

Given: $(\lambda x.t, \rho, c.\theta) \rightarrow (t, c.\rho, \theta)$

We induct on θ

(a) Case $\theta = \epsilon$:

Since we are given that D_s i.e $(\lambda x.t, \rho, c.\epsilon)$ is well typed

Therefore from Theorem 42 $\overline{\langle (\lambda x.t, \rho, c.\epsilon) \rangle}$ is well-typed

From Definition 40 $\overline{\langle (\lambda x.t, \rho) \langle c \rangle, ., \epsilon \rangle}$ is well-typed

Again from Definition 40 $\overline{\langle \lambda x.t, \rho \rangle \langle c \rangle}$ is well-typed

From Definition 39 we have

$\overline{\langle (\lambda x_1 \dots x_n. \lambda x.t) \langle \mathbf{C}_1 \rangle \dots \langle \mathbf{C}_n \rangle \langle c \rangle \rangle}$ is well-typed

Therefore from Theorem 22 we know that

$\overline{\langle D_s \rangle} =$

$\overline{\langle (\lambda x_1 \dots x_n. \lambda x.t) \langle \mathbf{C}_1 \rangle \dots \langle \mathbf{C}_n \rangle \langle \mathbf{C} \rangle \rangle} =$

$\lambda p. \text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t1} \ a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c$

where

$E_c = b \ (\text{coerce1 } !e_{t2} \ c) \ d$

$e_{t1} = \overline{\langle (\lambda x_1 \dots x_n. \lambda x.t) \langle \mathbf{C}_1 \rangle \dots \langle \mathbf{C}_n \rangle \rangle}$

$e_{t2} = \overline{\langle \mathbf{C} \rangle} \quad (\text{S-A0})$

Since we are given that $\overline{\langle D_s \rangle} () \Downarrow v_a \Downarrow^j v_1$

therefore from the evaluation rules we know that

$\overline{\langle t \rangle} [\overline{\langle \mathbf{C} \rangle} () / x] [\overline{\langle \mathbf{C}_1 \rangle} () / x_1] \dots [\overline{\langle \mathbf{C}_1 \rangle} () / x_1] \Downarrow - \Downarrow^j v_1 \quad (\text{S-A0.1})$

Similarly since we are given that E_s i.e (t, c, ρ, ϵ) is well-typed

Therefore from Theorem 42 $((t, c, \rho, \epsilon))$ is well-typed

From Definition 40 $((t, c, \rho))$ is well-typed

From Definition 39 we have $((\lambda x, x_1 \dots x_n.t) (\mathbb{C}) (\mathbb{C}_1) \dots (\mathbb{C}_n))$ is well-typed

Therefore from Theorem 22 we know that

$$\overline{(E_s)} = \overline{((\lambda x x_1 \dots x_n.t) (\mathbb{C}) (\mathbb{C}_1) \dots (\mathbb{C}_n))} =$$

$$\lambda p.\text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t_1} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c$$

where

$$E_c = b (\text{coerce1 } !e_{t_2} c) d$$

$$e_{t_1} = \overline{((\lambda x x_1 \dots x_n.t) (\mathbb{C}), (\mathbb{C}_1) \dots (\mathbb{C}_{n-1}))}$$

$$e_{t_2} = \overline{(\mathbb{C}_n)} \quad (\text{S-A1})$$

From (SA-0.1) we know that

$$\overline{(E_s)} () \Downarrow - \Downarrow^j v_1$$

And finally from Theorem 47 we have $\forall s.v_1 \overset{s}{\approx}_{aV} v_1$

(b) Case $\theta = \mathbb{C}'.\theta'$:

Let $\theta' = \mathbb{C}_{\theta_1} \dots \mathbb{C}_{\theta_n}$ and $\rho = \mathbb{C}_{\rho_1} \dots \mathbb{C}_{\rho_n}$

Since we are given that D_s i.e $(\lambda x.t, \rho, \mathbb{C}.\mathbb{C}'.\theta')$ is well typed

Therefore from Theorem 42 we know that $((\lambda x.t, \rho, \mathbb{C}.\mathbb{C}'.\theta'))$ is well-typed

From Definition 40 we also have $((\lambda x.t, \rho) (\mathbb{C}), \dots, \mathbb{C}'.\theta')$ is well-typed

which further means that $((\lambda x.t, \rho) (\mathbb{C}) (\mathbb{C}'), \dots, \theta')$ is well-typed

which further means that $((\lambda x.t, \rho) (\mathbb{C}) (\mathbb{C}') (\mathbb{C}_{\theta_1}) \dots (\mathbb{C}_{\theta_1}))$ is well-typed

which further means that $(\lambda x_1 \dots x_n.\lambda x.t) (\mathbb{C}_{\rho_1}) \dots (\mathbb{C}_{\rho_n}) (\mathbb{C}) (\mathbb{C}') (\mathbb{C}_{\theta_1}) \dots (\mathbb{C}_{\theta_m})$ is well-typed

From Theorem 22 we have

$$\overline{(D_s)} = \overline{(\lambda x_1 \dots x_n.\lambda x.t) (\mathbb{C}_{\rho_1}) \dots (\mathbb{C}_{\rho_n}) (\mathbb{C}) (\mathbb{C}') (\mathbb{C}_{\theta_1}) \dots (\mathbb{C}_{\theta_m})} =$$

$$\lambda p.\text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t_1} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c$$

where

$$E_c = b (\text{coerce1 } !e_{t_2} c) d$$

$$e_{t_1} = \overline{(\lambda x_1 \dots x_n.\lambda x.t) (\mathbb{C}_{\rho_1}) \dots (\mathbb{C}_{\rho_n}) (\mathbb{C}) (\mathbb{C}') (\mathbb{C}_{\theta_1}) \dots (\mathbb{C}_{\theta_{m-1}})}$$

$$e_{t_2} = \overline{(\mathbb{C}_{\theta_m})} \quad (\text{S-A2})$$

Since we are given that $\overline{(D_s)} () \Downarrow v_a \Downarrow^j v_1$

therefore from the evaluation rules we know that

$$\exists e', j_1. \overline{(t)} [\overline{(\mathbb{C})}() / x] [\overline{(\mathbb{C}_1)}() / x_1] \dots [\overline{(\mathbb{C}_1)}() / x_1] \Downarrow - \Downarrow^{j_1} \lambda x' x_1 \dots x_m.e'$$

s.t

$$\lambda x' x_1 \dots x_m.e' \overline{(t)} [\overline{(\mathbb{C}')}() / x] [\overline{(\mathbb{C}_{\theta_1})}() / x_1] \dots [\overline{(\mathbb{C}_{\theta_m})}() / x_m] \Downarrow - \Downarrow^{j_2} v_1$$

$$\text{and } j_1 + j_2 = j \quad (\text{S-A2.1})$$

Similarly since we are given that E_s i.e $(t, \mathbb{C}.\rho, \mathbb{C}'.\theta')$ is well typed

Therefore from Theorem 42 we know that $((t, \mathbb{C}.\rho, \mathbb{C}'.\theta'))$ is well-typed

From Definition 40 we also have $\langle\langle (t, \mathbf{C}.\rho) \langle\mathbf{C}'\rangle, \cdot, \theta' \rangle\rangle$ is well-typed
which further means that $\langle\langle (t, \mathbf{C}.\rho) \langle\mathbf{C}'\rangle \langle\mathbf{C}_{\theta_1}\rangle \dots \langle\mathbf{C}_{\theta_m}\rangle \rangle\rangle$ is well-typed
which further means that $\langle\langle (\lambda x, x_1 \dots x_n.t) \langle\mathbf{C}\rangle \langle\mathbf{C}_{\rho_1}\rangle \dots \langle\mathbf{C}_{\rho_n}\rangle \langle\mathbf{C}'\rangle \langle\mathbf{C}_{\theta_1}\rangle \dots \langle\mathbf{C}_{\theta_m}\rangle \rangle\rangle$ is well-typed

From Theorem 22 we have

$$\overline{\langle E_s \rangle} = \overline{(\lambda x, x_1 \dots x_n.t) \langle\mathbf{C}\rangle \langle\mathbf{C}_{\rho_1}\rangle \dots \langle\mathbf{C}_{\rho_n}\rangle \langle\mathbf{C}'\rangle \langle\mathbf{C}_{\theta_1}\rangle \dots \langle\mathbf{C}_{\theta_m}\rangle} =$$

$$\lambda p.\text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t_1} \ a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c$$

where

$$E_c = b \ (\text{coerce1 } !e_{t_2} \ c) \ d$$

$$e_{t_1} = \overline{(\lambda x, x_1 \dots x_n.t) \langle\mathbf{C}\rangle \langle\mathbf{C}_{\rho_1}\rangle \dots \langle\mathbf{C}_{\rho_n}\rangle \langle\mathbf{C}'\rangle \langle\mathbf{C}_{\theta_1}\rangle \dots \langle\mathbf{C}_{\theta_{m-1}}\rangle}$$

$$e_{t_2} = \overline{\langle\mathbf{C}_{\theta_m}\rangle} \quad (\text{S-A3})$$

From (S-A2.1) it is clear that

$$\overline{\langle E_s \rangle} () \Downarrow - \Downarrow^j v_1$$

And finally from Theorem 47 we have $\forall s.v_1 \overset{s}{\approx}_{aV} v_1$

$|D_s| > |E_s|$ holds directly from the Definition of $| - |$

3. Fix:

Given: $(\text{fix } x.t, \rho, \theta) \rightarrow (t, (\text{fix } x.t, \rho).\rho, \theta)$

Let $D'_s = (\text{fix } x.t, \rho, \epsilon)$ and $E'_s = (t, (\text{fix } x.t, \rho).\rho, \epsilon)$

Since we are given that D_s and E_s are well-typed therefore from Lemma 53 we know that D'_s and E'_s are well-typed too.

Also since we know that $e_t () \Downarrow v_a \Downarrow^j v_1$ therefore from Lemma 54 we also know that

$$\exists j_e. \overline{\langle D'_s \rangle} \Downarrow - \Downarrow^{j_e} v_e$$

From Lemma 57 we know that $\overline{\langle E'_s \rangle} () \Downarrow v'_e \Downarrow^{j_e} v_e$

And then from Lemma 55 we know that $\overline{\langle E_s \rangle} \Downarrow v_b \Downarrow^j v_2$ s.t $\forall s.v_1 \overset{s}{\approx}_{aV} v_2$

$|D_s| > |E_s|$ holds directly from the Definition of $| - |$

4. Var:

Given: $D_s = (x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \theta)$ and $E_s = (t_x, \rho_x, \theta)$

Let $D'_s = (x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \epsilon)$ and $E'_s = (t_x, \rho_x, \epsilon)$

Since we are given that D_s and E_s are well-typed therefore from Lemma 53 we know that D'_s and E'_s are well-typed too.

Also since we know that $e_t () \Downarrow - \Downarrow^j v_1$ therefore from Lemma 54 we also know that

$$\exists j_e. \overline{\langle D'_s \rangle} \Downarrow - \Downarrow^{j_e} v_e$$

From Lemma 59 we know that $\overline{\langle E'_s \rangle} \Downarrow - \Downarrow^{j_e-1} v_e$

And then from Lemma 58 we know that $\overline{\langle E_s \rangle} \Downarrow - \Downarrow^{j-1} v_2$ s.t. $\forall s.v_1 \stackrel{s}{\approx}_{aV} v_2$

$|E_s| < |D_s| + |e_s|$ holds directly from the Definition of $| - |$ and from Lemma 4.2 in [3]

□

Lemma 53 (ϵ typing). $\forall \Theta, \Delta, I, e, \rho, \theta.$

$\Theta; \Delta \vdash_- (e, \rho, \theta) : - \implies \Theta; \Delta \vdash_- (e, \rho, \epsilon) : -$

Proof. Main derivation:

$$\frac{\frac{\overline{\Theta; \Delta \vdash_I (e, \rho, \theta) : \tau}}{\Theta; \Delta \vdash_J (e, \rho) : \sigma} \text{ Given}}{\Theta; \Delta \vdash_J (e, \rho, \epsilon) : \sigma} \text{ By inversion} \quad \frac{}{\Theta; \Delta \vdash_0 \epsilon : (\sigma, \sigma)}$$

□

Lemma 54 (ϵ reduction). $\forall e, \rho, \theta.$

(e, ρ, θ) is well typed $\wedge \overline{\langle (e, \rho, \theta) \rangle} () \Downarrow - \Downarrow^- - \implies \overline{\langle (e, \rho, \epsilon) \rangle} () \Downarrow - \Downarrow^- -$

Proof. Since (e, ρ, θ) is well typed therefore from Lemma 53 we also know that

(e, ρ, ϵ) is well typed

From Theorem 42 we know that $\langle (e, \rho, \epsilon) \rangle$ is also well typed

From Definition 40 we know that $\langle (e, \rho, \epsilon) \rangle = \langle (e, \rho) \rangle$

Let $\theta = \mathbf{C}_1 \dots \mathbf{C}_n$

Similarly from Definition 40 we also know that

$$\begin{aligned} \langle (e, \rho, \theta) \rangle &= \langle (e, \rho, \mathbf{C}_1 \dots \mathbf{C}_n) \rangle = \\ \langle \langle (e, \rho) \rangle \langle \mathbf{C}_1 \rangle, \square, \mathbf{C}_2 \dots \mathbf{C}_n \rangle &= \\ \langle \langle (e, \rho) \rangle \langle \mathbf{C}_1 \rangle \dots \langle \mathbf{C}_n \rangle, \square, \epsilon \rangle &= \\ \langle \langle (e, \rho) \rangle \langle \mathbf{C}_1 \rangle \dots \langle \mathbf{C}_n \rangle \rangle & \end{aligned}$$

From Theorem 22 we know that

$$\begin{aligned} \overline{\langle (e, \rho, \theta) \rangle} &= \\ \overline{\langle \langle (e, \rho) \rangle \langle \mathbf{C}_1 \rangle \dots \langle \mathbf{C}_n \rangle \rangle} &= \end{aligned}$$

$\lambda p.\text{release } - = p$ in bind $a = \text{store}()$ in bind $b = e_{t1}$ a in bind $c = \text{store}!$ in bind $d = \text{store}()$ in E_c where

$$\begin{aligned} E_c &= b (\text{coerce1 } !e_{t2} c) d \\ e_{t1} &= \overline{\langle \langle (e, \rho) \rangle \langle \mathbf{C}_1 \rangle \dots \langle \mathbf{C}_{n-1} \rangle \rangle} \\ e_{t2} &= \langle \mathbf{C}_n \rangle \quad (\text{E0}) \end{aligned}$$

Since $\overline{\langle \langle (e, \rho) \rangle \langle \mathbf{C}_1 \rangle \dots \langle \mathbf{C}_n \rangle \rangle} \Downarrow - \Downarrow^- -$, therefore we also know that $\overline{\langle (e, \rho) \rangle} \Downarrow - \Downarrow^- -$

□

Lemma 55 (Lemma for fix : non-empty stack). $\forall t, \rho, \theta, j, j', j'', v_{\epsilon 1}, v_{\epsilon 2}, v_{\theta 1}.$

$(\text{fix } x.t, \rho, \epsilon)$ and $(t, (\text{fix } x.t, \rho).\rho, \epsilon)$ are well-typed

$(\text{fix } x.t, \rho, \theta)$ and $(t, (\text{fix } x.t, \rho).\rho, \theta)$ are well-typed

$$\begin{aligned} \overline{\langle (\text{fix } x.t, \rho, \epsilon) \rangle} () \Downarrow - \Downarrow^j v_{\epsilon 1} \wedge \overline{\langle (t, (\text{fix } x.t, \rho).\rho, \epsilon) \rangle} () \Downarrow - \Downarrow^{j'} v_{\epsilon 1} \wedge \forall s.v_{\epsilon 1} \stackrel{s}{\approx}_{aV} v_{\epsilon 2} \wedge \\ \overline{\langle (\text{fix } x.t, \rho, \theta) \rangle} () \Downarrow - \Downarrow^{j''} v_{\theta 1} \wedge \end{aligned}$$

\implies

$$\exists v_{\theta 2}, j'''. \overline{\langle (t, (\text{fix } x.t, \rho).\rho, \theta) \rangle} () \Downarrow - \Downarrow^{j'''} v_{\theta 2} \wedge \forall s.v_{\theta 1} \stackrel{s}{\approx}_{aV} v_{\theta 2} \wedge (j - j') = (j'' - j''')$$

Proof. We prove this by induction on θ

1. Case $\theta = \epsilon$:

Directly from given

2. Case $\theta = \mathbf{C}'.\theta'$:

Let $\theta' = \mathbf{C}'_1 \dots \mathbf{C}'_n$ and $\theta'' = \mathbf{C}'_1 \dots \mathbf{C}'_{n-1}$

Given:

$(\text{fix } x.t, \rho, \mathbf{C}'.\theta')$ and $(t, (\text{fix } x.t, \rho).\rho, \mathbf{C}'.\theta')$ are well-typed \wedge

$$\overline{\langle (\text{fix } x.t, \rho, \mathbf{C}'.\theta') \rangle} () \Downarrow - \Downarrow^{j''} v_{\theta 1}$$

We need to prove that

$$\overline{\langle (t, (\text{fix } x.t, \rho).\rho, \mathbf{C}'.\theta') \rangle} () \Downarrow - \Downarrow^{j'''} v_{\theta 2} \wedge \forall s.v_{\theta 1} \stackrel{s}{\approx}_{aV} v_{\theta 2} \wedge (j - j') = (j'' - j''') \quad (\text{ET-0})$$

From IH we know

$(\text{fix } x.t, \rho, \mathbf{C}'.\theta'')$ and $(t, (\text{fix } x.t, \rho).\rho, \mathbf{C}'.\theta'')$ are well-typed,

$$\overline{\langle (\text{fix } x.t, \rho, \mathbf{C}'.\theta'') \rangle} () \Downarrow - \Downarrow^{j''_1} v_{\theta 11} \implies$$

$$\overline{\langle (t, (\text{fix } x.t, \rho).\rho, \mathbf{C}'.\theta'') \rangle} () \Downarrow - \Downarrow^{j''_1} v_{\theta 22} \wedge \forall s.v_{\theta 11} \stackrel{s}{\approx}_{aV} v_{\theta 22} \wedge (j - j') = (j''_1 - j''_1)$$

(ET-IH)

From Definition 39 and Definition 40 we know that

$$\langle (\text{fix } x.t, \rho, \mathbf{C}'.\theta') \rangle = \langle (\text{fix } x.t, \rho) \rangle \langle \mathbf{C}' \rangle \dots \langle \mathbf{C}_{n-1} \rangle \langle \mathbf{C}_n \rangle \quad (\text{ET-1})$$

Since $(\text{fix } x.t, \rho, \mathbf{C}'.\theta')$ is well typed therefore we know that

$$\overline{\langle (\text{fix } x.t, \rho, \mathbf{C}'.\theta') \rangle} = \overline{\langle (\text{fix } x.t, \rho) \rangle \langle \mathbf{C}' \rangle \dots \langle \mathbf{C}_{n-1} \rangle \langle \mathbf{C}_n \rangle} =$$

$\lambda p.\text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t1} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c$

where

$$E_c = b (\text{coerce1 } !e_{t2} c) d$$

$$e_{t1} = \overline{\langle (\text{fix } x.t, \rho) \rangle \langle \mathbf{C}' \rangle \dots \langle \mathbf{C}_{n-1} \rangle}$$

$$e_{t2} = \overline{\langle \mathbf{C}_n \rangle} \quad (\text{ET-1.1})$$

Since we know that $\overline{\langle (\text{fix } x.t, \rho, \mathbf{C}'.\theta') \rangle} () \Downarrow - \Downarrow^{j''} v_{\theta 1}$ therefore we also know that

$$\exists j''_1, v'_1.e_{t1}() \Downarrow - \Downarrow^{j''_1} v_{\theta 11}$$

Also since we know that

$(\text{fix } x.t, \rho, \mathbf{C}'.\theta')$ and $(t, (\text{fix } x.t, \rho).\rho, \mathbf{C}'.\theta')$ are well-typed

therefore from Lemma 56 we also know that

$(\text{fix } x.t, \rho, \mathbf{C}'.\theta'')$ and $(t, (\text{fix } x.t, \rho).\rho, \mathbf{C}'.\theta'')$ are well-typed

Therefore from (ET-IH) we have

$$\exists v_{\theta 22}, j''_1. \overline{\langle (t, (\text{fix } x.t, \rho).\rho, \mathbf{C}'.\theta'') \rangle} \Downarrow - \Downarrow^{j''_1} v_{\theta 22} \wedge \forall s.v_{\theta 11} \stackrel{s}{\approx}_{aV} v_{\theta 22} \wedge (j - j') = (j''_1 - j''_1)$$

(ET-2)

From Definition 39 we know that

$$\overline{\langle (t, (\text{fix } x.t, \rho).\rho, \mathcal{C}'.\theta') \rangle} = \overline{\langle (t, (\text{fix } x.t, \rho).\rho) \rangle \langle \mathcal{C}' \rangle \dots \langle \mathcal{C}_{n-1} \rangle \langle \mathcal{C}_n \rangle}$$

Since $(t, (\text{fix } x.t, \rho).\rho, \mathcal{C}'.\theta')$ is well typed therefore we know that

$$\overline{\langle (t, (\text{fix } x.t, \rho).\rho, \mathcal{C}'.\theta') \rangle} =$$

$$\overline{\langle (t, (\text{fix } x.t, \rho).\rho) \rangle \langle \mathcal{C}' \rangle \dots \langle \mathcal{C}_{n-1} \rangle \langle \mathcal{C}_n \rangle} =$$

$\lambda p.\text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b' = e'_{t_1} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c$

where

$$E_c = b' (\text{coerce1 } !e'_{t_2} c) d$$

$$e'_{t_1} = \overline{\langle (t, (\text{fix } x.t, \rho).\rho) \rangle \langle \mathcal{C}' \rangle \dots \langle \mathcal{C}_{n-1} \rangle \langle \mathcal{C}_{n-1} \rangle}$$

$$e'_{t_2} = \overline{\langle \mathcal{C}_n \rangle}$$

Since from (ET-2) we know that $\overline{\langle (t, (\text{fix } x.t, \rho).\rho, \mathcal{C}'.\theta'') \rangle} \Downarrow - \Downarrow^{j''}_1 v_{\theta 22}$

Therefore it suffices to prove that

$$v_{\theta 22} (\text{coerce1 } !e'_{t_2} c) d \Downarrow - \Downarrow^{j''-j''}_1 v_{\theta 22} \text{ and } \forall s.v_{\theta 1} \overset{s}{\approx}_{aV} v_{\theta 2} \quad (\text{ET-p})$$

Since we are given that $\overline{\langle (\text{fix } x.t, \rho, \mathcal{C}'.\theta') \rangle}$ this means from (ET-1.1) we have

$$\lambda p.\text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t_1} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c \Downarrow - \Downarrow^{j''} v_{\theta 1}$$

where

$$E_c = b (\text{coerce1 } !e_{t_2} c) d$$

This means

$$1) e_{t_1} () \Downarrow - \Downarrow^{j''}_1 v_{\theta 11} \text{ and}$$

$$2) \text{ This means } v_{\theta 11} (\text{coerce1 } !e_{t_2} c) d \Downarrow - \Downarrow^y v_{\theta 1} \text{ for some } y \text{ s.t } y + j''_1 = j''$$

Since from (ET-2) we know that $\forall s.v_{\theta 11} \overset{s}{\approx}_{aV} v_{\theta 22}$ and since $e_{t_2} = e'_{t_2} = \overline{\langle \mathcal{C}_n \rangle}$ therefore from Definition 43 and Lemma 49 we have

$$v_{\theta 22} (\text{coerce1 } !e'_{t_2} c) d \Downarrow - \Downarrow^{j''-j''}_1 v_{\theta 22} \text{ and } \forall s.v_{\theta 1} \overset{s}{\approx}_{aV} v_{\theta 2}$$

This means

$$j'' - j''_1 = j'' - j''_1 =$$

$$j'' - j'' = j''_1 - j''_1 =$$

$$j'' - j'' = j - j' \text{ (From IH)}$$

□

Lemma 56. $\forall \mathcal{C}, \theta.$

$\theta.C$ is well-typed $\implies \theta$ is well-typed

Proof. Proof by induction on θ

1. Base case $\theta = \epsilon$:

Directly from the typing rule for ϵ

2. Case $\theta = \mathcal{C}'.\theta'$

This means we have $\mathcal{C}'.\theta'.\mathcal{C}$ is well-typed. This means from the stack typing rule for closure we know that $\theta'.\mathcal{C}$ is well-typed.

From IH we know that θ' is well-typed.

Since \mathcal{C}' is well typed and θ' is well-typed therefore $\mathcal{C}'.\theta'$ is well-typed.

□

Lemma 57 (Lemma for fix : empty stack). $\forall t, \rho, \theta$.

$$\begin{aligned} & \llbracket (\text{fix } x.t, \rho, \epsilon) \rrbracket \text{ is well-typed } \wedge \\ & \llbracket (t, (\text{fix } x.t, \rho). \rho, \epsilon) \rrbracket \text{ is well-typed } \wedge \\ & \frac{\llbracket (\text{fix } x.t, \rho, \epsilon) \rrbracket () \Downarrow - \Downarrow^j v_1 \implies}{\llbracket (t, (\text{fix } x.t, \rho). \rho, \epsilon) \rrbracket () \Downarrow - \Downarrow^j v_2 \wedge \forall s. v_1 \overset{s}{\approx}_{aV} v_2} \end{aligned}$$

Proof. Let $\rho = (\mathcal{C}_1, \dots, \mathcal{C}_n)$

Since we know that $\llbracket (\text{fix } x.t, (\mathcal{C}_1, \dots, \mathcal{C}_n), \epsilon) \rrbracket$ is well-typed and

$$\llbracket (\text{fix } x.t, (\mathcal{C}_1, \dots, \mathcal{C}_n), \epsilon) \rrbracket = ((\lambda x_1 \dots x_n. \text{fix } x.t) \llbracket \mathcal{C}_1 \rrbracket \dots \llbracket \mathcal{C}_n \rrbracket)$$

Therefore from Theorem 22 we know that

$$\frac{\llbracket (\text{fix } x.t, (\mathcal{C}_1, \dots, \mathcal{C}_n), \epsilon) \rrbracket =}{\llbracket (\lambda x_1 \dots x_n. \text{fix } x.t) \llbracket \mathcal{C}_1 \rrbracket \dots \llbracket \mathcal{C}_n \rrbracket \rrbracket =}$$

$\lambda p. \text{release} - = p$ in bind $a = \text{store}()$ in bind $b = e_{t1}$ a in bind $c = \text{store}!$ in bind $d = \text{store}()$ in E_c where

$$\begin{aligned} E_c &= b \text{ (coerce1 !} e_{t2} \text{ c) } d \\ e_{t1} &= \frac{\llbracket (\lambda x_1 \dots x_n. \text{fix } x.t) \llbracket \mathcal{C}_1 \rrbracket \dots \llbracket \mathcal{C}_{n-1} \rrbracket \rrbracket}{\llbracket \mathcal{C}_n \rrbracket} \\ e_{t2} &= \llbracket \mathcal{C}_n \rrbracket \quad (\text{F1}) \end{aligned}$$

Since we know that

$$\llbracket (\lambda x_1 \dots x_n. \text{fix } x.t) \llbracket \mathcal{C}_1 \rrbracket \dots \llbracket \mathcal{C}_n \rrbracket \rrbracket () \Downarrow - \Downarrow^j v_1$$

Therefore from E-release, E-store, E-bind, E-subExpE and E-app we know that

$$\bar{t}[\text{fix } x. \bar{t}[\llbracket \mathcal{C}_1 \rrbracket () / x_1] \dots [\llbracket \mathcal{C}_n \rrbracket () / x_n] () / x] \Downarrow - \Downarrow^j v_1 \quad (\text{F2})$$

Similarly since we know that $\llbracket (t, (\text{fix } x.t, (\mathcal{C}_1, \dots, \mathcal{C}_n)). (\mathcal{C}_1, \dots, \mathcal{C}_n), \epsilon) \rrbracket$ is well-typed and

$$\llbracket (t, (\text{fix } x.t, (\mathcal{C}_1, \dots, \mathcal{C}_n)). (\mathcal{C}_1, \dots, \mathcal{C}_n), \epsilon) \rrbracket = ((\lambda x, x_1 \dots x_n. t) \llbracket \text{fix } x.t, (\mathcal{C}_1, \dots, \mathcal{C}_n) \rrbracket \llbracket \mathcal{C}_1 \rrbracket \dots \llbracket \mathcal{C}_n \rrbracket)$$

Therefore from Theorem 22 we know that

$$\frac{\llbracket (t, (\text{fix } x.t, (\mathcal{C}_1, \dots, \mathcal{C}_n)). (\mathcal{C}_1, \dots, \mathcal{C}_n), \epsilon) \rrbracket =}{\llbracket (\lambda x, x_1 \dots x_n. t) \llbracket \text{fix } x.t, (\mathcal{C}_1, \dots, \mathcal{C}_n) \rrbracket \llbracket \mathcal{C}_1 \rrbracket \dots \llbracket \mathcal{C}_n \rrbracket \rrbracket =}$$

$\lambda p. \text{release} - = p$ in bind $a = \text{store}()$ in bind $b = e'_{t1}$ a in bind $c = \text{store}!$ in bind $d = \text{store}()$ in E_c where

$$\begin{aligned} E_c &= b \text{ (coerce1 !} e'_{t2} \text{ c) } d \\ e'_{t1} &= \frac{\llbracket (\lambda x, x_1 \dots x_n. t) \llbracket \text{fix } x.t, (\mathcal{C}_1, \dots, \mathcal{C}_n) \rrbracket \llbracket \mathcal{C}_1 \rrbracket \dots \llbracket \mathcal{C}_{n-1} \rrbracket \rrbracket}{\llbracket \mathcal{C}_n \rrbracket} \\ e'_{t2} &= \llbracket \mathcal{C}_n \rrbracket \quad (\text{F3}) \end{aligned}$$

We need to prove that

$$\llbracket (\lambda x, x_1 \dots x_n. t) \llbracket \text{fix } x.t, \rho \rrbracket \llbracket \mathcal{C}_1 \rrbracket \dots \llbracket \mathcal{C}_n \rrbracket \rrbracket \Downarrow - \Downarrow^j v_2$$

This means it suffices to prove that

$$\bar{t}[\text{fix } x. \bar{t}[\llbracket \mathcal{C}_1 \rrbracket () / x_1] \dots [\llbracket \mathcal{C}_n \rrbracket () / x_n] () / x] \Downarrow - \Downarrow^j v_2$$

We get this directly from (F2) and Lemma 48

□

Lemma 58 (Lemma for var : non-empty stack). $\forall t, \rho, \theta, j, j', j'', v_{\epsilon_1}, v_{\epsilon_2}, v_{\theta_1}$.
 $(x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \epsilon)$ and (t_x, ρ_x, ϵ) are well-typed
 $(x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \theta)$ and $(t, (\text{fix } x.t, \rho). \rho, \theta)$ are well-typed
 $\overline{\langle\langle x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \epsilon \rangle\rangle} () \Downarrow - \Downarrow^j v_{\epsilon_1} \wedge \overline{\langle\langle t_x, \rho_x, \epsilon \rangle\rangle} () \Downarrow - \Downarrow^{j'} v_{\epsilon_1} \wedge$
 $\forall s. v_{\epsilon_1} \stackrel{s}{\approx}_{aV} v_{\epsilon_2} \wedge$
 $\overline{\langle\langle x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \theta \rangle\rangle} () \Downarrow - \Downarrow^{j''} v_{\theta_1} \wedge$
 \implies
 $\exists v_{\theta_2}, j'''. \overline{\langle\langle t_x, \rho_x, \theta \rangle\rangle} () \Downarrow - \Downarrow^{j'''} v_{\theta_2} \wedge \forall s. v_{\theta_1} \stackrel{s}{\approx}_{aV} v_{\theta_2} \wedge (j - j') = (j'' - j''')$

Proof. We prove this by induction on θ

1. Case $\theta = \epsilon$:

Directly from given

2. Case $\theta = \mathbf{C}'.\theta'$:

Let $\theta' = \mathbf{C}'_1 \dots \mathbf{C}'_n$ and $\theta'' = \mathbf{C}'_1 \dots \mathbf{C}'_{n-1}$

Given:

$(x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathbf{C}'.\theta')$ and $(t_x, \rho_x, \mathbf{C}'.\theta')$ are well-typed \wedge
 $\overline{\langle\langle x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathbf{C}'.\theta' \rangle\rangle} () \Downarrow - \Downarrow^{j''} v_{\theta_1}$

We need to prove that

$$\overline{\langle\langle t_x, \rho_x, \mathbf{C}'.\theta' \rangle\rangle} () \Downarrow - \Downarrow^{j'''} v_{\theta_2} \wedge \forall s. v_{\theta_1} \stackrel{s}{\approx}_{aV} v_{\theta_2} \wedge (j - j') = (j'' - j''') \quad (\text{ET-0})$$

From IH we know

$$\begin{aligned} & (x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathbf{C}'.\theta'') \text{ and } (t_x, \rho_x, \mathbf{C}'.\theta'') \text{ are well-typed,} \\ & \overline{\langle\langle x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathbf{C}'.\theta'' \rangle\rangle} () \Downarrow - \Downarrow^{j''_1} v_{\theta_{11}} \implies \\ & \overline{\langle\langle t_x, \rho_x, \mathbf{C}'.\theta'' \rangle\rangle} () \Downarrow - \Downarrow^{j''_1} v_{\theta_{22}} \wedge \forall s. v_{\theta_{11}} \stackrel{s}{\approx}_{aV} v_{\theta_{22}} \wedge (j - j') = (j''_1 - j''_1) \quad (\text{ET-IH}) \end{aligned}$$

From Definition 39 and Definition 40 we know that

$$\overline{\langle\langle x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathbf{C}'.\theta' \rangle\rangle} = \overline{\langle\langle x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n) \rangle\rangle} \langle\langle \mathbf{C}' \rangle\rangle \dots \langle\langle \mathbf{C}_{n-1} \rangle\rangle \langle\langle \mathbf{C}_n \rangle\rangle \quad (\text{ET-1})$$

Since $(x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathbf{C}'.\theta')$ is well typed therefore we know that

$$\overline{\langle\langle x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathbf{C}'.\theta' \rangle\rangle} = \overline{\langle\langle x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n) \rangle\rangle} \langle\langle \mathbf{C}' \rangle\rangle \dots \langle\langle \mathbf{C}_{n-1} \rangle\rangle \langle\langle \mathbf{C}_n \rangle\rangle$$

$\lambda p. \text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t_1} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c$
where

$$E_c = b \text{ (coerce1 !} e_{t_2} c) d$$

$$e_{t_1} = \overline{\langle\langle x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n) \rangle\rangle} \langle\langle \mathbf{C}' \rangle\rangle \dots \langle\langle \mathbf{C}_{n-1} \rangle\rangle$$

$$e_{t_2} = \overline{\langle\langle \mathbf{C}_n \rangle\rangle} \quad (\text{ET-1.1})$$

Since we know that $\overline{\langle\langle x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathbf{C}'.\theta' \rangle\rangle} () \Downarrow - \Downarrow^{j''} v_{\theta_1}$ therefore we also know that

$$\exists j_1'', v_1'. e_{t1}() \Downarrow - \Downarrow^{j_1''} v_1'$$

Also since we know that

$(x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathbf{C}'.\theta')$ and $(t, (\text{fix } x.t, \rho).\rho, \mathbf{C}'.\theta')$ are well-typed therefore from Lemma 56 we also know that $(x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathbf{C}'.\theta'')$ and $(t_x, \rho_x, \mathbf{C}'.\theta'')$ are well-typed

Therefore from (ET-IH) we have

$$\exists v_{\theta 22}, j_1'''. \overline{\langle (t_x, \rho_x, \mathbf{C}'.\theta'') \rangle} \Downarrow - \Downarrow^{j_1'''} v_{\theta 22} \wedge \forall s.v_{\theta 11} \stackrel{s}{\approx}_{aV} v_{\theta 22} \wedge (j - j') = (j_1'' - j_1''')$$

(ET-2)

From Definition 39 we know that

$$\overline{\langle (t_x, \rho_x, \mathbf{C}'.\theta') \rangle} = \overline{\langle (t_x, \rho_x) \langle \mathbf{C}' \rangle \dots \langle \mathbf{C}_{n-1} \rangle \langle \mathbf{C}_n \rangle \rangle}$$

Since $(t_x, \rho_x, \mathbf{C}'.\theta')$ is well typed therefore we know that

$$\overline{\langle (t_x, \rho_x, \mathbf{C}'.\theta') \rangle} = \overline{\langle (t_x, \rho_x) \langle \mathbf{C}' \rangle \dots \langle \mathbf{C}_{n-1} \rangle \langle \mathbf{C}_n \rangle \rangle} =$$

$\lambda p.\text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b' = e'_{t1} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c$
where

$$\begin{aligned} E_c &= b' (\text{coerce1 } !e'_{t2} c) d \\ e'_{t1} &= \overline{\langle (t_x, \rho_x) \langle \mathbf{C}' \rangle \dots \langle \mathbf{C}_{n-1} \rangle \langle \mathbf{C}_{n-1} \rangle \rangle} \\ e'_{t2} &= \overline{\langle \mathbf{C}_n \rangle} \end{aligned}$$

Since from (ET-2) we know that $\overline{\langle (t_x, \rho_x, \mathbf{C}'.\theta'') \rangle} \Downarrow - \Downarrow^{j_1'''} v_{\theta 22}$

Therefore it suffices to prove that

$$v_{\theta 22} (\text{coerce1 } !e'_{t2} c) d \Downarrow - \Downarrow^{j_1''' - j_1''} v_{\theta 2} \text{ and } \forall s.v_{\theta 1} \stackrel{s}{\approx}_{aV} v_{\theta 2} \quad (\text{ET-p})$$

Since we are given that $\langle (x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathbf{C}'.\theta') \rangle \Downarrow - \Downarrow^{j''} v_{\theta 1}$ this means from (ET-1.1) we have

$$\lambda p.\text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t1} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c \Downarrow - \Downarrow^{j''} v_{\theta 1}$$

where

$$E_c = b (\text{coerce1 } !e_{t2} c) d$$

This means

- 1) $\overline{\langle (x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \mathbf{C}'.\theta'') \rangle} () \Downarrow - \Downarrow^{j_1''} v_{\theta 11}$ and
- 2) This means $v_{\theta 11} (\text{coerce1 } !e_{t2} c) d \Downarrow - \Downarrow^y v_{\theta 1}$ for some y s.t $y + j_1'' = j''$

Since from (ET-2) we have $\forall s.v_{\theta 11} \stackrel{s}{\approx}_{aV} v_{\theta 22} \wedge$ and since $e_{t2} = e'_{t2} = \overline{\langle \mathbf{C}_n \rangle}$ therefore from Definition 43 and Lemma 49 we have

$v_{\theta 22} (\text{coerce1 } !e'_{t_2} c) d \Downarrow - \Downarrow^{j''-j'_1} v_{\theta 2}$ and $\forall s.v_{\theta 1} \stackrel{s}{\approx}_{aV} v_{\theta 2}$

This means

$$\begin{aligned} j'' - j'_1 &= j''' - j'''_1 = \\ j'' - j''' &= j'_1 - j'''_1 = \\ j'' - j''' &= j - j' \text{ (From IH)} \end{aligned}$$

□

Lemma 59 (Lemma for var : empty stack). $\forall t, \rho, \theta$.

$$\begin{array}{l} \Theta; \Delta; \cdot \vdash - \llbracket (x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \epsilon) \rrbracket : - \wedge \\ \Theta; \Delta; \cdot \vdash - \llbracket (t_x, \rho_x, \epsilon) \rrbracket : - \wedge \\ \hline \llbracket (x, (t_0, \rho_0) \dots (t_x, \rho_x) \dots (t_n, \rho_n), \epsilon) \rrbracket () \Downarrow - \Downarrow^j v \implies \\ \llbracket (t_x, \rho_x, \epsilon) \rrbracket () \Downarrow - \Downarrow^{j-1} v \end{array}$$

Proof. From Definition 40 we also have

$$\begin{aligned} &\llbracket (x, (t_0, \rho_0), \dots (t_x, \rho_x), \dots (t_n, \rho_n), \epsilon) \rrbracket \\ &= \llbracket (x, (t_0, \rho_0), \dots (t_x, \rho_x), \dots (t_n, \rho_n)) \rrbracket \\ &= (\lambda x_1 \dots x \dots x_n.x) \llbracket (t_0, \rho_0) \rrbracket \dots \llbracket (t_n, \rho_n) \rrbracket \end{aligned}$$

Similarly from Definition 40 we also have

$$\llbracket (t_x, \rho_x, \epsilon) \rrbracket = \llbracket (t_x, \rho_x) \rrbracket \quad (\text{S-V1})$$

Therefore from Theorem 22 we know that

$$\begin{aligned} &\llbracket (x, ((t_1, \rho_1), \dots (t_x, \rho_x) \dots (t_n, \rho_n)), \epsilon) \rrbracket = \\ &\frac{\llbracket (\lambda x_1 \dots x \dots x_n.x) \llbracket (t_1, \rho_1) \rrbracket \dots \llbracket (t_x, \rho_x) \rrbracket \dots \llbracket (t_n, \rho_n) \rrbracket \rrbracket}{\lambda p.\text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t_1, n} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c} \\ &\text{where} \end{aligned}$$

$$\begin{aligned} E_c &= b (\text{coerce1 } !e_{t_2, n} c) d \\ e_{t_1, n} &= \frac{\llbracket (\lambda x_1 \dots x \dots x_n.x) \llbracket (t_1, \rho_1) \rrbracket \dots \llbracket (t_x, \rho_x) \rrbracket \dots \llbracket (t_{n-1}, \rho_{n-1}) \rrbracket \rrbracket}{\llbracket (t_n, \rho_n) \rrbracket} \quad (\text{V4}) \\ e_{t_2, n} &= \llbracket (t_n, \rho_n) \rrbracket \end{aligned}$$

Simialrly

$$\begin{aligned} e_{t_1, n} &= \\ \lambda p.\text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t_1, n-1} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c \\ &\text{where} \end{aligned}$$

$$\begin{aligned} E_c &= b (\text{coerce1 } !e_{t_2, n-1} c) d \\ e_{t_1, n-1} &= \frac{\llbracket (\lambda x_1 \dots x \dots x_n.x) \llbracket (t_1, \rho_1) \rrbracket \dots \llbracket (t_x, \rho_x) \rrbracket \dots \llbracket (t_{n-2}, \rho_{n-2}) \rrbracket \rrbracket}{\llbracket (t_{n-1}, \rho_{n-1}) \rrbracket} \\ e_{t_2, n-1} &= \llbracket (t_{n-1}, \rho_{n-1}) \rrbracket \end{aligned}$$

In the same way we have

$$\begin{aligned} e_{t_1, 1} &= \\ \lambda p.\text{release } - = p \text{ in bind } a = \text{store}() \text{ in bind } b = e_{t_1, 1} a \text{ in bind } c = \text{store}!() \text{ in bind } d = \text{store}() \text{ in } E_c \\ &\text{where} \end{aligned}$$

$$\begin{aligned} E_c &= b (\text{coerce1 } !e_{t_2, 1} c) d \\ e_{t_1, 1} &= \frac{\llbracket (\lambda x_1 \dots x \dots x_n.x) \rrbracket}{\llbracket (t_1, \rho_1) \rrbracket} \\ e_{t_2, 1} &= \llbracket (t_1, \rho_1) \rrbracket \end{aligned}$$

Simialrly we also get

$$e_{t_1, 1} =$$

$\lambda p_1. \text{ret } \lambda y. \lambda p_2. \text{let } !x = y \text{ in release } - = p_1 \text{ in release } - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_{l,1} a$
 where

$$e_{l,1} = \overline{((\lambda x_2 \dots x \dots x_n. x))}$$

and

$$e_{l,n} =$$

$\lambda p_1. \text{ret } \lambda y. \lambda p_2. \text{let } !x = y \text{ in release } - = p_1 \text{ in release } - = p_2 \text{ in bind } a = \text{store}() \text{ in } e_{l,n} a$
 where

$$e_{l,n} = \bar{x} = \lambda p. \text{release } - = p \text{ in bind } - = \uparrow^1 \text{ in } x$$

Since we know that

$$\overline{((\lambda x_1 \dots x \dots x_n. x) \llbracket (t_0, \rho_0) \rrbracket \dots \llbracket (t_n, \rho_n) \rrbracket \rrbracket)} () \Downarrow - \Downarrow^j v$$

this means from E-release, E-bind, E-store, E-app that

$$(\text{bind } - = \uparrow^1 \text{ in } \overline{\llbracket (t_x, \rho_x) \rrbracket}) () \Downarrow - \Downarrow^j v$$

Therefore from E-bind, E-step and E-app we know that $\overline{\llbracket (t_x, \rho_x) \rrbracket} () \Downarrow - \Downarrow^{j-1} v$ \square

Theorem 60 (Rederiving dlPCF's soundness). $\forall t, I, \tau, \rho.$

$$\vdash_I (t, \epsilon, \epsilon) : \tau \wedge (t, \epsilon, \epsilon) \xrightarrow{n} (v, \rho, \epsilon) \implies n \leq |t| * (I + 1)$$

Proof. Let us rename t to t_1 and v to t_{n+1} then we know that

$$(t_1, \epsilon, \epsilon) \rightarrow (t_2, \rho_2, \theta_2) \dots (t_n, \rho_n, \theta_n) \rightarrow (t_{n+1}, \rho, \epsilon)$$

Since we are given that (t, ϵ, ϵ) is well-typed therefore from dlPCF's subject reduction we know that (t_2, ρ_2, θ_2) to (t_n, ρ_n, θ_n) and $(t_{n+1}, \rho, \epsilon)$ are all well-typed.

From Theorem 63 we know that $\forall 1 \leq i \leq n. \llbracket (t_i, \rho_i, \theta_i) \rrbracket \xrightarrow{*} -$

Also from Theorem 42 we know that $\forall 1 \leq i \leq n. \llbracket (t_i, \rho_i, \theta_i) \rrbracket$ is well typed

So now we can apply Theorem 36 and from Definition 34 to get

$$\forall 1 \leq i \leq n + 1. \exists j_i. \overline{\llbracket (t_i, \rho_i, \theta_i) \rrbracket} () \Downarrow - \Downarrow^{j_i} -$$

Next we apply Theorem 52 for every step of the reduction starting from $(t_1, \epsilon, \epsilon)$ and we know that either the cost reduces by 1 and the size increases by $|t|$ or cost remains the same and the size reduces.

Thus we know that size can vary from t to 1 and cost can vary from j_1 to 0. Therefore, the number of reduction steps are bounded by $|t| * (j_1 + 1)$

From Theorem 20 we know that $j_1 < I$ therefore we have $n \leq |t| * (I + 1)$ \square

1.5.4 Cross-language model: Krivine to dlPCF

Definition 61 (Cross language logical reation: Krivine to dlPCF).

$$(v_k, \rho, \epsilon) \sim_v v_d \triangleq v_d = v_k \rho$$

$$(e_k, \rho, \theta) \sim_e e_d \triangleq \forall v_k, \rho'. (e_k, \rho, \theta) \xrightarrow{*} (v_k, \rho', \epsilon) \implies \exists v_d. e_d \xrightarrow{*} v_d \wedge (v_k, \rho', \epsilon) \sim_v v_d$$

Lemma 62. $\forall e_k, \rho, \theta, e'_k, \rho', \theta'.$

$$(e_k, \rho, \theta) \xrightarrow{*} (e'_k, \rho', \theta') \implies \exists e'_d. \llbracket (e_k, \rho, \theta) \rrbracket \xrightarrow{*} e'_d \wedge e'_d = \llbracket (e'_k, \rho', \theta') \rrbracket$$

Proof. Given: $(e_k, \rho, \theta) \xrightarrow{*} (e'_k, \rho', \theta')$

To prove: $\exists e'_d. \llbracket (e_k, \rho, \theta) \rrbracket \xrightarrow{*} e'_d \wedge e'_d = \llbracket (e'_k, \rho', \theta') \rrbracket$

Lets assume it takes n steps for $(e_k, \rho, \theta) \xrightarrow{n} (e'_k, \rho', \theta')$

We induct on n

Base case ($n = 1$)

1. App1:

In this case we are given $(t u, \rho, \theta) \rightarrow (t, \rho, (u, \rho).\theta)$

Let $\rho = \mathbf{C}_{\rho_1} \dots \mathbf{C}_{\rho_n}$ and $\theta = \mathbf{C}_{\theta_1} \dots \mathbf{C}_{\theta_m}$

From Definition 40 we know that

$$\begin{aligned} \llbracket (e_k, \rho, \theta) \rrbracket &= \\ (\lambda x_1 \dots x_n. t u) \llbracket \mathbf{C}_{\rho_1} \rrbracket \dots \llbracket \mathbf{C}_{\rho_n} \rrbracket \llbracket \mathbf{C}_{\theta_1} \rrbracket \dots \llbracket \mathbf{C}_{\theta_m} \rrbracket \end{aligned}$$

From dlPCF's app rule we know that

$$\begin{aligned} (\lambda x_1 \dots x_n. t u) \mathbf{C}_{\rho_1} \dots \mathbf{C}_{\rho_n} \mathbf{C}_{\theta_1} \dots \mathbf{C}_{\theta_m} &\xrightarrow{*} \\ t[\llbracket \mathbf{C}_{\rho_1} \rrbracket / x_1] \dots [\llbracket \mathbf{C}_{\rho_n} \rrbracket / x_n] u[\llbracket \mathbf{C}_{\rho_1} \rrbracket / x_1] \dots [\llbracket \mathbf{C}_{\rho_n} \rrbracket / x_n] \llbracket \mathbf{C}_{\theta_1} \rrbracket \dots \llbracket \mathbf{C}_{\theta_m} \rrbracket \end{aligned}$$

We choose e'_d as $t[\llbracket \mathbf{C}_{\rho_1} \rrbracket / x_1] \dots [\llbracket \mathbf{C}_{\rho_n} \rrbracket / x_n] u[\llbracket \mathbf{C}_{\rho_1} \rrbracket / x_1] \dots [\llbracket \mathbf{C}_{\rho_n} \rrbracket / x_n] \llbracket \mathbf{C}_{\theta_1} \rrbracket \dots \llbracket \mathbf{C}_{\theta_m} \rrbracket$ and we get the desired from Definition 40

2. App2:

In this case we are given $(\lambda x. t, \rho, \mathbf{C}.\theta) \rightarrow (t, \mathbf{C}.\rho, \theta)$

Let $\rho = \mathbf{C}_{\rho_1} \dots \mathbf{C}_{\rho_n}$ and $\theta = \mathbf{C}_{\theta_1} \dots \mathbf{C}_{\theta_m}$

From Definition 40 we know that

$$\begin{aligned} \llbracket (\lambda x. t, \rho, \mathbf{C}.\theta) \rrbracket &= \\ (\lambda x_1 \dots x_n. \lambda x. t) \llbracket \mathbf{C}_{\rho_1} \rrbracket \dots \llbracket \mathbf{C}_{\rho_n} \rrbracket \llbracket \mathbf{C} \rrbracket \llbracket \mathbf{C}_{\theta_1} \rrbracket \dots \llbracket \mathbf{C}_{\theta_m} \rrbracket \end{aligned}$$

From dlPCF's app rule we know that

$$\begin{aligned} (\lambda x_1 \dots x_n. \lambda x. t) \llbracket \mathbf{C}_{\rho_1} \rrbracket \dots \llbracket \mathbf{C}_{\rho_n} \rrbracket \llbracket \mathbf{C} \rrbracket \llbracket \mathbf{C}_{\theta_1} \rrbracket \dots \llbracket \mathbf{C}_{\theta_m} \rrbracket &\xrightarrow{*} \\ t[\llbracket \mathbf{C}_{\rho_1} \rrbracket / x_1] \dots [\llbracket \mathbf{C}_{\rho_n} \rrbracket / x_n] [\llbracket \mathbf{C} \rrbracket / x] \mathbf{C}_{\theta_1} \dots \mathbf{C}_{\theta_m} \end{aligned}$$

We choose e'_d as $t[\llbracket \mathbf{C}_{\rho_1} \rrbracket / x_1] \dots [\llbracket \mathbf{C}_{\rho_n} \rrbracket / x_n] [\llbracket \mathbf{C} \rrbracket / x] \mathbf{C}_{\theta_1} \dots \mathbf{C}_{\theta_m}$ and we get the desired from Definition 40

3. Var:

In this case we are given $(x, (t_0, \rho_0) \dots (t_n, \rho_n), \theta) \rightarrow (t_x, \rho_x, \theta)$

Let $\theta = \mathbf{C}_{\theta_1} \dots \mathbf{C}_{\theta_m}$

From Definition 40 we know that

$$\begin{aligned} \llbracket (x, (t_0, \rho_0) \dots (t_n, \rho_n), \theta) \rrbracket &= \\ (\lambda x_1 \dots x_n. \lambda x. t) \llbracket \mathbf{C}_{\rho_1} \rrbracket \dots \llbracket \mathbf{C}_{\rho_n} \rrbracket \llbracket \mathbf{C}_{\theta_1} \rrbracket \dots \llbracket \mathbf{C}_{\theta_m} \rrbracket \end{aligned}$$

From dlPCF's app rule we know that

$$(\lambda x_1 \dots x_n. x) \langle (t_0, \rho_0) \rangle \dots \langle (t_n, \rho_n) \rangle \langle \mathbf{C}_{\theta_1} \rangle \dots \langle \mathbf{C}_{\theta_m} \rangle \xrightarrow{*} \langle (t_x, \rho_x) \rangle \mathbf{C}_{\theta_1} \dots \mathbf{C}_{\theta_m}$$

Let $\rho_x = \mathbf{C}_{x_1} \dots \mathbf{C}_{x_k}$ therefore from Definition 40 we know that

$$\langle (t_x, \rho_x) \rangle \mathbf{C}_{\theta_1} \dots \mathbf{C}_{\theta_m} = \lambda x_{x_1} \dots x_{x_k}. t_x \langle \mathbf{C}_{x_1} \rangle \dots \langle \mathbf{C}_{x_k} \rangle \mathbf{C}_{\theta_1} \dots \mathbf{C}_{\theta_m}$$

Therefore from dlPCF's app rule we know that

$$\langle (t_x, \rho_x) \rangle \mathbf{C}_{\theta_1} \dots \mathbf{C}_{\theta_m} \xrightarrow{*} t_x [\langle \mathbf{C}_{x_1} \rangle / x_1] \dots [\langle \mathbf{C}_{x_k} \rangle / x_k] \mathbf{C}_{\theta_1} \dots \mathbf{C}_{\theta_m}$$

We choose e'_d as $t_x [\langle \mathbf{C}_{x_1} \rangle / x_1] \dots [\langle \mathbf{C}_{x_k} \rangle / x_k] \mathbf{C}_{\theta_1} \dots \mathbf{C}_{\theta_m}$ and we get the desired from Definition 40

4. Fix:

In this case we are given $(\text{fix } x.t, \rho, \theta) \rightarrow (t, (\text{fix } x.t, \rho). \rho, \theta)$

Let $\rho = \mathbf{C}_{\rho_1} \dots \mathbf{C}_{\rho_n}$ and $\theta = \mathbf{C}_{\theta_1} \dots \mathbf{C}_{\theta_m}$

From Definition 40 we know that

$$\langle (\text{fix } x.t, \rho, \theta) \rangle = (\lambda x_1 \dots x_n. \text{fix } x.t) \langle \mathbf{C}_{\rho_1} \rangle \dots \langle \mathbf{C}_{\rho_n} \rangle \langle (\text{fix } x.t, \rho) \rangle \langle \mathbf{C}_{\theta_1} \rangle \dots \langle \mathbf{C}_{\theta_m} \rangle$$

From dlPCF's app and fix rule we know that

$$\begin{aligned} & (\lambda x_1 \dots x_n. \text{fix } x.t) \langle \mathbf{C}_{\rho_1} \rangle \dots \langle \mathbf{C}_{\rho_n} \rangle \langle \mathbf{C} \rangle \langle \mathbf{C}_{\theta_1} \rangle \dots \langle \mathbf{C}_{\theta_m} \rangle \xrightarrow{*} \\ & \text{fix } x.t [\langle \mathbf{C}_{\rho_1} \rangle / x_1] \dots [\langle \mathbf{C}_{\rho_n} \rangle / x_n] [\langle (\text{fix } x.t, \rho) \rangle / x] \mathbf{C}_{\theta_1} \dots \mathbf{C}_{\theta_m} \rightarrow \\ & t [\langle \mathbf{C}_{\rho_1} \rangle / x_1] \dots [\langle \mathbf{C}_{\rho_n} \rangle / x_n] [\langle (\text{fix } x.t, \rho) \rangle / x] \mathbf{C}_{\theta_1} \dots \mathbf{C}_{\theta_m} \end{aligned}$$

We choose e'_d as $t [\langle \mathbf{C}_{\rho_1} \rangle / x_1] \dots [\langle \mathbf{C}_{\rho_n} \rangle / x_n] [\langle (\text{fix } x.t, \rho) \rangle / x] \mathbf{C}_{\theta_1} \dots \mathbf{C}_{\theta_m}$ and we get the desired from Definition 40

Inductive case

We get this directly from IH and the base case

□

Theorem 63 (Fundamental theorem). $\forall e_k, \rho, \theta. (e_k, \rho, \theta) \sim_e \langle (e_k, \rho, \theta) \rangle$

Proof. From Definition 61 it suffices to prove that

$$\forall v_k, \rho'. (e_k, \rho, \theta) \xrightarrow{*} (v_k, \rho', \epsilon) \implies \exists v_d. e_d \xrightarrow{*} v_d \wedge (v_k, \rho', \epsilon) \sim_v v_d$$

This means that given some v_k, ρ' s.t. $(e_k, \rho, \theta) \xrightarrow{*} (v_k, \rho', \epsilon)$ it suffices to prove that

$$\exists v_d. e_d \xrightarrow{*} v_d \wedge (v_k, \rho', \epsilon) \sim_v v_d$$

From Lemma 62 we know that

$$\exists e'_d. \langle (e_k, \rho, \theta) \rangle \xrightarrow{*} e'_d \wedge e'_d = \langle (v_k, \rho', \epsilon) \rangle$$

Let $\rho' = \mathbf{C}_1 \dots \mathbf{C}_n$ therefore from Definition 40 we know that

$$\langle (v_k, \rho', \epsilon) \rangle = (\lambda x_1 \dots x_n. v_k) \langle \mathbf{C}_1 \rangle \dots \langle \mathbf{C}_n \rangle$$

Therefore from dlPCF's app rule we know that

$$\langle (v_k, \rho', \epsilon) \rangle \xrightarrow{*} v_k [\langle \mathbf{C}_1 \rangle / x_1] \dots [\langle \mathbf{C}_n \rangle / x_n]$$

We choose v_d as $v_k [\langle \mathbf{C}_1 \rangle / x_1] \dots [\langle \mathbf{C}_n \rangle / x_n]$ and we get the desired from Definition 61

□

2 Development for univariate RAML's embedding

2.1 Syntax

Expressions	$e ::= v \mid e_1 e_2 \mid \langle\langle e_1, e_2 \rangle\rangle \mid \text{let}\langle\langle x, y \rangle\rangle = e_1 \text{ in } e_2 \mid \langle e, e \rangle \mid \text{fst}(e) \mid \text{snd}(e) \mid \text{inl}(e) \mid \text{inr}(e) \mid \text{case } e \text{ of } e; e \mid \text{let! } x = e_1 \text{ in } e_2 \mid e :: e \mid e \square \mid e; x.e$
Values	$v ::= x \mid () \mid c \mid \lambda x.e \mid \langle\langle v_1, v_2 \rangle\rangle \mid \langle v, v \rangle \mid \text{inl}(e) \mid \text{inr}(e) \mid !e \mid \text{nil} \mid \Lambda.e \mid \text{ret } e \mid \text{bind } x = e_1 \text{ in } e_2 \mid \uparrow^I \mid \text{release } x = e_1 \text{ in } e_2 \mid \text{store } e$ (No value forms for $[I] \tau$)
Index	$I ::= i \mid N \mid R \mid I + I \mid I - I \mid \sum_{i < I} I \mid \lambda_s i : S . I \mid I I$
Sort	$S ::= \mathbb{N} \mid \mathbb{R}^+ \mid S \rightarrow S$
Kind	$K ::= \text{Type} \mid S \rightarrow K$
Types	$\tau ::= \mathbf{1} \mid \mathbf{b} \mid \tau_1 \multimap \tau_2 \mid \tau_1 \otimes \tau_2 \mid \tau_1 \& \tau_2 \mid \tau_1 \oplus \tau_2 \mid !\tau \mid [I] \tau \mid \mathbb{M} I \tau \mid L^I \tau$ $\alpha \mid \forall \alpha : K . \tau \mid \forall i : S . \tau \mid \lambda_t i : S . \tau \mid \tau I \mid \exists i : S . \tau \mid c \Rightarrow \tau \mid c \& \tau$
Constraints	$c ::= I = I \mid I < I \mid c \wedge c$
Lin. context for term variables	$\Gamma ::= . \mid \Gamma, x : \tau$
Unres. context for term variables	$\Omega ::= . \mid \Omega, x : \tau$
Unres. context for index variables	$\Theta ::= . \mid \Theta, i : S$
Unres. context for type variables	$\Psi ::= . \mid \Psi, \alpha : K$

Definition 64 (Binary sum of multiplicity context).

$$\Omega_1 \oplus \Omega_2 \triangleq \begin{cases} \Omega_2 & \Omega_1 = . \\ (\Omega'_1 \oplus \Omega_2), x : \tau & \Omega_1 = \Omega'_1, x : \tau \wedge (x : -) \notin \Omega_2 \\ \text{undefined} & \Omega_1 = \Omega'_1, x : \tau \wedge (x : \tau) \in \Omega_2 \end{cases}$$

Definition 65 (Binary sum of affine context).

$$\Gamma_1 \oplus \Gamma_2 \triangleq \begin{cases} \Gamma_2 & \Gamma_1 = . \\ (\Gamma'_1 \oplus \Gamma_2), x : \tau & \Gamma_1 = \Gamma'_1, x : \tau \wedge (x : -) \notin \Gamma_2 \\ \text{undefined} & \Gamma_1 = \Gamma'_1, x : \tau \wedge (x : -) \in \Gamma_2 \end{cases}$$

2.2 Typesystem

Typing $\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau$

$$\begin{array}{c}
\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma, x : \tau \vdash x : \tau} \text{T-var1} \qquad \frac{}{\Psi; \Theta; \Delta; \Omega, x : \tau; \Gamma \vdash x : \tau} \text{T-var2} \\
\\
\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash () : \mathbf{1}} \text{T-unit} \qquad \frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash c : \mathbf{b}} \text{T-base} \qquad \frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{nil} : L^0 \tau} \text{T-nil} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e_1 : \tau \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2 \vdash e_2 : L^n \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash e_1 :: e_2 : L^{n+1} \tau} \text{T-cons} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : L^n \tau \quad \Psi; \Theta; \Delta, n > 0; \Omega; \Gamma_2, h : \tau, t : L^{n-1} \tau \vdash e_2 : \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \text{match } e \text{ with } | \text{nil} \mapsto e_1 \mid h :: t \mapsto e_2 : \tau'} \text{T-match} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau[n/s] \quad \Theta; \Delta \vdash n : S}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \exists s : S . \tau} \text{T-existI} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : \exists s . \tau \quad \Psi; \Theta, s; \Delta; \Omega; \Gamma_2, x : \tau \vdash e' : \tau' \quad \Psi; \Theta; \Delta \vdash \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash e; x.e' : \tau'} \text{T-existE} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma, x : \tau_1 \vdash e : \tau_2}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \lambda x. e : (\tau_1 \multimap \tau_2)} \text{T-lam} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e_1 : (\tau_1 \multimap \tau_2) \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2 \vdash e_2 : \tau_1}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash e_1 e_2 : \tau_2} \text{T-app} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \quad \Psi; \Theta; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau'} \text{T-sub} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \quad \Psi; \Theta; \Delta \models \Gamma' \sqsubseteq \Gamma \quad \Psi; \Theta; \Delta \models \Omega' \sqsubseteq \Omega}{\Psi; \Theta; \Delta; \Omega'; \Gamma' \vdash e : \tau} \text{T-weaken} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e_1 : \tau_1 \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2 \vdash e_2 : \tau_1}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \langle\langle e_1, e_2 \rangle\rangle : (\tau_1 \otimes \tau_2)} \text{T-tensorI} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : (\tau_1 \otimes \tau_2) \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : \tau_1, y : \tau_2 \vdash e' : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \text{let}\langle\langle x, y \rangle\rangle = e \text{ in } e' : \tau} \text{T-tensorE} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e_1 : \tau_1 \quad \Psi; \Theta; \Delta; \Omega; \Gamma \vdash e_2 : \tau_1}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \langle e_1, e_2 \rangle : (\tau_1 \& \tau_2)} \text{T-withI}
\end{array}$$

$$\begin{array}{c}
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\tau_1 \& \tau_2)}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{fst}(e) : \tau_1} \text{T-fst} \qquad \frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\tau_1 \& \tau_2)}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{snd}(e) : \tau_2} \text{T-snd} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau_1}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{inl}(e) : \tau_1 \oplus \tau_2} \text{T-inl} \qquad \frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau_2}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{inr}(e) : \tau_1 \oplus \tau_2} \text{T-inr} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : (\tau_1 \oplus \tau_2) \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : \tau_1 \vdash e_1 : \tau \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2, y : \tau_2 \vdash e_2 : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \text{case } e \text{ of } e_1; e_2 : \tau} \text{T-case} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \cdot \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \cdot \vdash !e : !\tau} \text{T-ExpI} \qquad \frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : !\tau \quad \Psi; \Theta; \Delta; \Omega, x : \tau; \Gamma_2 \vdash e' : \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \text{let } !x = e \text{ in } e' : \tau'} \text{T-ExpE} \\
\\
\frac{\Psi, \alpha : K; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \Lambda.e : (\forall \alpha : K . \tau)} \text{T-tabs} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\forall \alpha : K . \tau) \quad \Psi; \Theta; \Delta \vdash \tau' : K}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e [] : (\tau[\tau'/\alpha])} \text{T-tapp} \\
\\
\frac{\Psi; \Theta, i : S; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \Lambda.e : (\forall i : S . \tau)} \text{T-iabs} \qquad \frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\forall i : S . \tau) \quad \Theta; \Delta \vdash I : S}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e [] : (\tau[I/i])} \text{T-iapp} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega, x : \tau; \cdot \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \cdot \vdash \text{fix } x.e : \tau} \text{T-fix} \qquad \frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{ret } e : \mathbb{M} 0 \tau} \text{T-ret} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e_1 : \mathbb{M} n_1 \tau_1 \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : \tau_1 \vdash e_2 : \mathbb{M} n_2 \tau_2 \quad \Theta; \Delta \vdash I_1 : \mathbb{R}^+ \quad \Theta; \Delta \vdash I_2 : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \text{bind } x = e_1 \text{ in } e_2 : \mathbb{M}(I_1 + I_2) \tau_2} \text{T-bind} \\
\\
\frac{\Theta; \Delta \vdash I : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \uparrow^I : \mathbb{M} I \mathbf{1}} \text{T-tick} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e_1 : [I_1] \tau_1 \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : \tau_1 \vdash e_2 : \mathbb{M}(I_1 + I_2) \tau_2 \quad \Theta; \Delta \vdash I_1 : \mathbb{R}^+ \quad \Theta; \Delta \vdash I_2 : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \text{release } x = e_1 \text{ in } e_2 : \mathbb{M} I_2 \tau_2} \text{T-release} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \quad \Theta; \Delta \vdash I : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{store } e : \mathbb{M} I ([I] \tau)} \text{T-store} \qquad \frac{\Psi; \Theta; \Delta, c; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \Lambda. e : (c \Rightarrow \tau)} \text{T-CI} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (c \Rightarrow \tau) \quad \Theta; \Delta \Vdash c}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e [] : \tau} \text{T-CE} \qquad \frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \quad \Theta; \Delta \Vdash c}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (c \& \tau)} \text{T-CAndI} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (c \& \tau) \quad \Psi; \Theta; \Delta, c; \Omega; \Gamma, x : \tau \vdash e' : \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{clet } x = e \text{ in } e' : \tau'} \text{T-CAndE}
\end{array}$$

Figure 9: Typing rules for λ -Amor

$$\begin{array}{c}
\frac{}{\Psi; \Theta; \Delta \vdash \tau <: \tau} \text{sub-refl} \qquad \frac{\Psi; \Theta; \Delta \vdash \tau'_1 <: \tau_1 \quad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau'_2}{\Psi; \Theta; \Delta \vdash \tau_1 \multimap \tau_2 <: \tau'_1 \multimap \tau'_2} \text{sub-arrow} \\
\\
\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau'_1 \quad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau'_2}{\Psi; \Theta; \Delta \vdash \tau_1 \otimes \tau_2 <: \tau'_1 \otimes \tau'_2} \text{sub-tensor} \\
\\
\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau'_1 \quad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau'_2}{\Psi; \Theta; \Delta \vdash \tau_1 \& \tau_2 <: \tau'_1 \& \tau'_2} \text{sub-with} \\
\\
\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau'_1 \quad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau'_2}{\Psi; \Theta; \Delta \vdash \tau_1 \oplus \tau_2 <: \tau'_1 \oplus \tau'_2} \text{sub-sum} \\
\\
\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau' \quad \Theta; \Delta \models I' \leq I}{\Psi; \Theta; \Delta \vdash [I] \tau <: [I'] \tau'} \text{sub-potential} \\
\\
\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau' \quad \Theta; \Delta \models I \leq I'}{\Psi; \Theta; \Delta \vdash \mathbb{M} I \tau <: \mathbb{M} I' \tau'} \text{sub-monad} \qquad \frac{\Psi; \Theta; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta \vdash !\tau <: !\tau'} \text{sub-Exp} \\
\\
\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta \vdash L^n \tau <: L^n \tau'} \text{sub-list} \qquad \frac{\Psi; \Theta; \Delta, s \vdash \tau <: \tau'}{\Psi; \Theta; \Delta \vdash \exists s. \tau <: \exists s. \tau'} \text{sub-exist} \\
\\
\frac{\Psi, \alpha; \Theta; \Delta \vdash \tau_1 <: \tau_2}{\Psi; \Theta; \Delta \vdash \forall \alpha. \tau_1 <: \forall \alpha. \tau_2} \text{sub-typePoly} \qquad \frac{\Psi; \Theta, i; \Delta \vdash \tau_1 <: \tau_2}{\Psi; \Theta; \Delta \vdash \forall i. \tau_1 <: \forall i. \tau_2} \text{sub-indexPoly} \\
\\
\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau_2 \quad \Theta; \Delta \models c_2 \implies c_1}{\Psi; \Theta; \Delta \vdash c_1 \implies \tau_1 <: c_2 \implies \tau_2} \text{sub-constraint} \\
\\
\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau_2 \quad \Theta; \Delta \models c_1 \implies c_2}{\Psi; \Theta; \Delta \vdash c_1 \& \tau_1 <: c_2 \& \tau_2} \text{sub-CAnd} \\
\\
\frac{\Psi; \Theta, i : S; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta \vdash \lambda_{\ell} i : S . \tau <: \lambda_{\ell} i : S . \tau'} \text{sub-familyAbs} \qquad \frac{\Theta; \Delta \vdash I : S}{\Psi; \Theta; \Delta \vdash \lambda_{\ell} i : S . \tau I <: \tau [I/i]} \text{sub-familyApp1} \\
\\
\frac{\Theta; \Delta \vdash I : S}{\Psi; \Theta; \Delta \vdash \tau [I/i] <: \lambda_{\ell} i : S . \tau I} \text{sub-familyApp2}
\end{array}$$

Figure 10: Subtyping

$$\begin{array}{c}
\overline{\Psi; \Theta; \Delta \vdash \Omega \sqsubseteq .} \text{ sub-mBase} \\
\frac{x : \tau' \in \Omega_1 \quad \Psi; \Theta; \Delta \vdash \tau' <: \tau \quad \Psi; \Theta; \Delta \vdash \Omega_1/x \sqsubseteq \Omega_2}{\Psi; \Theta; \Delta \vdash \Omega_1 \sqsubseteq \Omega_2, x : \tau} \text{ sub-mInd}
\end{array}$$

Figure 11: Ω Subtyping

$$\begin{array}{c}
\overline{\Psi; \Theta; \Delta \vdash \Gamma \sqsubseteq .} \text{ sub-lBase} \\
\frac{x : \tau' \in \Gamma_1 \quad \Psi; \Theta; \Delta \vdash \tau' <: \tau \quad \Psi; \Theta; \Delta \vdash \Gamma_1/x \sqsubseteq \Gamma_2}{\Psi; \Theta; \Delta \vdash \Gamma_1 \sqsubseteq \Gamma_2, x : \tau} \text{ sub-lBase}
\end{array}$$

Figure 12: Γ Subtyping

$$\begin{array}{c}
\overline{\Theta, i : S; \Delta \vdash i : S} \text{ S-var} \quad \overline{\Theta; \Delta \vdash N : \mathbb{N}} \text{ S-nat} \quad \overline{\Theta; \Delta \vdash R : \mathbb{R}^+} \text{ S-real} \quad \frac{\Theta; \Delta \vdash i : \mathbb{N}}{\Theta; \Delta \vdash i : \mathbb{R}^+} \text{ S-real1} \\
\frac{\Theta; \Delta \vdash I_1 : \mathbb{N} \quad \Theta; \Delta \vdash I_2 : \mathbb{N}}{\Theta; \Delta \vdash I_1 + I_2 : \mathbb{N}} \text{ S-add-Nat} \quad \frac{\Theta; \Delta \vdash I_1 : \mathbb{R}^+ \quad \Theta; \Delta \vdash I_2 : \mathbb{R}^+}{\Theta; \Delta \vdash I_1 + I_2 : \mathbb{R}^+} \text{ S-add-Real} \\
\frac{\Theta; \Delta \vdash I_1 : \mathbb{N} \quad \Theta; \Delta \vdash I_2 : \mathbb{N} \quad \Theta; \Delta \models I_1 \geq I_2}{\Theta; \Delta \vdash I_1 - I_2 : \mathbb{N}} \text{ S-minus-Nat} \\
\frac{\Theta; \Delta \vdash I_1 : \mathbb{R}^+ \quad \Theta; \Delta \vdash I_2 : \mathbb{R}^+ \quad \Theta; \Delta \models I_1 \geq I_2}{\Theta; \Delta \vdash I_1 - I_2 : \mathbb{R}^+} \text{ S-minus-Real} \\
\frac{\Theta; \Delta \vdash I_1 : \mathbb{N} \quad \Theta; \Delta, a : \mathbb{N} \vdash I_2 : \mathbb{N}}{\Theta; \Delta \vdash \sum_{a < I_1} I_2 : \mathbb{N}} \text{ S-bSum-Nat} \\
\frac{\Theta; \Delta \vdash I_1 : \mathbb{N} \quad \Theta; \Delta, a : \mathbb{N} \vdash I_2 : \mathbb{R}^+}{\Theta; \Delta \vdash \sum_{a < I_1} I_2 : \mathbb{R}^+} \text{ S-bSum-Real} \quad \frac{\Theta, i : S; \Delta \vdash I : S'}{\Theta; \Delta \vdash \lambda_s i. I : S \rightarrow S'} \text{ S-family}
\end{array}$$

Figure 13: Typing rules for sorts

$$\begin{array}{c}
\frac{}{\Psi; \Theta; \Delta \vdash \mathbf{1} : Type} \text{K-unit} \qquad \frac{}{\Psi; \Theta; \Delta \vdash \mathbf{b} : Type} \text{K-base} \\
\\
\frac{\Psi; \Theta; \Delta \vdash \tau : K \quad \Theta; \Delta \vdash I : S}{\Psi; \Theta; \Delta \vdash L^I \tau : K} \text{K-List} \qquad \frac{\Psi; \Theta; \Delta \vdash \tau_1 : K \quad \Psi; \Theta; \Delta \vdash \tau_2 : K}{\Psi; \Theta; \Delta \vdash \tau_1 \multimap \tau_2 : K} \text{K-arrow} \\
\\
\frac{\Psi; \Theta; \Delta \vdash \tau_1 : K \quad \Psi; \Theta; \Delta \vdash \tau_2 : K}{\Psi; \Theta; \Delta \vdash \tau_1 \otimes \tau_2 : K} \text{K-tensor} \\
\\
\frac{\Psi; \Theta; \Delta \vdash \tau_1 : K \quad \Psi; \Theta; \Delta \vdash \tau_2 : K}{\Psi; \Theta; \Delta \vdash \tau_1 \& \tau_2 : K} \text{K-with} \qquad \frac{\Psi; \Theta; \Delta \vdash \tau_1 : K \quad \Psi; \Theta; \Delta \vdash \tau_2 : K}{\Psi; \Theta; \Delta \vdash \tau_1 \oplus \tau_2 : K} \text{K-or} \\
\\
\frac{\Psi; \Theta; \Delta \vdash \tau : K}{\Psi; \Theta; \Delta \vdash !\tau : K} \text{K-Exp} \qquad \frac{\Psi; \Theta; \Delta \vdash \tau : K \quad \Theta; \Delta \vdash I : \mathbb{R}^+}{\Psi; \Theta; \Delta \vdash [I]\tau : K} \text{K-lab} \\
\\
\frac{\Psi; \Theta; \Delta \vdash \tau : K \quad \Theta; \Delta \vdash I : \mathbb{R}^+}{\Psi; \Theta; \Delta \vdash \mathbb{M} I \tau : K} \text{K-monad} \qquad \frac{\Psi, \alpha : K'; \Theta; \Delta \vdash \tau : K}{\Psi; \Theta; \Delta \vdash \forall \alpha. \tau : K} \text{K-tabs} \\
\\
\frac{\Psi; \Theta, i : S; \Delta \vdash \tau : K}{\Psi; \Theta; \Delta \vdash \forall i. \tau : K} \text{K-iabs} \qquad \frac{\Psi; \Theta; \Delta, c \vdash \tau : K}{\Psi; \Theta; \Delta \vdash c \Rightarrow \tau : K} \text{K-constraint} \\
\\
\frac{\Psi; \Theta; \Delta \vdash \tau : K \quad \Theta; \Delta \models c}{\Psi; \Theta; \Delta \vdash c \& \tau : K} \text{K-consAnd} \qquad \frac{\Psi; \Theta, i : S; \Delta \vdash \tau : K}{\Psi; \Theta; \Delta \vdash \lambda i. \tau : S \rightarrow K} \text{K-family} \\
\\
\frac{\Psi; \Theta; \Delta \vdash \tau : S \rightarrow K \quad \Theta; \Delta \vdash I : S}{\Psi; \Theta; \Delta \vdash \tau I : K} \text{K-iapp}
\end{array}$$

Figure 14: Kind rules for types

2.3 Semantics

Pure reduction, $e \Downarrow_t v$ Forcing reduction, $e \Downarrow_t^c v$

$$\begin{array}{c}
\frac{e_1 \Downarrow_{t_1} v \quad e_2 \Downarrow_{t_2} l}{e_1 :: e_2 \Downarrow_{t_1+t_2+1} v :: l} \text{E-cons} \qquad \frac{e_1 \Downarrow_{t_1} nil \quad e_2 \Downarrow_{t_2} v}{\text{match } e_1 \text{ with } |nil \mapsto e_2| h :: t \mapsto e_3 \Downarrow_{t_1+t_2+1} v} \text{E-matchNil} \\
\\
\frac{e_1 \Downarrow_{t_1} v_h :: l \quad e_3[v_h/h][l/t] \Downarrow_{t_2} v}{\text{match } e_1 \text{ with } |nil \mapsto e_2| h :: t \mapsto e_3 \Downarrow_{t_1+t_2+1} v} \text{E-matchCons} \\
\\
\frac{e_1 \Downarrow_{t_1} v \quad e_2[v/x] \Downarrow_{t_2} v'}{e_1; x.e_2 \Downarrow_{t_1+t_2+1} v'} \text{E-exist} \qquad \frac{e_1 \Downarrow_{t_1} \lambda x.e' \quad e'[e_2/x] \Downarrow_{t_2} v'}{e_1 e_2 \Downarrow_{t_1+t_2+1} v'} \text{E-app} \\
\\
\frac{e_1 \Downarrow_{t_1} v_1 \quad e_2 \Downarrow_{t_2} v_2}{\langle\langle e_1, e_2 \rangle\rangle \Downarrow_{t_1+t_2+1} \langle\langle v_1, v_2 \rangle\rangle} \text{E-TI} \qquad \frac{e \Downarrow_{t_1} \langle\langle v_1, v_2 \rangle\rangle \quad e'[v_1/x][v_2/y] \Downarrow_{t_2} v}{\text{let } \langle\langle x, y \rangle\rangle = e \text{ in } e' \Downarrow_{t_1+t_2+1} v} \text{E-TE} \\
\\
\frac{e_1 \Downarrow_{t_1} v_1 \quad e_2 \Downarrow_{t_2} v_2}{\langle e_1, e_2 \rangle \Downarrow_{t_1+t_2+1} \langle v_1, v_2 \rangle} \text{E-WI} \qquad \frac{e \Downarrow_t \langle v_1, v_2 \rangle}{\text{fst}(e) \Downarrow_{t+1} v_1} \text{E-fst} \qquad \frac{e \Downarrow_t \langle v_1, v_2 \rangle}{\text{snd}(e) \Downarrow_{t+1} v_2} \text{E-snd} \\
\\
\frac{e \Downarrow_t v}{\text{inl}(e) \Downarrow_{t+1} \text{inl}(v)} \text{E-inl} \qquad \frac{e \Downarrow_t v}{\text{inr}(e) \Downarrow_{t+1} \text{inr}(v)} \text{E-inr} \qquad \frac{e \Downarrow_{t_1} \text{inl}(v) \quad e'[v/x] \Downarrow_{t_2} v'}{\text{case } e \text{ of } e'; e'' \Downarrow_{t_1+t_2+1} \text{inl}(v')} \text{E-case1} \\
\\
\frac{e \Downarrow_{t_1} \text{inr}(v) \quad e''[v/y] \Downarrow_{t_2} v''}{\text{case } e \text{ of } e'; e'' \Downarrow_{t_1+t_2+1} \text{inl}(v'')} \text{E-case2} \qquad \frac{}{!e \Downarrow_0 !e} \text{E-expI} \\
\\
\frac{e \Downarrow_{t_1} !e'' \quad e'[e''/x] \Downarrow_{t_2} v}{\text{let } !x = e \text{ in } e' \Downarrow_{t_1+t_2+1} v} \text{E-expE} \qquad \frac{e[\text{fix } x.e'/x] \Downarrow_t v}{\text{fix } x.e \Downarrow_{t+1} v} \text{E-fix} \\
\\
\frac{v \in \{(), x, nil, \lambda y.e, \Lambda.e, \text{ret } e, \text{bind } x = e_1 \text{ in } e_2, \uparrow^\kappa, \text{release } x = e_1 \text{ in } e_2, \text{store } e\}}{v \Downarrow_0 v} \text{E-val}
\end{array}$$

$$\begin{array}{c}
\frac{e \Downarrow_{t_1} \Lambda.e' \quad e' \Downarrow_{t_2} v}{e \square \Downarrow_{t_1+t_2+1} v} \text{E-tapp} \qquad \frac{e \Downarrow_{t_1} \Lambda.e' \quad e' \Downarrow_{t_2} v}{e \square \Downarrow_{t_1+t_2+1} v} \text{E-iapp} \qquad \frac{e \Downarrow_{t_1} \Lambda.e' \quad e' \Downarrow_{t_1} v}{e \square \Downarrow_{t_1+t_2+1} v} \text{E-CE} \\
\\
\frac{e_1 \Downarrow_{t_1} v \quad e_2[v/x] \Downarrow_{t_2} v'}{\text{clet } x = e_1 \text{ in } e_2 \Downarrow_{t_1+t_2+1} v'} \text{E-CandE} \qquad \frac{e \Downarrow_t v}{\text{ret } e \Downarrow_{t+1}^0 v} \text{E-return} \\
\\
\frac{e_1 \Downarrow_{t_1} v_1 \quad v_1 \Downarrow_{t_2}^{c_1} v'_1 \quad e_2[v'_1/x] \Downarrow_{t_3} v_2 \quad v_2 \Downarrow_{t_4}^{c_2} v'_2}{\text{bind } x = e_1 \text{ in } e_2 \Downarrow_{t_1+t_2+t_3+t_4+1}^{c_1+c_2} v'_2} \text{E-bind} \qquad \frac{}{\uparrow^\kappa \Downarrow_1^\kappa ()} \text{E-tick} \\
\\
\frac{e_1 \Downarrow_{t_1} v_1 \quad e_2[v_1/x] \Downarrow_{t_2} v_2 \quad v_2 \Downarrow_{t_3}^c v'_2}{\text{release } x = e_1 \text{ in } e_2 \Downarrow_{t_1+t_2+t_3+1}^c v'_2} \text{E-release} \qquad \frac{e \Downarrow_t v}{\text{store } e \Downarrow_{t+1}^0 v} \text{E-store}
\end{array}$$

Figure 15: Evaluation rules: pure and forcing

2.4 Model

Definition 66 (Value and expression relation).

$$\begin{aligned}
\llbracket \mathbf{1} \rrbracket &\triangleq \{(p, T, ())\} \\
\llbracket \mathbf{b} \rrbracket &\triangleq \{(p, T, v) \mid v \in \llbracket \mathbf{b} \rrbracket\} \\
\llbracket L^0 \tau \rrbracket &\triangleq \{(p, T, nil)\} \\
\llbracket L^{s+1} \tau \rrbracket &\triangleq \{(p, T, v :: l) \mid \exists p_1, p_2. p_1 + p_2 \leq p \wedge (p_1, T, v) \in \llbracket \tau \rrbracket \wedge (p_2, T, l) \in \llbracket L^s \tau \rrbracket\} \\
\llbracket \tau_1 \otimes \tau_2 \rrbracket &\triangleq \{(p, T, \langle\langle v_1, v_2 \rangle\rangle) \mid \exists p_1, p_2. p_1 + p_2 \leq p \wedge (p_1, T, v_1) \in \llbracket \tau_1 \rrbracket \wedge (p_2, T, v_2) \in \llbracket \tau_2 \rrbracket\} \\
\llbracket \tau_1 \& \tau_2 \rrbracket &\triangleq \{(p, T, \langle v_1, v_2 \rangle) \mid (p, T, v_1) \in \llbracket \tau_1 \rrbracket \wedge (p, T, v_2) \in \llbracket \tau_2 \rrbracket\} \\
\llbracket \tau_1 \oplus \tau_2 \rrbracket &\triangleq \{(p, T, inl(v)) \mid (p, T, v) \in \llbracket \tau_1 \rrbracket\} \cup \{(p, T, inr(v)) \mid (p, T, v) \in \llbracket \tau_2 \rrbracket\} \\
\llbracket \tau_1 \multimap \tau_2 \rrbracket &\triangleq \{(p, T, \lambda x. e) \mid \forall p', e', T' < T. (p', T', e') \in \llbracket \tau_1 \rrbracket_{\mathcal{E}} \implies (p + p', T', e[e'/x]) \in \llbracket \tau_2 \rrbracket_{\mathcal{E}}\} \\
\llbracket !\tau \rrbracket &\triangleq \{(p, T, !e) \mid (0, T, e) \in \llbracket \tau \rrbracket_{\mathcal{E}}\} \\
\llbracket [n] \tau \rrbracket &\triangleq \{(p, T, v) \mid \exists p'. p' + n \leq p \wedge (p', T, v) \in \llbracket \tau \rrbracket\} \\
\llbracket \mathbb{M} n \tau \rrbracket &\triangleq \{(p, T, v) \mid \forall n', v', T' < T. v \Downarrow_{T'}^{n'} v' \implies \exists p'. n' + p' \leq p + n \wedge (p', T - T', v') \in \llbracket \tau \rrbracket\} \\
\llbracket \forall \alpha. \tau \rrbracket &\triangleq \{(p, T, \Lambda. e) \mid \forall \tau', T' < T. (p, T', e) \in \llbracket \tau[\tau'/\alpha] \rrbracket_{\mathcal{E}}\} \\
\llbracket \forall i. \tau \rrbracket &\triangleq \{(p, T, \Lambda. e) \mid \forall I, T' < T. (p, T', e) \in \llbracket \tau[I/i] \rrbracket_{\mathcal{E}}\} \\
\llbracket c \Rightarrow \tau \rrbracket &\triangleq \{(p, T, \Lambda. e) \mid . \models c \implies (p, T, e) \in \llbracket \tau \rrbracket_{\mathcal{E}}\} \\
\llbracket c \&\tau \rrbracket &\triangleq \{(p, T, v) \mid . \models c \wedge (p, T, v) \in \llbracket \tau \rrbracket\} \\
\llbracket \exists s. \tau \rrbracket &\triangleq \{(p, T, v) \mid \exists s'. (p, T, v) \in \llbracket \tau[s'/s] \rrbracket\} \\
\llbracket \lambda_i i. \tau \rrbracket &\triangleq f \text{ where } \forall I. f I = \llbracket \tau[I/i] \rrbracket \\
\llbracket \tau I \rrbracket &\triangleq \llbracket \tau \rrbracket I \\
\llbracket \tau \rrbracket_{\mathcal{E}} &\triangleq \{(p, T, e) \mid \forall T' < T, v. e \Downarrow_{T'} v \implies (p, T - T', v) \in \llbracket \tau \rrbracket\}
\end{aligned}$$

Definition 67 (Interpretation of typing contexts).

$$\begin{aligned}
\llbracket \Gamma \rrbracket_{\mathcal{E}} &= \{(p, T, \gamma) \mid \exists f : \text{Vars} \rightarrow \text{Pots}. \\
&\quad (\forall x \in \text{dom}(\Gamma). (f(x), T, \gamma(x)) \in \llbracket \Gamma(x) \rrbracket_{\mathcal{E}}) \wedge (\sum_{x \in \text{dom}(\Gamma)} f(x) \leq p)\} \\
\llbracket \Omega \rrbracket_{\mathcal{E}} &= \{(0, T, \delta) \mid (\forall x \in \text{dom}(\Omega). (0, T, \delta(x)) \in \llbracket \tau \rrbracket_{\mathcal{E}})\}
\end{aligned}$$

Definition 68 (Type and index substitutions). $\sigma : \text{TypeVar} \rightarrow \text{Type}$, $\iota : \text{IndexVar} \rightarrow \text{Index}$

Lemma 69 (Value monotonicity lemma). $\forall p, p', v, \tau$.

$$(p, T, v) \in \llbracket \tau \rrbracket \wedge p \leq p' \wedge T' \leq T \implies (p', T', v) \in \llbracket \tau \rrbracket$$

Proof. Proof by induction on τ □

Lemma 70 (Expression monotonicity lemma). $\forall p, p', v, \tau$.

$$(p, T, e) \in \llbracket \tau \rrbracket_{\mathcal{E}} \wedge p \leq p' \wedge T' \leq T \implies (p', T', e) \in \llbracket \tau \rrbracket_{\mathcal{E}}$$

Proof. From Definition 66 and Lemma 69 □

Theorem 71 (Fundamental theorem). $\forall \Theta, \Omega, \Gamma, e, \tau, T, p_l, \gamma, \delta, \sigma, \iota$.

$$\begin{aligned}
\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \wedge (p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}} \wedge (0, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}} \wedge . \models \Delta \iota \implies \\
(p_l, T, e \gamma \delta) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}.
\end{aligned}$$

Proof. Proof by induction on the typing judgment

1. T-var1:

$$\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma, x : \tau \vdash x : \tau} \text{T-var1}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma, x : \tau \sigma \iota \rrbracket_{\mathcal{E}}$ and $(0, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, x \delta \gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$

Since we are given that $(p_l, T, \gamma) \in \llbracket \Gamma, x : \tau \sigma \iota \rrbracket_{\mathcal{E}}$ therefore from Definition 67 we know that $\exists f. (f(x), T, \gamma(x)) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$ where $f(x) \leq p_l$

Therefore from Lemma 70 we get $(p_l, T, x \delta \gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$

2. T-var2:

$$\frac{}{\Psi; \Theta; \Delta; \Omega, x : \tau; \Gamma \vdash x : \tau} \text{T-var2}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma, \sigma \iota \rrbracket_{\mathcal{E}}$ and $(0, T, \delta) \in \llbracket (\Omega, x : \tau) \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, x \delta \gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$

Since we are given that $(0, T, \delta) \in \llbracket (\Omega, x : \tau) \sigma \iota \rrbracket_{\mathcal{E}}$ therefore from Definition 67 we know that

$(0, T, \delta(x)) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$

Therefore from Lemma 70 we get $(p_l, T, x \delta \gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$

3. T-unit:

$$\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash () : \mathbf{1}} \text{T-unit}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, () \delta \gamma) \in \llbracket \mathbf{1} \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall T' < T, v'. () \Downarrow_{T'} v' \implies (p_l, T - T', v') \in \llbracket \mathbf{1} \rrbracket$

This means given some $T' < T, v'$ s.t $() \Downarrow_{T'} v'$ it suffices to prove that

$(p_l, T - T', v') \in \llbracket \mathbf{1} \rrbracket$

From (E-val) we know that $T' = 0$ and $v' = ()$, therefore it suffices to prove that

$(p_l, T, ()) \in \llbracket \mathbf{1} \rrbracket$

We get this directly from Definition 66

4. T-base:

$$\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash c : \mathbf{b}} \text{T-base}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, c) \in \llbracket \mathbf{b} \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall T' < T, v'.c \Downarrow_{T'} v' \implies (p_l, T - T', v') \in \llbracket \mathbf{1} \rrbracket$$

This means given some $T' < T, v'$ s.t $c \Downarrow_{T'} v'$ it suffices to prove that $(p_l, T - T', v') \in \llbracket \mathbf{1} \rrbracket$

From (E-val) we know that $T' = 0$ and $v' = c$, therefore it suffices to prove that $(p_l, T, c) \in \llbracket \mathbf{b} \rrbracket$

We get this directly from Definition 66

5. T-nil:

$$\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash nil : L^0 \tau} \text{T-nil}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma, \sigma\iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma\iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, nil \delta\gamma) \in \llbracket L^0 \tau \sigma\iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall T' < T, v'.nil \Downarrow_{T'} v' \implies (p_l, T - T', v') \in \llbracket L^0 \tau \sigma\iota \rrbracket$$

This means given some $T' < T, v'$ s.t $nil \Downarrow_{T'} v'$ it suffices to prove that $(p_l, T - T', v') \in \llbracket L^0 \tau \sigma\iota \rrbracket$

From (E-val) we know that $T' = 0$ and $v' = nil$, therefore it suffices to prove that $(p_l, T, nil) \in \llbracket L^0 \tau \sigma\iota \rrbracket$

We get this directly from Definition 66

6. T-cons:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e_1 : \tau \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2 \vdash e_2 : L^n \tau \quad \Theta \vdash n : \mathbb{N}}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash e_1 :: e_2 : L^{n+1} \tau} \text{T-cons}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma_1 \oplus \Gamma_2) \sigma\iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket (\Omega) \sigma\iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, (e_1 :: e_2) \delta\gamma) \in \llbracket L^{n+1} \tau \sigma\iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T, v'.(e_1 :: e_2) \delta\gamma \Downarrow_t v' \implies (p_l, T - t, v') \in \llbracket L^{n+1} \tau \sigma\iota \rrbracket$$

This means given some $t < T, v'$ s.t $(e_1 :: e_2) \delta\gamma \Downarrow_t v'$, it suffices to prove that $(p_l, T - t, v') \in \llbracket L^{n+1} \tau \sigma\iota \rrbracket$

From (E-cons) we know that $\exists v_f, l.v' = v_f :: l$

Therefore from Definition 66 it suffices to prove that

$$\exists p_1, p_2.p_1 + p_2 \leq p_l \wedge (p_1, T - t, v_f) \in \llbracket \tau \sigma\iota \rrbracket \wedge (p_2, T - t, l) \in \llbracket L^n \tau \sigma\iota \rrbracket \quad (\text{F-C0})$$

From Definition 67 and Definition 65 we know that $\exists p_{l_1}, p_{l_2}.p_{l_1} + p_{l_2} = p_l$ s.t $(p_{l_1}, \gamma) \in \llbracket (\Gamma_1) \sigma\iota \rrbracket_{\mathcal{E}}$ and $(p_{l_2}, \gamma) \in \llbracket (\Gamma_2) \sigma\iota \rrbracket_{\mathcal{E}}$

IH1:

$$(p_{l1}, T, e_1 \delta\gamma) \in \llbracket \tau \sigma\iota \rrbracket_{\mathcal{E}}$$

Therefore from Definition 66 we have

$$\forall t1 < T. e_1 \delta\gamma \Downarrow_{t1} v_f \implies (p_{l1}, T - t1, v_f) \in \llbracket \tau \rrbracket$$

Since we are given that $(e_1 :: e_2) \delta\gamma \Downarrow_t v_f :: l$ therefore fom E-cons we also know that $\exists t1 < t. e_1 \delta\gamma \Downarrow_{t1} v_f$

$$\text{Since } t1 < t < T, \text{ therefore we have } (p_{l1}, T - t1, v_f) \in \llbracket \tau \sigma\iota \rrbracket \quad (\text{F-C1})$$

IH2:

$$(p_{l2}, T, e_2 \delta\gamma) \in \llbracket L^n \tau \sigma\iota \rrbracket_{\mathcal{E}}$$

Therefore from Definition 66 we have

$$\forall t2 < T. e_2 \delta\gamma \Downarrow_{t2} l \implies (p_{l2}, T - t2, l) \in \llbracket L^n \tau \sigma\iota \rrbracket$$

Since we are given that $(e_1 :: e_2) \delta\gamma \Downarrow_t v_f :: l$ therefore fom E-cons we also know that $\exists t2 < t - t1. e_2 \delta\gamma \Downarrow_{t2} l$

Since $t2 < t - t1 < t < T$, therefore we have

$$(p_{l2}, T - t2, l) \in \llbracket L^n \tau \sigma\iota \rrbracket \quad (\text{F-C2})$$

In order to prove (F-C0) we choose p_1 as p_{l1} and p_2 as p_{l2} and it suffices to prove that

$$(p_{l1}, T - t, v) \in \llbracket \tau \sigma\iota \rrbracket \wedge (p_{l2}, T - t, l) \in \llbracket L^n \tau \sigma\iota \rrbracket$$

Since $t = t_1 + t_2 + 1$ therefore from (F-C1) and Lemma 69 we get $(p_{l1}, T - t, v) \in \llbracket \tau \sigma\iota \rrbracket$

Similarly from (F-C2) and Lemma 69 we also get $(p_{l2}, T - t, l) \in \llbracket L^n \tau \sigma\iota \rrbracket$

7. T-match:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : L^n \tau \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2 \vdash e_1 : \tau' \quad \Psi; \Theta; \Delta; n > 0; \Omega; \Gamma_2, h : \tau, t : L^{n-1} \tau \vdash e_2 : \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \text{match } e \text{ with } | \text{nil} \mapsto e_1 \mid h :: t \mapsto e_2 : \tau'} \text{ T-match}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma_1 \oplus \Gamma_2) \sigma\iota \rrbracket_{\mathcal{E}}, (0, T, \delta) \in \llbracket \Omega \sigma\iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, (\text{match } e \text{ with } | \text{nil} \mapsto e_1 \mid h :: t \mapsto e_2) \delta\gamma) \in \llbracket \tau' \sigma\iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T, v_f. (\text{match } e \text{ with } | \text{nil} \mapsto e_1 \mid h :: t \mapsto e_2) \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in \llbracket \tau' \sigma\iota \rrbracket$$

This means given some $t < T, v_f$ s.t $(\text{match } e \text{ with } | \text{nil} \mapsto e_1 \mid h :: t \mapsto e_2) \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_l, T - t, v_f) \in \llbracket \tau' \sigma\iota \rrbracket \quad (\text{F-M0})$$

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2}. p_{l1} + p_{l2} = p_l$ s.t

$$(p_{l1}, \gamma) \in \llbracket (\Gamma_1) \sigma\iota \rrbracket_{\mathcal{E}} \text{ and } (p_{l2}, \gamma) \in \llbracket (\Gamma_2) \sigma\iota \rrbracket_{\mathcal{E}}$$

IH1

$$(p_{l1}, T, e \delta\gamma) \in \llbracket L^n \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 66 we have

$$\forall t' < T . e \delta\gamma \Downarrow_{t'} v_1 \implies (p_{l1}, T - t', v_1) \in \llbracket L^n \tau \sigma \iota \rrbracket$$

Since we know that (match e with $|nil \mapsto e_1 | h :: t \mapsto e_2$) $\delta\gamma \Downarrow_t v_f$ therefore from E-match we know that $\exists t' < t, v_1. e \delta\gamma \Downarrow_{t'} v_1$.

$$\text{Since } t' < t < T, \text{ therefore we have } (p_{l1}, T - t', v_1) \in \llbracket L^n \tau \sigma \iota \rrbracket$$

2 cases arise:

(a) $v_1 = nil$:

In this case we know that $n = 0$ therefore

IH2

$$(p_{l2}, T, e_1 \delta\gamma) \in \llbracket \tau' \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 66 we have

$$\forall t_1 < T . e_1 \delta\gamma \Downarrow_{t_1} v_f \implies (p_{l2}, T - t_1, v_f) \in \llbracket \tau' \sigma \iota \rrbracket$$

Since we know that (match e with $|nil \mapsto e_1 | h :: t \mapsto e_2$) $\delta\gamma \Downarrow_t v_f$ therefore from E-match we know that $\exists t_1 < t. e_1 \delta\gamma \Downarrow_{t_1} v_f$.

Since $t_1 < t < T$ therefore we have

$$(p_{l2}, T - t_1, v_f) \in \llbracket \tau' \sigma \iota \rrbracket$$

And from Lemma 69 we get

$$(p_{l2} + p_{l1}, T - t, v_f) \in \llbracket \tau' \sigma \iota \rrbracket_{\mathcal{E}}$$

And finally since $p_l = p_{l1} + p_{l2}$ therefore we get

$$(p_l, T - t, v_f) \in \llbracket \tau' \sigma \iota \rrbracket_{\mathcal{E}}$$

And we are done

(b) $v_1 = v :: l$:

In this case we know that $n > 0$

IH2

$$(p_{l2} + p_{l1}, T, e_2 \delta\gamma') \in \llbracket \tau' \sigma \iota \rrbracket_{\mathcal{E}}$$

where

$$\gamma' = \gamma \cup \{h \mapsto v\} \cup \{t \mapsto l\}$$

This means from Definition 66 we have

$$\forall t_2 < T . e_2 \delta\gamma' \Downarrow_{t_2} v_f \implies (p_{l2} + p_{l1}, T - t_2, v_f) \in \llbracket \tau' \sigma \iota \rrbracket$$

Since we know that (match e with $|nil \mapsto e_1 | h :: t \mapsto e_2$) $\delta\gamma \Downarrow_t v_f$ therefore from E-match we know that $\exists t_2 < t. e_2 \delta\gamma' \Downarrow_{t_2} v_f$.

Since $t_2 < t < T$ therefore we have

$$(p_{l2} + p_{l1}, T - t_2, v_f) \in \llbracket \tau' \sigma \iota \rrbracket$$

From Lemma 69 we get

$$(p_{l2} + p_{l1}, T - t, v_f) \in \llbracket \tau' \sigma \iota \rrbracket$$

And finally since $p_l = p_{l1} + p_{l2}$ therefore we get

$$(p_l, T - t, v_f) \in \llbracket \tau' \sigma \iota \rrbracket_{\mathcal{E}}$$

And we are done

8. T-existI:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau[n/s] \quad \Theta \vdash n : S}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \exists s : S . \tau} \text{T-existI}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma_l \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma_l \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, e \delta\gamma) \in \llbracket \exists s . \tau \sigma_l \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T, v_f . e \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f \delta\gamma) \in \llbracket \exists s . \tau \sigma_l \rrbracket$$

This means given some $t < T, v_f$ s.t $e \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_l, T - t, v_f) \in \llbracket \exists s . \tau \sigma_l \rrbracket$$

From Definition 66 it suffices to prove that

$$\exists s' . (p_l, T - t, v_f) \in \llbracket \tau[s'/s] \sigma_l \rrbracket \quad (\text{F-E0})$$

$$\underline{\text{IH}}: (p_l, T, e \delta\gamma) \in \llbracket \tau[n/s] \sigma_l \rrbracket_{\mathcal{E}}$$

This means from Definition 66 we have

$$\forall t' < T . e \delta\gamma \Downarrow_{t'} v_f \implies (p_l, T - t', v_f) \in \llbracket \tau[n/s] \sigma_l \rrbracket$$

Since we are given that $e \delta\gamma \Downarrow_t v_f$ therefore we get

$$(p_l, T - t, v_f) \in \llbracket \tau[n/s] \sigma_l \rrbracket \quad (\text{F-E1})$$

To prove (F-E0) we choose s' as n and we get the desired from (F-E1)

9. T-existsE:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : \exists s . \tau \quad \Psi; \Theta, s; \Delta; \Omega; \Gamma_2, x : \tau \vdash e' : \tau' \quad \Theta \vdash \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash e; x.e' : \tau'} \text{T-existsE}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma_1 \oplus \Gamma_2) \sigma_l \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket (\Omega) \sigma_l \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, (e; x.e') \delta\gamma) \in \llbracket \tau' \sigma_l \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T, v_f . (e; x.e') \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in \llbracket \tau' \sigma_l \rrbracket$$

This means given soem $t < T, v_f$ s.t $(e; x.e') \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_l, T - t, v_f) \in \llbracket \tau' \sigma_l \rrbracket \quad (\text{F-EE0})$$

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2} . p_{l1} + p_{l2} = p_l$ s.t

$(p_{l1}, \gamma) \in \llbracket (\Gamma_1) \sigma_l \rrbracket_{\mathcal{E}}$ and $(p_{l2}, \gamma) \in \llbracket (\Gamma_2) \sigma_l \rrbracket_{\mathcal{E}}$

IH1

$$(p_{l1}, T, e \delta\gamma) \in \llbracket \exists s . \tau \sigma_l \rrbracket_{\mathcal{E}}$$

This means from Definition 66 we have

$$\forall t_1 < T . e \delta\gamma \Downarrow_{t_1} v_1 \implies (p_{l_1}, T - t_1, v_1) \in \llbracket \exists s. \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

Since we know that $(e; x.e') \delta\gamma \Downarrow_t v_f$ therefore from E-existE we know that $\exists t_1 < t, v_1. e \delta\gamma \Downarrow_{t_1} v_1$. Therefore we have

$$(p_{l_1}, T - t_1, v_1) \in \llbracket \exists s. \tau \sigma \iota \rrbracket$$

Therefore from Definition 66 we have

$$\exists s'. (p_{l_1}, T - t_1, v_1) \in \llbracket \tau[s'/s] \sigma \iota \rrbracket \quad (\text{F-EE1})$$

IH2

$$(p_{l_1} + p_{l_2}, T, e' \delta'\gamma) \in \llbracket \tau' \sigma \iota' \rrbracket_{\mathcal{E}}$$

where

$$\delta' = \delta \cup \{x \mapsto e_1\} \text{ and } \iota' = \iota \cup \{s \mapsto s'\}$$

This means from Definition 66 we have

$$\forall t_2 < T . e' \delta'\gamma \Downarrow_{t_2} v_f \implies (p_{l_1} + p_{l_2}, T - t_2, v_f) \in \llbracket \tau' \sigma \iota' \rrbracket$$

Since we know that $(e; x.e') \delta\gamma \Downarrow_t v_f$ therefore from E-existE we know that $\exists t_2 < t. e' \delta'\gamma \Downarrow_{t_2} v_f$.

Since $t_2 < t < T$ therefore we have

$$(p_{l_1} + p_{l_2}, T - t_2, v_f) \in \llbracket \tau' \sigma \iota' \rrbracket$$

Since $p_l = p_{l_1} + p_{l_2}$ therefore we get

$$(p_l, T - t_2, v_f) \in \llbracket \tau' \sigma \iota' \rrbracket$$

From Lemma 69 we get

$$(p_l, T - t, v_f) \in \llbracket \tau' \sigma \iota' \rrbracket$$

And finally since we have $\Psi; \Theta \vdash \tau'$ therefore we also have

$$(p_l, T - t, v_f) \in \llbracket \tau' \sigma \iota \rrbracket$$

And we are done

10. T-lam:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma, x : \tau_1 \vdash e : \tau_2}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \lambda x. e : (\tau_1 \multimap \tau_2)} \text{T-lam}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma, \sigma \iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, (\lambda x. e) \delta\gamma) \in \llbracket (\tau_1 \multimap \tau_2) \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T, v_f. (\lambda x. e) \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in \llbracket (\tau_1 \multimap \tau_2) \sigma \iota \rrbracket$$

This means given some $t < T, v_f$ s.t $(\lambda x. e) \delta\gamma \Downarrow_t v_f$. From E-val we know that $t = 0$ and $v_f = (\lambda x. e) \delta\gamma$

Therefore it suffices to prove that

$$(p_l, T, (\lambda x.e) \delta\gamma) \in \llbracket (\tau_1 \multimap \tau_2) \sigma\iota \rrbracket$$

From Definition 66 it suffices to prove that

$$\forall p', e', T' < T. (p', T', e') \in \llbracket \tau_1 \sigma\iota \rrbracket_{\mathcal{E}} \implies (p_l + p', T', e[e'/x]) \in \llbracket \tau_2 \sigma\iota \rrbracket_{\mathcal{E}}$$

This means given some $p', e', T' < T$ s.t. $(p', T', e') \in \llbracket \tau_1 \sigma\iota \rrbracket_{\mathcal{E}}$ it suffices to prove that

$$(p_l + p', T', e[e'/x]) \in \llbracket \tau_2 \sigma\iota \rrbracket_{\mathcal{E}} \quad (\text{F-L1})$$

From IH we know that

$$(p_l + p', T, e \delta\gamma') \in \llbracket \tau_2 \sigma\iota \rrbracket_{\mathcal{E}}$$

where

$$\gamma' = \gamma \cup \{x \mapsto e'\}$$

Therefore from Lemma 70 we get the desired

11. T-app:

$$\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e_1 : (\tau_1 \multimap \tau_2) \quad \Psi; \Theta; \Delta; \Omega_2; \Gamma_2 \vdash e_2 : \tau_1}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash e_1 e_2 : \tau_2} \text{T-app}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma_1 \oplus \Gamma_2) \sigma\iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket (\Omega) \sigma\iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, e_1 e_2 \delta\gamma) \in \llbracket \tau_2 \sigma\iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T, v_f. (e_1 e_2) \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in \llbracket \tau_2 \sigma\iota \rrbracket$$

This means given some $t < T, v_f$ s.t. $(e_1 e_2) \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_l, T - t, v_f) \in \llbracket \tau_2 \sigma\iota \rrbracket \quad (\text{F-A0})$$

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2}. p_{l1} + p_{l2} = p_l$ s.t.

$(p_{l1}, \gamma) \in \llbracket (\Gamma_1) \sigma\iota \rrbracket_{\mathcal{E}}$ and $(p_{l2}, \gamma) \in \llbracket (\Gamma_2) \sigma\iota \rrbracket_{\mathcal{E}}$

IH1

$$(p_{l1}, T, e_1 \delta\gamma) \in \llbracket (\tau_1 \multimap \tau_2) \sigma\iota \rrbracket_{\mathcal{E}}$$

This means from Definition 66 we have

$$\forall t_1 < T. e_1 \Downarrow_{t_1} \lambda x.e \implies (p_{l1}, T - t_1, \lambda x.e) \in \llbracket (\tau_1 \multimap \tau_2) \sigma\iota \rrbracket$$

Since we know that $(e_1 e_2) \delta\gamma \Downarrow_t v_f$ therefore from E-app we know that $\exists t_1 < t. e_1 \Downarrow_{t_1} \lambda x.e$, therefore we have

$$(p_{l1}, T - t_1, \lambda x.e) \in \llbracket (\tau_1 \multimap \tau_2) \sigma\iota \rrbracket$$

Therefore from Definition 66 we have

$$\forall p', e_1, T_1 < T - t_1. (p', T_1, e_1) \in \llbracket \tau_1 \sigma\iota \rrbracket_{\mathcal{E}} \implies (p_{l1} + p', T_1, e[e_1'/x]) \in \llbracket \tau_2 \sigma\iota \rrbracket_{\mathcal{E}} \quad (\text{F-A1})$$

IH2

$$(p_{l2}, T - t_1 - 1, e_2 \delta\gamma) \in \llbracket \tau_1 \sigma\iota \rrbracket_{\mathcal{E}} \quad (\text{F-A2})$$

Instantiating (F-A1) with p_{l2} , $e_2 \delta\gamma$ and $T - t_1 - 1$ we get

$$(p_{l1} + p_{l2}, T - t_1 - 1, e[e_2 \delta\gamma/x]) \in \llbracket \tau_2 \sigma\iota \rrbracket_{\mathcal{E}}$$

This means from Definition 66 we have

$$\forall t_2 < T - t_1 - 1. e[e_2 \delta\gamma/x] \Downarrow_{t_2} v_f \implies (p_{l1} + p_{l2}, T - t_1 - 1 - t_2, v_f) \in \llbracket \tau_2 \sigma\iota \rrbracket$$

Since we know that $(e_1 e_2) \delta\gamma \Downarrow_t v_f$ therefore from E-app we know that $\exists t_2. e[e_2 \delta\gamma/x] \Downarrow_{t_2} v_f$ where $t_2 = t - t_1 - 1$, therefore we have

$$(p_{l1} + p_{l2}, T - t_1 - t_2 - 1, v_f) \in \llbracket \tau_2 \sigma\iota \rrbracket \text{ where } p_{l1} + p_{l2} = p_l$$

Since from E-app we know that $t = t_1 + t_2 + 1$, therefore we have proved (F-A0)

12. T-sub:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \quad \Theta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau'} \text{ T-sub}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma) \sigma\iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma\iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, e \delta\gamma) \in \llbracket \tau' \sigma\iota \rrbracket_{\mathcal{E}}$

IH $(p_l, T, e \delta\gamma) \in \llbracket \tau \sigma\iota \rrbracket_{\mathcal{E}}$

We get the desired directly from IH and Lemma 73

13. T-weaken:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \quad \Psi; \Theta; \Delta \Vdash \Gamma' <: \Gamma \quad \Psi; \Theta; \Delta \Vdash \Omega' <: \Omega}{\Psi; \Theta; \Delta; \Omega'; \Gamma' \vdash e : \tau} \text{ T-weaken}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma') \sigma\iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket (\Omega') \sigma\iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, e \delta\gamma) \in \llbracket \tau \sigma\iota \rrbracket_{\mathcal{E}}$

Since we are given that $(p_l, T, \gamma) \in \llbracket (\Gamma') \sigma\iota \rrbracket_{\mathcal{E}}$ therefore from Lemma 74 we also have $(p_l, T, \gamma) \in \llbracket (\Gamma) \sigma\iota \rrbracket_{\mathcal{E}}$

Similarly since we are given that $(0, T, \delta) \in \llbracket (\Omega') \sigma\iota \rrbracket_{\mathcal{E}}$ therefore from Lemma 76 we also have $(0, T, \delta) \in \llbracket (\Omega) \sigma\iota \rrbracket_{\mathcal{E}}$

IH:

$(p_l, T, e \delta\gamma) \in \llbracket \tau \sigma\iota \rrbracket_{\mathcal{E}}$

We get the desired directly from IH

14. T-tensorI:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e_1 : \tau_1 \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2 \vdash e_2 : \tau_2}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \langle\langle e_1, e_2 \rangle\rangle : (\tau_1 \otimes \tau_2)} \text{ T-tensorI}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma_1 \oplus \Gamma_2) \sigma\iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket (\Omega) \sigma\iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, \langle\langle e_1, e_2 \rangle\rangle \delta\gamma) \in \llbracket (\tau_1 \otimes \tau_2) \sigma\iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T . \langle\langle e_1, e_2 \rangle\rangle \delta\gamma \Downarrow_t \langle\langle v_{f_1}, v_{f_2} \rangle\rangle \implies (p_l, T - t, \langle\langle v_{f_1}, v_{f_2} \rangle\rangle) \in \llbracket (\tau_1 \otimes \tau_2) \sigma \iota \rrbracket$$

This means given some $t < T$ s.t $\langle\langle e_1, e_2 \rangle\rangle \delta\gamma \Downarrow_t \langle\langle v_{f_1}, v_{f_2} \rangle\rangle$ it suffices to prove that

$$(p_l, T - t, \langle\langle v_{f_1}, v_{f_2} \rangle\rangle) \in \llbracket (\tau_1 \otimes \tau_2) \sigma \iota \rrbracket \quad (\text{F-TI0})$$

From Definition 67 and Definition 65 we know that $\exists p_{l_1}, p_{l_2}. p_{l_1} + p_{l_2} = p_l$ s.t

$$(p_{l_1}, \gamma) \in \llbracket (\Gamma_1) \sigma \iota \rrbracket_{\mathcal{E}} \text{ and } (p_{l_2}, \gamma) \in \llbracket (\Gamma_2) \sigma \iota \rrbracket_{\mathcal{E}}$$

IH1:

$$(p_{l_1}, T, e_1 \delta\gamma) \in \llbracket \tau_1 \sigma \iota \rrbracket_{\mathcal{E}}$$

Therefore from Definition 66 we have

$$\forall t_1 < T . e_1 \delta\gamma \Downarrow_{t_1} v_{f_1} \implies (p_{l_1}, T - t_1, v_{f_1}) \in \llbracket \tau_1 \sigma \iota \rrbracket$$

Since we are given that $\langle\langle e_1, e_2 \rangle\rangle \delta\gamma \Downarrow_t \langle\langle v_{f_1}, v_{f_2} \rangle\rangle$ therefore fom E-TI we know that $\exists t_1 < t. e_1 \delta\gamma \Downarrow_{t_1} v_{f_1}$

$$\text{Hence we have } (p_{l_1}, T - t_1, v_{f_1}) \in \llbracket \tau_1 \sigma \iota \rrbracket \quad (\text{F-TI1})$$

IH2:

$$(p_{l_2}, T, e_2 \delta\gamma) \in \llbracket \tau_2 \sigma \iota \rrbracket_{\mathcal{E}}$$

Therefore from Definition 66 we have

$$\forall t_2 < T . e_2 \delta\gamma \Downarrow_{t_2} v_{f_2} \implies (p_{l_2}, T - t_2, v_{f_2}) \in \llbracket \tau_2 \sigma \iota \rrbracket$$

Since we are given that $\langle\langle e_1, e_2 \rangle\rangle \delta\gamma \Downarrow_t \langle\langle v_{f_1}, v_{f_2} \rangle\rangle$ therefore fom E-TI we also know that $\exists t_2 < t. e_2 \delta\gamma \Downarrow_{t_2} v_{f_2}$ s.t

Since $t_2 < t < T$ therefore we have

$$(p_{l_2}, T - t_2, v_{f_2}) \in \llbracket \tau_2 \sigma \iota \rrbracket \quad (\text{F-TI2})$$

Applying Lemma 69 on (F-TI1) and (F-TI2) and by using Definition 66 we get the desired.

15. T-tensorE:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : (\tau_1 \otimes \tau_2) \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : \tau_1, y : \tau_2 \vdash e' : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \text{let}\langle\langle x, y \rangle\rangle = e \text{ in } e' : \tau} \text{ T-tensorE}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma_1 \oplus \Gamma_2) \sigma \iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, (\text{let}\langle\langle x, y \rangle\rangle = e \text{ in } e') \delta\gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T, v_f. (\text{let}\langle\langle x, y \rangle\rangle = e \text{ in } e') \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in \llbracket \tau \sigma \iota \rrbracket$$

This means given some $t < T, v_f$ s.t $(\text{let}\langle\langle x, y \rangle\rangle = e \text{ in } e') \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_l, T - t, v_f) \in \llbracket \tau \sigma \iota \rrbracket \quad (\text{F-TE0})$$

From Definition 67 and Definition 65 we know that $\exists p_{l_1}, p_{l_2}. p_{l_1} + p_{l_2} = p_l$ s.t

$(p_{l1}, \gamma) \in \llbracket (\Gamma_1) \sigma \iota \rrbracket_{\mathcal{E}}$ and $(p_{l2}, \gamma) \in \llbracket (\Gamma_2) \sigma \iota \rrbracket_{\mathcal{E}}$

IH1

$(p_{l1}, T, e \delta \gamma) \in \llbracket (\tau_1 \otimes \tau_2) \sigma \iota \rrbracket_{\mathcal{E}}$

This means from Definition 66 we have

$\forall t_1 < T . e \delta \gamma \Downarrow_{t_1} \langle\langle v_1, v_2 \rangle\rangle \delta \gamma \implies (p_{l1}, T - t_1, \langle\langle v_1, v_2 \rangle\rangle) \in \llbracket (\tau_1 \otimes \tau_2) \sigma \iota \rrbracket$

Since we know that $(\text{let}\langle\langle x, y \rangle\rangle = e \text{ in } e') \delta \gamma \Downarrow_t v_f$ therefore from E-TE we know that $\exists t_1 < t, v_1, v_2 . e \delta \gamma \Downarrow_{t_1} \langle\langle v_1, v_2 \rangle\rangle$. Therefore we have

$(p_{l1}, T - t_1, \langle\langle v_1, v_2 \rangle\rangle) \in \llbracket (\tau_1 \otimes \tau_2) \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 66 we know that

$\exists p_1, p_2 . p_1 + p_2 \leq p_{l1} \wedge (p_1, T, v_1) \in \llbracket \tau_1 \sigma \iota \rrbracket \wedge (p_2, T, v_2) \in \llbracket \tau_2 \sigma \iota \rrbracket$ (F-TE1)

IH2

$(p_{l2} + p_1 + p_2, T, e' \delta \gamma') \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$

where

$\gamma' = \gamma \cup \{x \mapsto v_1\} \cup \{y \mapsto v_2\}$

This means from Definition 66 we have

$\forall t_2 < T . e' \delta \gamma' \Downarrow_{t_2} v_f \implies (p_{l2} + p_1 + p_2, T - t_2, v_f) \in \llbracket \tau \sigma \iota \rrbracket$

Since we know that $(\text{let}\langle\langle x, y \rangle\rangle = e \text{ in } e') \delta \gamma \Downarrow_t v_f$ therefore from E-TE we know that $\exists t_2 < t . e' \delta \gamma' \Downarrow_{t_2} v_f$. Therefore we have

$(p_{l2} + p_1 + p_2, T - t_2, v_f) \in \llbracket \tau \sigma \iota \rrbracket$

From Lemma 69 we get

$(p_l, T - t, v_f) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$

And we are done

16. T-withI:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e_1 : \tau_1 \quad \Psi; \Theta; \Delta; \Omega; \Gamma \vdash e_2 : \tau_2}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \langle e_1, e_2 \rangle : (\tau_1 \& \tau_2)} \text{T-withI}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, \langle e_1, e_2 \rangle \delta \gamma) \in \llbracket (\tau_1 \& \tau_2) \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$\forall t < T . \langle e_1, e_2 \rangle \delta \gamma \Downarrow_t \langle v_{f1}, v_{f2} \rangle \implies (p_l, T - t, \langle v_{f1}, v_{f2} \rangle) \in \llbracket (\tau_1 \& \tau_2) \sigma \iota \rrbracket$

This means given $\langle e_1, e_2 \rangle \delta \gamma \Downarrow_t \langle v_{f1}, v_{f2} \rangle$ it suffices to prove that

$(p_l, T - t, \langle v_{f1}, v_{f2} \rangle) \in \llbracket (\tau_1 \& \tau_2) \sigma \iota \rrbracket$ (F-WI0)

IH1:

$$(p_l, T, e_1 \delta\gamma) \in \llbracket \tau_1 \sigma_l \rrbracket_{\mathcal{E}}$$

Therefore from Definition 66 we have

$$\forall t_1 < T . e_1 \delta\gamma \Downarrow_{t_1} v_{f_1} \implies (p_l, T - t_1, v_{f_1}) \in \llbracket \tau_1 \sigma_l \rrbracket$$

Since we are given that $\langle e_1, e_2 \rangle \delta\gamma \Downarrow_t \langle v_{f_1}, v_{f_2} \rangle$ therefore fom E-WI we know that $\exists t_1 < t . e_1 \delta\gamma \Downarrow_{t_1} v_{f_1}$

Since $t_1 < t < T$, therefore we have

$$(p_l, T - t_1, v_{f_1}) \in \llbracket \tau_1 \sigma_l \rrbracket \quad (\text{F-WI1})$$

IH2:

$$(p_l, T, e_2 \delta\gamma) \in \llbracket \tau_2 \sigma_l \rrbracket_{\mathcal{E}}$$

Therefore from Definition 66 we have

$$\forall t_2 < T . e_2 \delta\gamma \Downarrow_{t_2} v_{f_2} \implies (p_l, T - t_2, v_{f_2}) \in \llbracket \tau_2 \sigma_l \rrbracket$$

Since we are given that $\langle e_1, e_2 \rangle \delta\gamma \Downarrow_t \langle v_{f_1}, v_{f_2} \rangle$ therefore fom E-WI we also know that $\exists t_2 < t . e_2 \delta\gamma \Downarrow_{t_2} v_{f_2}$

Since $t_2 < t < T$, therefore we have

$$(p_l, T - t_2, v_{f_2}) \in \llbracket \tau_2 \sigma_l \rrbracket \quad (\text{F-WI2})$$

Applying Lemma 69 on (F-W1) and (F-W2) we get the desired.

17. T-fst:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\tau_1 \& \tau_2)}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{fst}(e) : \tau_1} \text{T-fst}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma) \sigma_l \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma_l \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, (\text{fst}(e)) \delta\gamma) \in \llbracket \tau_1 \sigma_l \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T, v_f . (\text{fst}(e)) \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in \llbracket \tau_1 \sigma_l \rrbracket$$

This means given some $t < T, v_f$ s.t $(\text{fst}(e)) \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_l, T - t, v_f) \in \llbracket \tau_1 \sigma_l \rrbracket \quad (\text{F-F0})$$

IH

$$(p_l, T, e \delta\gamma) \in \llbracket (\tau_1 \& \tau_2) \sigma_l \rrbracket_{\mathcal{E}}$$

This means from Definition 66 we have

$$\forall t_1 < T . e \delta\gamma \Downarrow_{t_1} \langle v_1, v_2 \rangle \implies (p_l, T - t_1, \langle v_1, v_2 \rangle) \in \llbracket (\tau_1 \& \tau_2) \sigma_l \rrbracket$$

Since we know that $(\text{fst}(e)) \delta\gamma \Downarrow_t v_f$ therefore from E-fst we know that $\exists t_1 < t . v_1, v_2 . e \delta\gamma \Downarrow_{t_1} \langle v_1, v_2 \rangle$.

Since $t_1 < t < T$, therefore we have

$$(p_l, T - t_1, \langle v_1, v_2 \rangle) \in \llbracket (\tau_1 \& \tau_2) \sigma_l \rrbracket$$

From Definition 66 we know that

$$(p_l, T -t_1, v_1) \in \llbracket \tau_1 \sigma \iota \rrbracket$$

Finally using Lemma 69 we also have

$$(p_l, T -t, v_1) \in \llbracket \tau_1 \sigma \iota \rrbracket$$

Since from E-fst we know that $v_f = v_1$, therefore we are done.

18. T-snd:

Similar reasoning as in T-fst case above.

19. T-inl:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau_1}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{inl}(e) : \tau_1 \oplus \tau_2} \text{ T-inl}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, \text{inl}(e) \delta \gamma) \in \llbracket (\tau_1 \oplus \tau_2) \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T . \text{inl}(e) \delta \gamma \Downarrow_t \text{inl}(v) \implies (p_l, T -t, \text{inl}(v)) \in \llbracket (\tau_1 \oplus \tau_2) \sigma \iota \rrbracket$$

This means given some $t < T$ s.t $\text{inl}(e) \delta \gamma \Downarrow_t \text{inl}(v)$ it suffices to prove that

$$(p_l, T -t, \text{inl}(v)) \in \llbracket (\tau_1 \oplus \tau_2) \sigma \iota \rrbracket \quad (\text{F-IL0})$$

IH:

$$(p_l, T, e_1 \delta \gamma) \in \llbracket \tau_1 \sigma \iota \rrbracket_{\mathcal{E}}$$

Therefore from Definition 66 we have

$$\forall t_1 < T . e_1 \delta \gamma \Downarrow_{t_1} v_{f1} \implies (p_l, T -t_1, v_{f1}) \in \llbracket \tau_1 \sigma \iota \rrbracket$$

Since we are given that $\text{inl}(e) \delta \gamma \Downarrow_t \text{inl}(v)$ therefore fom E-inl we know that $\exists t_1 < t . e \delta \gamma \Downarrow_{t_1} v$

$$\text{Hence we have } (p_l, T -t_1, v) \in \llbracket \tau_1 \sigma \iota \rrbracket$$

$$\text{From Lemma 69 we get } (p_l, T -t, v) \in \llbracket \tau_1 \sigma \iota \rrbracket$$

And finally from Definition 66 we get (F-IL0)

20. T-inr:

Similar reasoning as in T-inr case above.

21. T-case:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : (\tau_1 \oplus \tau_2) \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : \tau_1 \vdash e_1 : \tau \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2, y : \tau_2 \vdash e_2 : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \text{case } e \text{ of } e_1; e_2 : \tau} \text{ T-case}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma_1 \oplus \Gamma_2) \sigma \iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, (\text{case } e \text{ of } e_1; e_2) \delta \gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T, v_f. (\text{case } e \text{ of } e_1; e_2) \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in \llbracket \tau \sigma \iota \rrbracket$$

This means given some $t < T, v_f$ s.t $(\text{case } e \text{ of } e_1; e_2) \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_l, T - t, v_f) \in \llbracket \tau \sigma \iota \rrbracket \quad (\text{F-C0})$$

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2}. p_{l1} + p_{l2} = p_l$ s.t

$$(p_{l1}, \gamma) \in \llbracket (\Gamma_1) \sigma \iota \rrbracket_{\mathcal{E}} \text{ and } (p_{l2}, \gamma) \in \llbracket (\Gamma_2) \sigma \iota \rrbracket_{\mathcal{E}}$$

IH1

$$(p_{l1}, T, e \delta\gamma) \in \llbracket (\tau_1 \oplus \tau_2) \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 66 we have

$$\forall t' < T . e \delta\gamma \Downarrow_{t'} v_1 \delta\gamma \implies (p_{l1}, T - t', v_1) \in \llbracket (\tau_1 \oplus \tau_2) \sigma \iota \rrbracket$$

Since we know that $(\text{case } e \text{ of } e_1; e_2) \delta\gamma \Downarrow_t v_f$ therefore from E-case we know that $\exists t' < t, v_1. e \delta\gamma \Downarrow_{t'} v_1$.

Since $t' < t < T$, therefore we have

$$(p_{l1}, T - t', v_1) \in \llbracket (\tau_1 \oplus \tau_2) \sigma \iota \rrbracket$$

2 cases arise:

(a) $v_1 = \text{inl}(v)$:

IH2

$$(p_{l2} + p_{l1}, T - t', e_1 \delta\gamma') \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

where

$$\gamma' = \gamma \cup \{x \mapsto v\}$$

This means from Definition 66 we have

$$\forall t_1 < T - t'. e_1 \delta\gamma' \Downarrow_{t_1} v_f \implies (p_{l2}, T - t' - t_1, v_f) \in \llbracket \tau \sigma \iota \rrbracket$$

Since we know that $(\text{case } e \text{ of } e_1; e_2) \delta\gamma \Downarrow_t v_f$ therefore from E-case we know that $\exists t_1. e_1 \delta\gamma' \Downarrow_{t_1} v_f$ where $t_1 = t - t' - 1$.

Since $t_1 = t - t' - 1 < T - t'$ therefore we have

$$(p_{l2}, T - t' - t_1, v_f) \in \llbracket \tau \sigma \iota \rrbracket$$

From Lemma 69 we get

$$(p_{l2} + p_{l1}, T - t, v_f) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

And finally since $p_l = p_{l1} + p_{l2}$ therefore we get

$$(p_l, T - t, v_f) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

And we are done

(b) $v_1 = \text{inr}(v)$:

Similar reasoning as in the inl case above.

22. T-ExpI:

$$\frac{\Psi; \Theta; \Delta; \Omega; . \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; . \vdash !e : !\tau} \text{T-ExpI}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma_l \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma_l \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, !e \delta \gamma) \in \llbracket !\tau \sigma_l \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T. !(e) \delta \gamma \Downarrow_t !(e) \delta \gamma \implies (p_l, T - t, !(e) \delta \gamma) \in \llbracket !\tau \sigma_l \rrbracket$$

This means given some $t < T$ s.t $!(e) \delta \gamma \Downarrow_t !(e) \delta \gamma$ it suffices to prove that

$$(p_l, T - t, !(e) \delta \gamma) \in \llbracket !\tau \sigma_l \rrbracket$$

From Definition 66 it suffices to prove that

$$(0, T - t, e \delta \gamma) \in \llbracket \tau \sigma_l \rrbracket_{\mathcal{E}}$$

$$\underline{\text{IH}}: (0, T - t, e \delta \gamma) \in \llbracket \tau \sigma_l \rrbracket_{\mathcal{E}}$$

We get the desired directly from IH

23. T-ExpE:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : !\tau \quad \Psi; \Theta; \Delta; \Omega; x : \tau; \Gamma_2 \vdash e' : \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \text{let } !x = e \text{ in } e' : \tau'} \text{T-ExpE}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma_1 \oplus \Gamma_2) \sigma_l \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket (\Omega) \sigma_l \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, (\text{let } !x = e \text{ in } e') \delta \gamma) \in \llbracket \tau' \sigma_l \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T, v_f. (\text{let } !x = e \text{ in } e') \delta \gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in \llbracket \tau' \sigma_l \rrbracket$$

This means given some $t < T, v_f$ s.t $(\text{let } !x = e \text{ in } e') \delta \gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_l, T - t, v_f) \in \llbracket \tau' \sigma_l \rrbracket \quad (\text{F-E0})$$

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2}. p_{l1} + p_{l2} = p_l$ s.t

$$(p_{l1}, \gamma) \in \llbracket (\Gamma_1) \sigma_l \rrbracket_{\mathcal{E}} \text{ and } (p_{l2}, \gamma) \in \llbracket (\Gamma_2) \sigma_l \rrbracket_{\mathcal{E}}$$

IH1

$$(p_{l1}, T, e \delta \gamma) \in \llbracket !\tau \sigma_l \rrbracket_{\mathcal{E}}$$

This means from Definition 66 we have

$$\forall t_1 < T. e \delta \gamma \Downarrow_{t_1} !e_1 \delta \gamma \implies (p_{l1}, T - t_1, !e_1 \delta \gamma) \in \llbracket !\tau \sigma_l \rrbracket$$

Since we know that $(\text{let } !x = e \text{ in } e') \delta \gamma \Downarrow_t v_f$ therefore from (E-ExpE) we know that

$$\exists t_1 < t, e_1. e \delta \gamma \Downarrow_{t_1} !e_1 \delta \gamma.$$

Since $t_1 < t < T$, therefore we have

$$(p_{l1}, T - t_1, !e_1 \delta\gamma) \in \llbracket !\tau \sigma\iota \rrbracket_{\mathcal{E}}$$

This means from Definition 66 we have

$$(0, T - t_1, e_1 \delta\gamma) \in \llbracket \tau \rrbracket_{\mathcal{E}} \quad (\text{F-E1})$$

IH2

$$(p_{l2}, T - t_1, e' \delta'\gamma) \in \llbracket \tau' \sigma\iota \rrbracket_{\mathcal{E}}$$

where

$$\delta' = \delta \cup \{x \mapsto e_1\}$$

This means from Definition 66 we have

$$\forall t_2 < T - t_1. e' \delta'\gamma \Downarrow_{t_2} v_f \implies (p_{l2}, T - t_1 - t_2, v_f) \in \llbracket \tau' \sigma\iota \rrbracket$$

Since we know that $(\text{let } !x = e \text{ in } e') \delta\gamma \Downarrow_t v_f$ therefore from (E-ExpE) we know that $\exists t_2. e' \delta'\gamma \Downarrow v_f$ where $t_2 = t - t_1 - 1$.

Since $t_2 = t - t_1 - 1 < T - t_1$, therefore we have

$$(p_{l2}, T - t_1 - t_2, v_f) \in \llbracket \tau' \sigma\iota \rrbracket$$

From Lemma 70 we get

$$(p_{l2} + p_{l1}, T - t, v_f) \in \llbracket \tau' \sigma\iota \rrbracket$$

And finally since $p_l = p_{l1} + p_{l2}$ therefore we get

$$(p_l, T - t, v_f) \in \llbracket \tau' \sigma\iota \rrbracket$$

And we are done

24. T-tabs:

$$\frac{\Psi, \alpha : K; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \Lambda.e : (\forall \alpha : K. \tau)} \text{T-tabs}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma, \sigma\iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma\iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, (\Lambda.e) \delta\gamma) \in \llbracket (\forall \alpha. \tau) \sigma\iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T, v_f. (\Lambda.e) \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in \llbracket (\forall \alpha. \tau) \sigma\iota \rrbracket$$

This means given some $t < T, v_f$ s.t $(\Lambda.e) \delta\gamma \Downarrow_t v_f$. From E-val we know that $t = 0$ and $v_f = (\Lambda.e) \delta\gamma$

Therefore it suffices to prove that

$$(p_l, T, (\Lambda.e) \delta\gamma) \in \llbracket (\forall \alpha. \tau) \sigma\iota \rrbracket$$

From Definition 66 it suffices to prove that

$$\forall \tau', T' < T. (p_l, T', e) \in \llbracket \tau[\tau'/\alpha] \sigma\iota \rrbracket_{\mathcal{E}}$$

This means given some $\tau', T' < T$ it suffices to prove that

$$(p_l, T', e) \in \llbracket \tau[\tau'/\alpha] \sigma\iota \rrbracket_{\mathcal{E}} \quad (\text{F-TAB0})$$

From IH we know that

$$(p_l, T, e \delta\gamma) \in \llbracket \tau \sigma' \iota \rrbracket_{\mathcal{E}}$$

where

$$\sigma' = \gamma \cup \{\alpha \mapsto \tau'\}$$

Therefore from Lemma 70 we get the desired

25. T-tapp:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\forall \alpha. \tau) \quad \Psi; \Theta \Delta \vdash \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e \square : (\tau[\tau'/\alpha])} \text{T-tapp}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, e \square \delta\gamma) \in \llbracket \tau[\tau'/\alpha] \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T, v_f. (e \square) \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in \llbracket \tau[\tau'/\alpha] \sigma \iota \rrbracket$$

This means given some $t < T, v_f$ s.t $(e \square) \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_l, T - t, v_f) \in \llbracket \tau[\tau'/\alpha] \sigma \iota \rrbracket \quad (\text{F-A0})$$

IH

$$(p_l, T, e \delta\gamma) \in \llbracket (\forall \alpha. \tau) \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 66 we have

$$\forall t_1 < T. e \Downarrow_{t_1} \Lambda.e \implies (p_l, T - t_1, \Lambda.e) \in \llbracket (\forall \alpha. \tau) \sigma \iota \rrbracket$$

Since we know that $(e \square) \delta\gamma \Downarrow_t v_f$ therefore from E-tapp we know that $\exists t_1 < t. e \Downarrow_{t_1} \Lambda.e$, therefore we have

$$(p_l, T - t_1, \Lambda.e) \in \llbracket (\forall \alpha. \tau) \sigma \iota \rrbracket$$

Therefore from Definition 66 we have

$$\forall \tau'', T_1 < T - t_1. (p_l, T_1, e) \in \llbracket \tau[\tau''/\alpha] \sigma \iota \rrbracket_{\mathcal{E}} \quad (\text{F-A1})$$

Instantiating (F-A1) with the given τ' and $T - t_1 - 1$ we get

$$(p_l, T - t_1 - 1, e) \in \llbracket \tau[\tau'/\alpha] \sigma \iota \rrbracket_{\mathcal{E}}$$

From Definition 66 we have

$$\forall t_2 < T - t_1 - 1. e \Downarrow_{t_2} v_f \implies (p_l, T - t_1 - t_2 - 1, v_f) \in \llbracket \tau[\tau'/\alpha] \sigma \iota \rrbracket$$

Since we know that $(e \square) \delta\gamma \Downarrow_t v_f$ therefore from E-tapp we know that $\exists t_2. e \Downarrow_{t_2} v_f$ where $t_2 = t - t_1 - 1$

Since $t_2 = t - t_1 - 1 < T - t_1 - 1$, therefore we have

$$(p_l, T - t_1 - t_2 - 1, v_f) \in \llbracket \tau[\tau'/\alpha] \sigma \iota \rrbracket \text{ and we are done.}$$

26. T-iabs:

$$\frac{\Psi; \Theta, i : S; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \Lambda.e : (\forall i : S . \tau)} \text{T-iabs}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma, \sigma_l \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma_l \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, (\Lambda.e) \delta\gamma) \in \llbracket (\forall i.\tau) \sigma_l \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T, v_f. (\Lambda.e) \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in \llbracket (\forall i.\tau) \sigma_l \rrbracket$$

This means given some $t < T, v_f$ s.t $(\Lambda.e) \delta\gamma \Downarrow_t v_f$. From E-val we know that $t = 0$ and $v_f = (\Lambda.e) \delta\gamma$

Therefore it suffices to prove that

$$(p_l, T, (\Lambda.e) \delta\gamma) \in \llbracket (\forall i.\tau) \sigma_l \rrbracket$$

From Definition 66 it suffices to prove that

$$\forall I, T' < T . (p_l, T', e) \in \llbracket \tau[I/i] \sigma_l \rrbracket_{\mathcal{E}}$$

This means given some $I, T' < T$ it suffices to prove that

$$(p_l, T', e) \in \llbracket \tau[I/i] \sigma_l \rrbracket_{\mathcal{E}} \quad (\text{F-IAB0})$$

From IH we know that

$$(p_l, T, e \delta\gamma) \in \llbracket \tau \sigma_l' \rrbracket_{\mathcal{E}}$$

where

$$l' = \gamma \cup \{i \mapsto I\}$$

Therefore from Lemma 70 we get the desired

27. T-iapp:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (\forall i : S . \tau) \quad \Psi; \Theta; \Delta \vdash I : S}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e \square : (\tau[I/i])} \text{T-iapp}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma_l \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma_l \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, e \square \delta\gamma) \in \llbracket \tau[I/i] \sigma_l \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T, v_f. (e \square) \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in \llbracket \tau[I/i] \sigma_l \rrbracket$$

This means given some $t < T, v_f$ s.t $(e \square) \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_l, T - t, v_f) \in \llbracket \tau[I/i] \sigma_l \rrbracket \quad (\text{F-A0})$$

IH

$$(p_l, T, e \delta\gamma) \in \llbracket (\forall i.\tau) \sigma_l \rrbracket_{\mathcal{E}}$$

This means from Definition 66 we have

$$\forall t_1 < T . e \Downarrow_{t_1} \Lambda.e \implies (p_l, T - t_1, \Lambda.e) \in \llbracket (\forall i.\tau) \sigma \iota \rrbracket$$

Since we know that $(e \Downarrow) \delta\gamma \Downarrow_t v_f$ therefore from E-tapp we know that $\exists t_1 < t.e \Downarrow_{t_1} \Lambda.e$, therefore we have

$$(p_l, T - t_1, \Lambda.e) \in \llbracket (\forall i.\tau) \sigma \iota \rrbracket$$

Therefore from Definition 66 we have

$$\forall I, T_1 < T - t_1. (p_l, T_1, e) \in \llbracket \tau[I/i] \sigma \iota \rrbracket_{\mathcal{E}} \quad (\text{F-IAP1})$$

Instantiating (F-IAP1) with the given I and $T - t_1 - 1$ we get

$$(p_l, T - t_1 - 1, e) \in \llbracket \tau[I/i] \sigma \iota \rrbracket_{\mathcal{E}}$$

From Definition 66 we have

$$\forall t_2 < T - t_1 - 1. e \Downarrow_{t_2} v_f \implies (p_l, T - t_1 - t_2 - 1, v_f) \in \llbracket \tau[I/i] \sigma \iota \rrbracket$$

Since we know that $(e \Downarrow) \delta\gamma \Downarrow_t v_f$ therefore from E-iapp we know that $\exists t_2. e \Downarrow_{t_2} v_f$ where $t_2 = t - t_1 - 1$

Since $t_2 = t - t_1 - 1 < T - t_1 - 1$, therefore we have

$$(p_l, T - t_1 - t_2 - 1, v_f) \in \llbracket \tau[I/i] \sigma \iota \rrbracket \text{ and we are done.}$$

28. T-CI:

$$\frac{\Psi; \Theta; \Delta, c; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \Lambda.e : (c \Rightarrow \tau)} \text{ T-CI}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$ and $\models \Delta \iota$

To prove: $(p_l, T, \Lambda.e \delta\gamma) \in \llbracket (c \Rightarrow \tau) \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall v, t < T . \Lambda.e \delta\gamma \Downarrow_t v \implies (p_l, T - t, v) \in \llbracket (c \Rightarrow \tau) \sigma \iota \rrbracket$$

This means given some $v, t < T$ s.t $\Lambda.e \delta\gamma \Downarrow_t v$ and from (E-val) we know that $v = \Lambda.e \delta\gamma$ and $t = 0$ therefore it suffices to prove that

$$(p_l, T, \Lambda.e \delta\gamma) \in \llbracket (c \Rightarrow \tau) \sigma \iota \rrbracket$$

From Definition 66 it suffices to prove that

$$\cdot \models c \iota \implies (p_l, T, e \delta\gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

This means given that $\cdot \models c \iota$ it suffices to prove that

$$(p_l, T, e \delta\gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

$$\underline{\text{IH}} (p_l, T, e \delta\gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

We get the desired directly from IH

29. T-CE:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (c \Rightarrow \tau) \quad \Theta; \Delta \models c}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e \square : \tau} \text{T-CE}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$ and $\models \Delta \iota$

To prove: $(p_l, T, e \square \delta \gamma) \in \llbracket (\tau) \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall v_f, t < T . (e \square) \delta \gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in \llbracket (\tau) \sigma \iota \rrbracket$$

This means given some $v_f, t < T$ s.t $(e \square) \delta \gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_l, T - t, v_f) \in \llbracket (\tau) \sigma \iota \rrbracket \quad (\text{F-Tap0})$$

IH

$$(p_l, T, e \delta \gamma) \in \llbracket (c \Rightarrow \tau) \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 66 we have

$$\forall v', t' < T . e \delta \gamma \Downarrow_{t'} v' \implies (p_l + p_m, v') \in \llbracket (c \Rightarrow \tau) \sigma \iota \rrbracket$$

Since we know that $(e \square) \delta \gamma \Downarrow_t v_f$ therefore from E-CE we know that $\exists t' < t . e \delta \gamma \Downarrow_{t'} \Lambda . e'$, and since $t' < t < T$ therefore we have

$$(p_l, T - t', \Lambda . e') \in \llbracket (c \Rightarrow \tau) \sigma \iota \rrbracket$$

Therefore from Definition 66 we have

$$. \models c \iota \implies (p_l, T - t', e' \delta \gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

Since we are given $\Theta; \Delta \models c$ and $. \models \Delta \iota$ therefore we know that $. \models c \iota$. Hence we get

$$(p_l, T - t', e' \delta \gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 66 we have

$$\forall v'_f, t'' < T - t' . (e') \delta \gamma \Downarrow_{t''} v'_f \implies (p_l, T - t' - t'', v'_f) \in \llbracket (\tau) \sigma \iota \rrbracket \quad (\text{F-CE1})$$

Since from E-CE we know that $e' \delta \gamma \Downarrow_t v_f$ therefore we know that $\exists t'' . e' \delta \gamma \Downarrow_{t''} v_f$ s.t $t = t' + t'' + 1$

Therefore instantiating (F-CE1) with the given v_f and t'' we get

$$(p_l, T - t' - t'', v_f) \in \llbracket (\tau) \sigma \iota \rrbracket$$

Since $t = t' + t'' + 1$ therefore from Lemma 69 we get the desired.

30. T-CAndI:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \quad \Theta; \Delta \models c}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : (c \& \tau)} \text{T-CAndI}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, e \delta \gamma) \in \llbracket c \& \tau \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall v_f, t < T . e \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f \delta\gamma) \in \llbracket c\&\tau \sigma_l \rrbracket$$

This means given some $v_f, t < T$ s.t $e \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_l, T - t, v_f) \in \llbracket c\&\tau \sigma_l \rrbracket$$

From Definition 66 it suffices to prove that

$$. \models c_l \wedge (p_l, T - t, v_f) \in \llbracket \tau \sigma_l \rrbracket$$

Since we are given that $. \models \Delta_l$ and $\Theta; \Delta \models c$ therefore it suffices to prove that

$$(p_l, T - t, v_f) \in \llbracket \tau \sigma_l \rrbracket \quad (\text{F-CAI0})$$

$$\underline{\text{IH}}: (p_l, T, e \delta\gamma) \in \llbracket \tau \sigma_l \rrbracket_{\mathcal{E}}$$

This means from Definition 66 we have

$$\forall t' < T . e \delta\gamma \Downarrow_{t'} v_f \implies (p_l, T - t', v_f) \in \llbracket \tau \sigma_l \rrbracket$$

Since we are given that $e \delta\gamma \Downarrow_t v_f$ therefore we get

$$(p_l, T - t, v_f) \in \llbracket \tau \sigma_l \rrbracket \quad (\text{F-CAI1})$$

We get the desired from (F-CAI1)

31. T-CAndE:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : (c\&\tau) \quad \Psi; \Theta; \Delta, c; \Omega; \Gamma_2, x : \tau \vdash e' : \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \text{clet } x = e \text{ in } e' : \tau'} \text{ T-CAndE}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma_1 \oplus \Gamma_2) \sigma_l \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket (\Omega) \sigma_l \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, (\text{clet } x = e \text{ in } e') \delta\gamma) \in \llbracket \tau' \sigma_l \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall v_f, t < T . (\text{clet } x = e \text{ in } e') \delta\gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in \llbracket \tau' \sigma_l \rrbracket$$

This means given soem $v_f, t < T$ s.t $(\text{clet } x = e \text{ in } e') \delta\gamma \Downarrow_t v_f$ it suffices to prove that

$$(p_l, T - t, v_f) \in \llbracket \tau' \sigma_l \rrbracket \quad (\text{F-CAE0})$$

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2}. p_{l1} + p_{l2} = p_l$ s.t

$$(p_{l1}, T, \gamma) \in \llbracket (\Gamma_1) \sigma_l \rrbracket_{\mathcal{E}} \text{ and } (p_{l2}, T, \gamma) \in \llbracket (\Gamma_2) \sigma_l \rrbracket_{\mathcal{E}}$$

IH1

$$(p_{l1}, T, e \delta\gamma) \in \llbracket c\&\tau \sigma_l \rrbracket_{\mathcal{E}}$$

This means from Definition 66 we have

$$\forall t_1 < T . e \delta\gamma \Downarrow_{t_1} v_1 \implies (p_{l1}, T - t_1, v_1) \in \llbracket c\&\tau \sigma_l \rrbracket_{\mathcal{E}}$$

Since we know that $(\text{clet } x = e \text{ in } e') \delta\gamma \Downarrow_t v_f$ therefore from E-CAndE we know that

$\exists v_1, t_1 < t. e \delta\gamma \Downarrow_{t_1} v_1$. Therefore we have

$$(p_{l1}, T - t_1, v_1) \in \llbracket c\&\tau \sigma_l \rrbracket$$

Therefore from Definition 66 we have

$$\cdot \models c_l \wedge (p_{l1}, T - t_1, v_1) \in \llbracket \tau \sigma_l \rrbracket \quad (\text{F-CAE1})$$

IH2

$$(p_{l2} + p_{l1}, T, e' \delta\gamma') \in \llbracket \tau' \sigma_l \rrbracket_{\mathcal{E}}$$

where

$$\gamma' = \gamma \cup \{x \mapsto v_1\}$$

This means from Definition 66 we have

$$\forall t_2 < T. e' \delta\gamma' \Downarrow_{t_2} v_f \implies (p_{l2} + p_{l1}, T - t_2, v_f) \in \llbracket \tau' \sigma_l \rrbracket$$

Since we know that $(\text{clet } x = e \text{ in } e') \delta\gamma \Downarrow_t v_f$ therefore from E-CAndE we know that $\exists t_2 < t. e' \delta'\gamma \Downarrow_{t_2} v_f$.

Therefore we have

$$(p_{l2} + p_{l1}, T - t_2, v_f) \in \llbracket \tau' \sigma_l \rrbracket$$

Since $p_l = p_{l1} + p_{l2}$ therefore we get

$$(p_l, T - t_2, v_f) \in \llbracket \tau' \sigma_l \rrbracket$$

And finally from From Lemma 69 we get

$$(p_l, T - t, v_f) \in \llbracket \tau' \sigma_l \rrbracket$$

And we are done.

32. T-fix:

$$\frac{\Psi; \Theta; \Delta; \Omega, x : \tau; \cdot \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \cdot \vdash \text{fix } x.e : \tau} \text{ T-fix}$$

Given: $(0, T, \gamma) \in \llbracket \cdot \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma_l \rrbracket_{\mathcal{E}}$

To prove: $(0, T, (\text{fix } x.e) \delta\gamma) \in \llbracket \tau \sigma_l \rrbracket_{\mathcal{E}} \quad (\text{F-FX0})$

We induct on T

Base case, $T = 1$:

It suffices to prove that $(0, 1, (\text{fix } x.e) \delta\gamma) \in \llbracket \tau \sigma_l \rrbracket$

This means from Definition 66 it suffices to prove

$$\forall t < 1. (\text{fix } x.e) \delta\gamma \Downarrow_t v \implies (0, 1 - t, v) \in \llbracket \tau \rrbracket$$

This further means that given $t < 1$ s.t. $(\text{fix } x.e) \delta\gamma \Downarrow_t v$ it suffices to prove that

$$(0, 1 - t, v) \in \llbracket \tau \rrbracket$$

Since from E-fix we know that minimum value of t can be 1 therefore $t < 1$ is not possible and the goal holds vacuously.

Inductive case:

IH: $(0, T - 1, (\text{fix } x.e) \delta\gamma) \in \llbracket \tau \sigma_l \rrbracket_{\mathcal{E}}$

Therefore from Definition 67 we have

$$(0, T - 1, \delta') \in \llbracket \Omega, x : \tau \sigma \iota \rrbracket_{\mathcal{E}} \text{ where } \delta' = \delta \cup \{x \mapsto \text{fix}x.e \delta\}$$

Applying Definition 66 on (F-FX0) it suffices to prove that

$$\forall t < T. (\text{fix}x.e) \delta \gamma \Downarrow_t v_f \implies (0, T - t, v_f) \in \llbracket \tau \sigma \iota \rrbracket$$

This means given some $t < T$ s.t $\text{fix}x.e \delta \gamma \Downarrow_t v_f$ it suffices to prove that

$$(0, T - t, v_f) \in \llbracket \tau \sigma \iota \rrbracket \quad (\text{F-FX0.0})$$

Now from IH of outer induction we have

$$(0, T - 1, e \delta' \gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 66 we have

$$\forall t' < T - 1. e \delta' \gamma \Downarrow_{t'} v_f \implies (0, T - 1 - t', v_f) \in \llbracket \tau \sigma \iota \rrbracket$$

Since we know that $\text{fix}x.e \delta \gamma \Downarrow_t v_f$ therefore from E-fix we know that $\exists t' = t - 1$ s.t $e \delta' \gamma \Downarrow_{t'} v_f$

Since $t < T$ therefore $t' = t - 1 < T - 1$ hence we have

$$(0, T - t, v_f) \in \llbracket \tau \sigma \iota \rrbracket$$

Therefore we are done

33. T-ret:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{ret } e : \mathbb{M} 0 \tau} \text{ T-ret}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, \text{ret } e \delta \gamma) \in \llbracket \mathbb{M} 0 \tau \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T, v_f. (\text{ret } e) \delta \gamma \Downarrow_t v_f \implies (p_l, T - t, v_f) \in \llbracket \mathbb{M} 0 \tau \sigma \iota \rrbracket$$

It means we are given some $t < T, v_f$ s.t $(\text{ret } e) \delta \gamma \Downarrow_t v_f$. From E-val we know that $t = 0$ and $v_f = (\text{ret } e) \delta \gamma$.

Therefore it suffices to prove that

$$(p_l, T, (\text{ret } e) \delta \gamma) \in \llbracket \mathbb{M} 0 \tau \sigma \iota \rrbracket$$

From Definition 66 it further suffices to prove that

$$\forall t' < T. (\text{ret } e) \delta \gamma \Downarrow_{t'}^{n'} v_f \implies \exists p'. n' + p' \leq p_l \wedge (p', T - t', v_f) \in \llbracket \tau \sigma \iota \rrbracket$$

This means given some $t' < T$ s.t $(\text{ret } e) \delta \gamma \Downarrow_{t'}^{n'} v_f$ it suffices to prove that

$$\exists p'. n' + p' \leq p_l \wedge (p', T - t', v_f) \in \llbracket \tau \sigma \iota \rrbracket$$

From (E-ret) we know that $n' = 0$ therefore we choose p' as p_l and it suffices to prove that

$$(p_l, T - t', v_f) \in \llbracket \tau \sigma \iota \rrbracket \quad (\text{F-R0})$$

IH

$$(p_l, T, e \delta\gamma) \in \llbracket \tau \sigma\iota \rrbracket_{\mathcal{E}}$$

This means from Definition 66 we have

$$\forall t_1 < T . (e) \delta\gamma \Downarrow_t v_f \implies (p_l, T - t_1, v_f) \in \llbracket \tau \sigma\iota \rrbracket$$

Since we know that $(\text{ret } e) \delta\gamma \Downarrow_{t'}^0 v_f$ therefore from (E-ret) we know that $\exists t_1 . e \delta\gamma \Downarrow_{t_1} v_f$

Since $t_1 < t < T$ therefore we have

$$(p_l, T - t_1, v_f) \in \llbracket \tau \sigma\iota \rrbracket$$

And finally from Lemma 69 we get

$$(p_l, T - t, v_f) \in \llbracket \tau \sigma\iota \rrbracket$$

and we are done.

34. T-bind:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e_1 : \mathbb{M} n_1 \tau_1 \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : \tau_1 \vdash e_2 : \mathbb{M} n_2 \tau_2 \quad \Theta \vdash n_1 : \mathbb{R}^+ \quad \Theta \vdash n_2 : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \text{bind } x = e_1 \text{ in } e_2 : \mathbb{M}(n_1 + n_2) \tau_2} \text{T-bind}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma_1 \oplus \Gamma_2) \sigma\iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket (\Omega) \sigma\iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, \text{bind } x = e_1 \text{ in } e_2 \delta\gamma) \in \llbracket \mathbb{M}(n_1 + n_2) \tau_2 \sigma\iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T, v. (\text{bind } x = e_1 \text{ in } e_2) \delta\gamma \Downarrow_t v \implies (p_l, T - t, v) \in \llbracket \mathbb{M}(n_1 + n_2) \tau_2 \sigma\iota \rrbracket$$

This means given some $t < T, v$ s.t $(\text{bind } x = e_1 \text{ in } e_2) \delta\gamma \Downarrow_t v$. From E-val we know that $t = 0$ and $v = (\text{bind } x = e_1 \text{ in } e_2 \delta\gamma)$

Therefore it suffices to prove that

$$(p_l, T, (\text{bind } x = e_1 \text{ in } e_2 \delta\gamma)) \in \llbracket \mathbb{M}(n_1 + n_2) \tau_2 \sigma\iota \rrbracket$$

This means from Definition 66 it suffices to prove that

$$\forall t' < T, v_f. (\text{bind } x = e_1 \text{ in } e_2 \delta\gamma) \Downarrow_{t'}^{s'} v_f \implies \exists p'. s' + p' \leq p_l + n \wedge (p', T - t', v_f) \in \llbracket \tau_2 \sigma\iota \rrbracket$$

This means given some $t' < T, v_f$ s.t $(\text{bind } x = e_1 \text{ in } e_2 \delta\gamma) \Downarrow_{t'}^{s'} v_f$ and we need to prove that $\exists p'. s' + p' \leq p_l + n \wedge (p', T - t', v_f) \in \llbracket \tau_2 \sigma\iota \rrbracket$ (F-B0)

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2}. p_{l1} + p_{l2} = p_l$ s.t

$$(p_{l1}, \gamma) \in \llbracket (\Gamma_1) \sigma\iota \rrbracket_{\mathcal{E}} \text{ and } (p_{l2}, \gamma) \in \llbracket (\Gamma_2) \sigma\iota \rrbracket_{\mathcal{E}}$$

IH1

$$(p_{l1}, T, e_1 \delta\gamma) \in \llbracket \mathbb{M}(n_1) \tau_1 \sigma\iota \rrbracket_{\mathcal{E}}$$

From Definition 66 it means we have

$$\forall t_1 < T . (e_1) \delta\gamma \Downarrow_{t_1} v_{m1} \implies (p_{l1}, T - t_1, v_{m1}) \in \llbracket \mathbb{M}(n_1) \tau_1 \sigma\iota \rrbracket$$

Since we know that $(\text{bind } x = e_1 \text{ in } e_2) \delta\gamma \Downarrow_{t'}^{s'} v_f$ therefore from E-bind we know that $\exists t_1 < t', v_{m1}.(e_1) \delta\gamma \Downarrow_{t_1} v_{m1}$.

Since $t_1 < t' < T$, therefore we have

$$(p_{l1}, T - t_1, v_{m1}) \in \llbracket \mathbb{M}(n_1) \tau_1 \sigma \iota \rrbracket \quad (\text{F-B1})$$

This means from Definition 66 we are given that

$$\forall t'_1 < T - t_1. v_{m1} \Downarrow_{t'_1}^{s_1} v_1 \implies \exists p'_1. s_1 + p'_1 \leq p_{l1} + n_1 \wedge (p'_1, T - t_1 - t'_1, v_1) \in \llbracket \tau_1 \sigma \iota \rrbracket$$

Since we know that $(\text{bind } x = e_1 \text{ in } e_2) \delta\gamma \Downarrow_{t'} v_f$ therefore from E-bind we know that $\exists t'_1 < t - t_1. (e_1) \delta\gamma \Downarrow_{t'_1}^{s_1} v_1$.

Since $t'_1 < t - t_1 < T - t_1$ therefore means we have

$$\exists p'_1. s_1 + p'_1 \leq p_{l1} + n_1 \wedge (p'_1, T - t_1 - t'_1, v_1) \in \llbracket \tau_1 \sigma \iota \rrbracket \quad (\text{F-B1})$$

IH2

$$(p_{l2} + p'_1, T - t_1 - t'_1, e_2 \delta\gamma \cup \{x \mapsto v_1\}) \in \llbracket \mathbb{M}(n_2) \tau_2 \sigma \iota \rrbracket_{\mathcal{E}}$$

From Definition 66 it means we have

$$\forall t_2 < T - t_1 - t'_1. (e_2) \delta\gamma \cup \{x \mapsto v_1\} \Downarrow_{t_2} v_{m2} \implies (p_{l2} + p'_1, T - t_1 - t'_1 - t_2, v_{m2}) \in \llbracket \mathbb{M}(n_2) \tau_2 \sigma \iota \rrbracket$$

Since we know that $(\text{bind } x = e_1 \text{ in } e_2) \delta\gamma \Downarrow_{t'}^{s'} v_f$ therefore from E-bind we know that $\exists t_2 < t' - t_1 - t'_1. (e_2) \delta\gamma \cup \{x \mapsto v_1\} \Downarrow_{t_2} v_{m2}$.

Since $t_2 < t' - t_1 - t'_1 < T - t_1 - t'_1$ therefore we have

$$(p_{l2} + p'_1, T - t_1 - t'_1 - t_2, v_{m2}) \in \llbracket \mathbb{M}(n_2) \tau_2 \sigma \iota \rrbracket$$

This means from Definition 66 we are given that

$$\forall t'_2 < T - t_1 - t'_1 - t_2. v_{m2} \Downarrow_{t'_2}^{s_2} v_2 \implies \exists p'_2. s_2 + p'_2 \leq p_{l2} + p'_1 + n_2 \wedge (p'_2, T - t_1 - t'_1 - t_2, v_2) \in \llbracket \tau_2 \sigma \iota \rrbracket$$

Since we know that $(\text{bind } x = e_1 \text{ in } e_2) \delta\gamma \Downarrow_{t'}^{s'} v_f$ therefore from E-bind we know that $\exists t'_2 < t' - t_1 - t'_1 - t_2. s_2, v_2. v_{m2} \Downarrow_{t'_2}^{s_2} v_2$.

This means we have

$$\exists p'_2. s_2 + p'_2 \leq p_{l2} + p'_1 + n_2 \wedge (p'_2, T - t_1 - t'_1 - t_2, v_2) \in \llbracket \tau_2 \sigma \iota \rrbracket \quad (\text{F-B2})$$

In order to prove (F-B0) we choose p' as p'_2 and it suffices to prove

(a) $s' + p'_2 \leq p_l + n$:

Since from (F-B2) we know that

$$s_2 + p'_2 \leq p_{l2} + p'_1 + n_2$$

Adding s_1 on both sides we get

$$s_1 + s_2 + p'_2 \leq p_{l2} + s_1 + p'_1 + n_2$$

Since from (F-B1) we know that

$$s_1 + p'_1 \leq p_{l1} + n_1$$

therefore we also have

$$s_1 + s_2 + p'_2 \leq p_{l2} + p_{l1} + n_1 + n_2$$

And finally since we know that $n = n_1 + n_2$, $s' = s_1 + s_2$ and $p_l = p_{l1} + p_{l2}$ therefore we get the desired

(b) $(p'_2, T - t_1 - t'_1 - t_2 - t'_2, v_f) \in \llbracket \tau_2 \sigma \iota \rrbracket$:

From E-bind we know that $v_f = v_2$ therefore we get the desired from (F-B2)

35. T-tick:

$$\frac{\Theta \vdash n : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \uparrow^n : \mathbb{M} n \mathbf{1}} \text{ T-tick}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, \uparrow^n \delta \gamma) \in \llbracket \mathbb{M} n \mathbf{1} \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T, v. (\uparrow^n) \delta \gamma \downarrow_t v \implies (p_l, T - t, v) \in \llbracket \mathbb{M} n \mathbf{1} \sigma \iota \rrbracket$$

This means we are given some $t < T, v$ s.t. $(\uparrow^n) \delta \gamma \downarrow_t v$. From E-val we know that $t = 0$ and $v = (\uparrow^n) \delta \gamma$

Therefore it suffices to prove that

$$(p_l, T, (\uparrow^n) \delta \gamma) \in \llbracket \mathbb{M} n \mathbf{1} \sigma \iota \rrbracket$$

From Definition 66 it suffices to prove that

$$\forall t' < T. (\uparrow^n) \delta \gamma \downarrow_{t'}^{n'} () \implies \exists p'. n' + p' \leq p_l + n \wedge (p', T - t', ()) \in \llbracket \mathbf{1} \rrbracket$$

This means given some $t' < T$ s.t. $(\uparrow^n) \delta \gamma \downarrow_{t'}^{n'} ()$ it suffices to prove that

$$\exists p'. n' + p' \leq p_l + n \wedge (p', T - t', ()) \in \llbracket \mathbf{1} \rrbracket$$

From (E-tick) we know that $n' = n$ therefore we choose p' as p_l and it suffices to prove that

$$(p_l, T - t', ()) \in \llbracket \mathbf{1} \rrbracket$$

We get this directly from Definition 66

36. T-release:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e_1 : [n_1] \tau_1 \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : \tau_1 \vdash e_2 : \mathbb{M}(n_1 + n_2) \tau_2 \quad \Theta \vdash n_1 : \mathbb{R}^+ \quad \Theta \vdash n_2 : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \text{release } x = e_1 \text{ in } e_2 : \mathbb{M} n_2 \tau_2} \text{ T-release}$$

Given: $(p_l, T, \gamma) \in \llbracket (\Gamma_1 \oplus \Gamma_2) \sigma \iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket (\Omega) \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, \text{release } x = e_1 \text{ in } e_2 \delta \gamma) \in \llbracket \mathbb{M}(n_2) \tau_2 \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T, v. (\text{release } x = e_1 \text{ in } e_2) \delta\gamma \Downarrow_t v \implies (p_l, T - t, v) \in \llbracket \mathbb{M}(n_2) \tau_2 \sigma \iota \rrbracket$$

This means given some $t < T, v$ s.t $(\text{release } x = e_1 \text{ in } e_2) \delta\gamma \Downarrow (\text{release } x = e_1 \text{ in } e_2) \delta\gamma$. From E-val we know that $t = 0$ and $v = (\text{release } x = e_1 \text{ in } e_2) \delta\gamma$

Therefore it suffices to prove that

$$(p_l, T, (\text{release } x = e_1 \text{ in } e_2) \delta\gamma) \in \llbracket \mathbb{M}(n_2) \tau_2 \sigma \iota \rrbracket$$

This means from Definition 66 it suffices to prove that

$$\forall t' < T, v_f. (\text{release } x = e_1 \text{ in } e_2) \delta\gamma \Downarrow_{v'}^{s'} v_f \implies \exists p'. s' + p' \leq p_l + n_2 \wedge (p', T - t', v_f) \in \llbracket \tau_2 \sigma \iota \rrbracket$$

This means given some $t' < T, v_f$ s.t $(\text{release } x = e_1 \text{ in } e_2) \delta\gamma \Downarrow_{v'}^{s'} v_f$ and we need to prove that

$$\exists p'. s' + p' \leq p_l + n_2 \wedge (p', T - t', v_f) \in \llbracket \tau_2 \sigma \iota \rrbracket \quad (\text{F-R0})$$

From Definition 67 and Definition 65 we know that $\exists p_{l1}, p_{l2}. p_{l1} + p_{l2} = p_l$ s.t

$$(p_{l1}, \gamma) \in \llbracket (\Gamma_1) \sigma \iota \rrbracket_{\mathcal{E}} \text{ and } (p_{l2}, \gamma) \in \llbracket (\Gamma_2) \sigma \iota \rrbracket_{\mathcal{E}}$$

IH1

$$(p_{l1}, T, e_1 \delta\gamma) \in \llbracket [n_1] \tau_1 \sigma \iota \rrbracket_{\mathcal{E}}$$

From Definition 66 it means we have

$$\forall t_1 < T. (e_1) \delta\gamma \Downarrow_{t_1} v_1 \implies (p_{l1}, T - t_1, v_1) \in \llbracket [n_1] \tau_1 \sigma \iota \rrbracket$$

Since we know that $(\text{release } x = e_1 \text{ in } e_2) \delta\gamma \Downarrow_{v'}^{s'} v_f$ therefore from E-rel we know that $\exists t_1 < t'. (e_1) \delta\gamma \Downarrow_{t_1} v_1$.

Since $t_1 < t' < T$, therefore we have

$$(p_{l1}, T - t_1, v_1) \in \llbracket [n_1] \tau_1 \sigma \iota \rrbracket$$

This means from Definition 66 we have

$$\exists p'_1. p'_1 + n_1 \leq p_{l1} \wedge (p'_1, T - t_1, v_1) \in \llbracket \tau_1 \rrbracket \quad (\text{F-R1})$$

IH2

$$(p_{l2} + p'_1, T - t_1, e_2 \delta\gamma \cup \{x \mapsto v_1\}) \in \llbracket \mathbb{M}(n_1 + n_2) \tau_2 \sigma \iota \rrbracket_{\mathcal{E}}$$

From Definition 66 it means we have

$$\forall t_2 < T - t_1. (e_2) \delta\gamma \cup \{x \mapsto v_1\} \Downarrow_{t_2} v_{m2} \cup \{x \mapsto v_1\} \implies (p_{l2} + p'_1, T - t_1 - t_2, v_{m2}) \in \llbracket \mathbb{M}(n_1 + n_2) \tau_2 \sigma \iota \rrbracket$$

Since we know that $(\text{release } x = e_1 \text{ in } e_2) \delta\gamma \Downarrow_{v'}^{s'} v_f$ therefore from E-rel we know that $\exists t_2 < t - t_1. (e_2) \delta\gamma \cup \{x \mapsto v_1\} \Downarrow_{t_2} v_{m2}$. This means we have

$$(p_{l2} + p'_1, T - t_1 - t_2, v_{m2}) \in \llbracket \mathbb{M}(n_1 + n_2) \tau_2 \sigma \iota \rrbracket$$

This means from Definition 66 we are given that

$$\forall t'_2 < T - t_1 - t_2. v_{m2} \Downarrow_{t'_2}^{s'_2} v_2 \implies \exists p'_2. s'_2 + p'_2 \leq p_{l2} + p'_1 + n_1 + n_2 \wedge (p'_2, T - t_1 - t_2 - t'_2, v_2) \in \llbracket \tau_2 \sigma \iota \rrbracket$$

Since we know that (release $x = e_1$ in e_2) $\delta\gamma \Downarrow_{t'}^{s'} v_f$ therefore from E-rel we know that $\exists t'_2.v_{m2} \Downarrow_{t'_2}^{s'_2} v_2$ s.t. $t'_2 = t' - t_1 - t_2 - 1$

Since $t'_2 = t' - t_1 - t_2 < T - t_1 - t_2$, therefore we have

$$\exists p'_2.s_2 + p'_2 \leq p_{l2} + p'_1 + n_1 + n_2 \wedge (p'_2, T - t_1 - t_2 - t'_2, v_2) \in \llbracket \tau_2 \sigma \iota \rrbracket \quad (\text{F-R2})$$

In order to prove (F-R0) we choose p' as p'_2 and it suffices to prove

(a) $s' + p'_2 \leq p_l + n_2$:

Since from (F-R2) we know that

$$s_2 + p'_2 \leq p_{l2} + p'_1 + n_1 + n_2$$

Since from (F-R1) we know that

$$p'_1 + n_1 \leq p_{l1}$$

therefore we also have

$$s_2 + p'_2 \leq p_{l2} + p_{l1} + p_{m1} + n_2$$

And finally since we know that $s' = s_2$, $p_l = p_{l1} + p_{l2}$ and $0 = p_{m1}$ therefore we get the desired

(b) $(p'_2, T - t_1 - t_2 - t'_2, v_f) \in \llbracket \tau_2 \sigma \iota \rrbracket$:

From E-rel we know that $v_f = v_2$ therefore we get the desired from (F-R2)

37. T-store:

$$\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau \quad \Theta \vdash n : \mathbb{R}^+}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash \text{store } e : \mathbb{M} n ([n] \tau)} \text{T-store}$$

Given: $(p_l, T, \gamma) \in \llbracket \Gamma \sigma \iota \rrbracket_{\mathcal{E}}$, $(0, T, \delta) \in \llbracket \Omega \sigma \iota \rrbracket_{\mathcal{E}}$

To prove: $(p_l, T, \text{store } e \delta\gamma) \in \llbracket \mathbb{M} n ([n] \tau) \sigma \iota \rrbracket_{\mathcal{E}}$

From Definition 66 it suffices to prove that

$$\forall t < T, v. (\text{store } e) \delta\gamma \Downarrow_t v \implies (p_l, T - t, v) \in \llbracket \mathbb{M} n ([n] \tau) \sigma \iota \rrbracket$$

This means we are given some $t < T, v$ s.t. $(\text{store } e) \delta\gamma \Downarrow_t v$. From E-val we know that $t = 0$ and $v = (\text{store } e) \delta\gamma$

Therefore it suffices to prove that

$$(p_l, T, (\text{store } e) \delta\gamma) \in \llbracket \mathbb{M} n ([n] \tau) \sigma \iota \rrbracket$$

From Definition 66 it suffices to prove that

$$\forall t' < T, v_f, n'. (\text{store } e) \delta\gamma \Downarrow_{t'}^{n'} v_f \implies \exists p'. n' + p' \leq p_l \wedge (p', T - t', v_f) \in \llbracket [n] \tau \sigma \iota \rrbracket$$

This means given some $t' < T, v_f$ s.t. $(\text{store } e) \delta\gamma \Downarrow_{t'}^{n'} v_f$ it suffices to prove that

$$\exists p'. n' + p' \leq p_l \wedge (p', T - t', v_f) \in \llbracket [n] \tau \sigma \iota \rrbracket$$

From (E-store) we know that $n' = 0$ therefore we choose p' as $p_l + n$ and it suffices to prove that

$$(p_l + n, T - t', v_f) \in \llbracket [n] \tau \sigma \iota \rrbracket$$

This further means that from Definition 66 we have

$$\exists p'' . p'' + n \leq p_l + n \wedge (p'', T - t', v_f) \in \llbracket \tau \sigma \iota \rrbracket$$

We choose p'' as p_l and it suffices to prove that

$$(p_l, T - t', v_f) \in \llbracket \tau \sigma \iota \rrbracket \quad (\text{F-S0})$$

IH

$$(p_l, T, e \delta\gamma) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$$

This means from Definition 66 we have

$$\forall t_1 < T . (e) \delta\gamma \Downarrow_{t_1} v_f \implies (p_l, T - t_1, v_f) \in \llbracket \tau \sigma \iota \rrbracket$$

Since we know that $(\text{store } e) \delta\gamma \Downarrow_{t'}^0 v_f$ therefore from (E-store) we know that $\exists t_1 < t' . e \delta\gamma \Downarrow_{t_1} v_f$

Since $t_1 < t' < T$ therefore we have

$$(p_l, T - t_1, v_f) \in \llbracket \tau \sigma \iota \rrbracket$$

and finally from Lemma 69 we have

$$(p_l, T - t', v_f) \in \llbracket \tau \sigma \iota \rrbracket$$

□

Lemma 72 (Value subtyping lemma). $\forall \Psi, \Theta, \tau \in \text{Type}, \tau'$.

$$\Psi; \Theta; \Delta \vdash \tau <: \tau' \wedge . \models \Delta \iota \implies \llbracket \tau \sigma \iota \rrbracket \subseteq \llbracket \tau' \sigma \iota \rrbracket$$

Proof. Proof by induction on the $\Psi; \Theta; \Delta \vdash \tau <: \tau'$ relation

1. sub-refl:

$$\frac{}{\Psi; \Theta; \Delta \vdash \tau <: \tau} \text{sub-refl}$$

$$\text{To prove: } \forall (p, T, v) \in \llbracket \tau \sigma \iota \rrbracket \implies (p, T, v) \in \llbracket \tau \sigma \iota \rrbracket$$

Trivial

2. sub-arrow:

$$\frac{\Psi; \Theta; \Delta \vdash \tau'_1 <: \tau_1 \quad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau'_2}{\Psi; \Theta; \Delta \vdash \tau_1 \multimap \tau_2 <: \tau'_1 \multimap \tau'_2} \text{sub-arrow}$$

$$\text{To prove: } \forall (p, T, \lambda x . e) \in \llbracket (\tau_1 \multimap \tau_2) \sigma \iota \rrbracket \implies (p, T, \lambda x . e) \in \llbracket (\tau'_1 \multimap \tau'_2) \sigma \iota \rrbracket$$

This means given some $(p, T, \lambda x . e) \in \llbracket (\tau_1 \multimap \tau_2) \sigma \iota \rrbracket$ we need to prove

$$(p, T, \lambda x . e) \in \llbracket (\tau'_1 \multimap \tau'_2) \sigma \iota \rrbracket$$

From Definition 66 we are given that

$$\forall T' < T, p', e' . (p', T', e') \in \llbracket \tau_1 \sigma \iota \rrbracket_{\mathcal{E}} \implies (p + p', T', e[e'/x]) \in \llbracket \tau_2 \sigma \iota \rrbracket_{\mathcal{E}} \quad (\text{F-SL0})$$

Also from Definition 66 it suffices to prove that

$$\forall T'' < T, p'', e''. (p'', T'', e'') \in \llbracket \tau'_1 \sigma \iota \rrbracket_{\mathcal{E}} \implies (p + p'', T'', e[e''/x]) \in \llbracket \tau'_2 \sigma \iota \rrbracket_{\mathcal{E}}$$

This means given some $T'' < T, p'', e''$ s.t. $(p'', T'', e'') \in \llbracket \tau'_1 \sigma \iota \rrbracket$ we need to prove $(p + p'', T'', e[e''/x]) \in \llbracket \tau'_2 \sigma \iota \rrbracket_{\mathcal{E}}$ (F-SL1)

$$\underline{\text{IH1}}: \llbracket \tau'_1 \sigma \iota \rrbracket \subseteq \llbracket \tau_1 \sigma \iota \rrbracket$$

Since we have $(p'', T'', e'') \in \llbracket \tau'_1 \sigma \iota \rrbracket$ therefore from IH1 we also have $(p'', T'', e'') \in \llbracket \tau_1 \sigma \iota \rrbracket$

Therefore instantiating (F-SL0) with p'', T'', e'' we get

$$(p + p'', T'', e[e''/x]) \in \llbracket \tau_2 \sigma \iota \rrbracket_{\mathcal{E}}$$

And finally from Lemma 73 we get the desired

3. sub-tensor:

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau'_1 \quad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau'_2}{\Psi; \Theta; \Delta \vdash \tau_1 \otimes \tau_2 <: \tau'_1 \otimes \tau'_2} \text{ sub-tensor}$$

To prove: $\forall (p, T, \langle\langle v_1, v_2 \rangle\rangle) \in \llbracket (\tau_1 \otimes \tau_2) \sigma \iota \rrbracket \implies (p, T, \langle\langle v_1, v_2 \rangle\rangle) \in \llbracket (\tau'_1 \otimes \tau'_2) \sigma \iota \rrbracket$

This means given $(p, T, \langle\langle v_1, v_2 \rangle\rangle) \in \llbracket (\tau_1 \otimes \tau_2) \sigma \iota \rrbracket$

It suffices prove that

$$(p, T, \langle\langle v_1, v_2 \rangle\rangle) \in \llbracket (\tau'_1 \otimes \tau'_2) \sigma \iota \rrbracket$$

This means from Definition 66 we are given that

$$\exists p_1, p_2. p_1 + p_2 \leq p \wedge (p_1, T, v_1) \in \llbracket \tau_1 \sigma \iota \rrbracket \wedge (p_2, T, v_2) \in \llbracket \tau_2 \sigma \iota \rrbracket$$

Also from Definition 66 it suffices to prove that

$$\exists p'_1, p'_2. p'_1 + p'_2 \leq p \wedge (p'_1, T, v_1) \in \llbracket \tau'_1 \sigma \iota \rrbracket \wedge (p'_2, T, v_2) \in \llbracket \tau'_2 \sigma \iota \rrbracket$$

$$\underline{\text{IH1}} \llbracket (\tau_1) \sigma \iota \rrbracket \subseteq \llbracket (\tau'_1) \sigma \iota \rrbracket$$

$$\underline{\text{IH2}} \llbracket (\tau_2) \sigma \iota \rrbracket \subseteq \llbracket (\tau'_2) \sigma \iota \rrbracket$$

Choosing p_1 for p'_1 and p_2 for p'_2 we get the desired from IH1 and IH2

4. sub-with:

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau'_1 \quad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau'_2}{\Psi; \Theta; \Delta \vdash \tau_1 \& \tau_2 <: \tau'_1 \& \tau'_2} \text{ sub-with}$$

To prove: $\forall (p, T, \langle v_1, v_2 \rangle) \in \llbracket (\tau_1 \& \tau_2) \sigma \iota \rrbracket \implies (p, T, \langle v_1, v_2 \rangle) \in \llbracket (\tau'_1 \& \tau'_2) \sigma \iota \rrbracket$

This means given $(p, T, \langle v_1, v_2 \rangle) \in \llbracket (\tau_1 \& \tau_2) \sigma \iota \rrbracket$

It suffices prove that

$$(p, T, \langle v_1, v_2 \rangle) \in \llbracket (\tau'_1 \& \tau'_2) \sigma \iota \rrbracket$$

This means from Definition 66 we are given that

$$(p, T, v_1) \in \llbracket \tau_1 \sigma \iota \rrbracket \wedge (p, T, v_2) \in \llbracket \tau_2 \sigma \iota \rrbracket \quad (\text{F-SW0})$$

Also from Definition 66 it suffices to prove that

$$(p, T, v_1) \in \llbracket \tau'_1 \sigma \iota \rrbracket \wedge (p, T, v_2) \in \llbracket \tau'_2 \sigma \iota \rrbracket$$

$$\underline{\text{IH1}} \llbracket (\tau_1) \sigma \iota \rrbracket \subseteq \llbracket (\tau'_1) \sigma \iota \rrbracket$$

$$\underline{\text{IH2}} \llbracket (\tau_2) \sigma \iota \rrbracket \subseteq \llbracket (\tau'_2) \sigma \iota \rrbracket$$

We get the desired from (F-SW0), IH1 and IH2

5. sub-sum:

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau'_1 \quad \Psi; \Theta; \Delta \vdash \tau_2 <: \tau'_2}{\Psi; \Theta; \Delta \vdash \tau_1 \oplus \tau_2 <: \tau'_1 \oplus \tau'_2} \text{sub-sum}$$

$$\text{To prove: } \forall (p, T, \langle v_1, v_2 \rangle) \in \llbracket (\tau_1 \oplus \tau_2) \sigma \iota \rrbracket \implies (p, T, \langle v_1, v_2 \rangle) \in \llbracket (\tau'_1 \oplus \tau'_2) \sigma \iota \rrbracket$$

This means given $(p, T, v) \in \llbracket (\tau_1 \oplus \tau_2) \sigma \iota \rrbracket$

It suffices prove that

$$(p, T, v) \in \llbracket (\tau'_1 \oplus \tau'_2) \sigma \iota \rrbracket$$

This means from Definition 66 two cases arise

(a) $v = \text{inl}(v')$:

$$\text{This means from Definition 66 we have } (p, T, v') \in \llbracket \tau_1 \sigma \iota \rrbracket \quad (\text{F-SS0})$$

Also from Definition 66 it suffices to prove that

$$(p, T, v') \in \llbracket \tau'_1 \sigma \iota \rrbracket$$

$$\underline{\text{IH}} \llbracket (\tau_1) \sigma \iota \rrbracket \subseteq \llbracket (\tau'_1) \sigma \iota \rrbracket$$

We get the desired from (F-SS0), IH

(b) $v = \text{inr}(v')$:

Symmetric reasoning as in the inl case

6. sub-list:

$$\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta \vdash L^n \tau <: L^n \tau'} \text{sub-list}$$

$$\text{To prove: } \forall (p, T, v) \in \llbracket L^n \tau \sigma \iota \rrbracket. (p, T, v) \in \llbracket L^n \tau' \sigma \iota \rrbracket$$

This means given $(p, T, v) \in \llbracket L^n \tau \sigma \iota \rrbracket$ and we need to prove

$$(p, T, v) \in \llbracket L^n \tau' \sigma \iota \rrbracket$$

We induct on $(p, T, v) \in \llbracket L^n \tau \sigma \iota \rrbracket$

(a) $(p, T, nil) \in \llbracket L^0 \tau \sigma \iota \rrbracket$:

We need to prove $(p, T, nil) \in \llbracket L^0 \tau' \sigma \iota \rrbracket$

We get this directly from Definition 66

(b) $(p, T, v' :: l') \in \llbracket L^{m+1} \tau \sigma \iota \rrbracket$:

In this case we are given $(p, T, v' :: l') \in \llbracket L^{m+1} \tau \sigma \iota \rrbracket$

and we need to prove $(p, T, v' :: l') \in \llbracket L^{m+1} \tau' \sigma \iota \rrbracket$

This means from Definition 66 are given

$$\exists p_1, p_2. p_1 + p_2 \leq p \wedge (p_1, T, v') \in \llbracket \tau \sigma \iota \rrbracket \wedge (p_2, T, l') \in \llbracket L^m \tau \sigma \iota \rrbracket \quad (\text{Sub-List0})$$

Similarly from Definition 66 we need to prove that

$$\exists p'_1, p'_2. p'_1 + p'_2 \leq p \wedge (p'_1, T, v') \in \llbracket \tau' \sigma \iota \rrbracket \wedge (p_2, T, l') \in \llbracket L^m \tau' \sigma \iota \rrbracket$$

We choose p'_1 as p_1 and p'_2 as p_2 and we get the desired from (Sub-List0) IH of outer induction and IH of inner induction

7. sub-exist:

$$\frac{\Psi; \Theta, s; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta \vdash \exists s. \tau <: \exists s. \tau'} \text{ sub-exist}$$

To prove: $\forall (p, T, v) \in \llbracket \exists s. \tau \sigma \iota \rrbracket. (p, T, v) \in \llbracket \exists s. \tau' \sigma \iota \rrbracket$

This means given some $(p, T, v) \in \llbracket \exists s. \tau \sigma \iota \rrbracket$ we need to prove

$(p, T, v) \in \llbracket \exists s. \tau' \sigma \iota \rrbracket$

From Definition 66 we are given that

$$\exists s'. (p, T, v) \in \llbracket \tau \sigma \iota[s'/s] \rrbracket \quad (\text{F-exist0})$$

$$\underline{\text{IH}}: \llbracket (\tau) \sigma \iota \cup \{s \mapsto s'\} \rrbracket \subseteq \llbracket (\tau') \sigma \iota \cup \{s \mapsto s'\} \rrbracket$$

Also from Definition 66 it suffices to prove that

$$\exists s''. (p, T, v) \in \llbracket \tau' \sigma \iota[s''/s] \rrbracket$$

We choose s'' as s' and we get the desired from IH

8. sub-potential:

$$\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau' \quad \Psi; \Theta; \Delta \vdash n' \leq n}{\Psi; \Theta; \Delta \vdash [n] \tau <: [n'] \tau'} \text{ sub-potential}$$

To prove: $\forall (p, T, v) \in \llbracket [n] \tau \sigma \iota \rrbracket. (p, T, v) \in \llbracket [n'] \tau' \sigma \iota \rrbracket$

This means given $(p, T, v) \in \llbracket [n] \tau \sigma \iota \rrbracket$ and we need to prove

$(p, T, v) \in \llbracket [n'] \tau' \sigma \iota \rrbracket$

This means from Definition 66 we are given

$$\exists p'. p' + n \leq p \wedge (p', T, v) \in \llbracket \tau \sigma \iota \rrbracket \quad (\text{F-SP0})$$

And we need to prove

$$\exists p''. p'' + n' \leq p \wedge (p'', T, v) \in \llbracket \tau' \sigma \iota \rrbracket \quad (\text{F-SP1})$$

In order to prove (F-SP1) we choose p'' as p'

Since from (F-SP0) we know that $p' + n \leq p$ and we are given that $n' \leq n$ therefore we also have $p' + n' \leq p$

$$\underline{\text{IH}} \llbracket \tau \sigma \iota \rrbracket \subseteq \llbracket \tau' \sigma \iota \rrbracket$$

We get the desired directly from IH

9. sub-monad:

$$\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau' \quad \Psi; \Theta; \Delta \vdash n \leq n'}{\Psi; \Theta; \Delta \vdash \mathbb{M} n \tau <: \mathbb{M} n' \tau'} \text{ sub-monad}$$

To prove: $\forall (p, T, v) \in \llbracket \mathbb{M} n \tau \sigma \iota \rrbracket. (p, T, v) \in \llbracket \mathbb{M} n' \tau' \sigma \iota \rrbracket$

This means given $(p, T, v) \in \llbracket \mathbb{M} n \tau \sigma \iota \rrbracket$ and we need to prove $(p, T, v) \in \llbracket \mathbb{M} n' \tau' \sigma \iota \rrbracket$

This means from Definition 66 we are given

$$\forall t' < T, n_1, v'. v \Downarrow_{v'}^{n_1} v' \implies \exists p'. n_1 + p' \leq p + n \wedge (p', T - t', v') \in \llbracket \tau \sigma \iota \rrbracket \quad (\text{F-SM0})$$

Again from Definition 66 we need to prove that

$$\forall t'' < T, n_2, v''. v \Downarrow_{v''}^{n_2} v'' \implies \exists p''. n_2 + p'' \leq p + n' \wedge (p'', T - t'', v'') \in \llbracket \tau' \sigma \iota \rrbracket$$

This means given some $t'' < T, v'', n_2$ s.t $v \Downarrow_{v''}^{n_2} v''$ it suffices to prove that

$$\exists p''. n_2 + p'' \leq p + n' \wedge (p'', T - t'', v'') \in \llbracket \tau' \sigma \iota \rrbracket \quad (\text{F-SM1})$$

Instantiating (F-SM0) with t'', n_2, v'' Since $v \Downarrow_{v''}^{n_2} v''$ therefore from (F-SM0) we know that

$$\exists p'. n_2 + p' \leq p + n \wedge (p', T - t'', v'') \in \llbracket \tau \sigma \iota \rrbracket \quad (\text{F-SM2})$$

$$\underline{\text{IH}} \llbracket \tau \sigma \iota \rrbracket \subseteq \llbracket \tau' \sigma \iota \rrbracket$$

In order to prove (F-SM1) we choose p'' as p' and we need to prove

$$(a) \ n_2 + p'' \leq p + n':$$

Since we are given that $n \leq n'$ therefore we get the desired from (F-SM2)

$$(b) \ (p', v'') \in \llbracket \tau' \sigma \iota \rrbracket$$

We get this directly from IH and (F-SM2)

10. sub-Exp:

$$\frac{\Psi; \Theta; \Delta \vdash \tau <: \tau'}{\Psi; \Theta; \Delta \vdash !\tau <: !\tau'} \text{ sub-Exp}$$

To prove: $\forall (p, T, v) \in \llbracket !\tau \ \sigma\iota \rrbracket . (p, T, v) \in \llbracket !\tau' \ \sigma\iota \rrbracket$

This means given $(p, T, !e) \in \llbracket !\tau \ \sigma\iota \rrbracket$ and we need to prove
 $(p, T, !e) \in \llbracket !\tau' \ \sigma\iota \rrbracket$

This means from Definition 66 we are given

$$(0, T, e) \in \llbracket \tau \ \sigma\iota \rrbracket_{\mathcal{E}} \quad (\text{F-SE0})$$

Again from Definition 66 we need to prove that

$$(0, T, e) \in \llbracket \tau' \ \sigma\iota \rrbracket_{\mathcal{E}} \quad (\text{F-SE1})$$

$$\underline{\text{IH}} \llbracket \tau \ \sigma\iota \rrbracket \subseteq \llbracket \tau' \ \sigma\iota \rrbracket$$

Therefore from (F-SE0) and IH we get $(0, T, e) \in \llbracket \tau' \ \sigma\iota \rrbracket$ and we are done.

11. sub-typePoly:

$$\frac{\Psi, \alpha; \Theta; \Delta \vdash \tau_1 <: \tau_2}{\Psi; \Theta; \Delta \vdash \forall \alpha. \tau_1 <: \forall i. \tau_2} \text{sub-typePoly}$$

To prove: $\forall (p, T, \Lambda.e) \in \llbracket (\forall i. \tau_1) \ \sigma\iota \rrbracket . (p, T, \Lambda.e) \in \llbracket (\forall i. \tau_2) \ \sigma\iota \rrbracket$

This means given some $(p, T, \Lambda.e) \in \llbracket (\forall \alpha. \tau_1) \ \sigma\iota \rrbracket$ we need to prove
 $(p, T, \Lambda.e) \in \llbracket (\forall \alpha. \tau_2) \ \sigma\iota \rrbracket$

From Definition 66 we are given that

$$\forall \tau, T' < T . (p, T', e) \in \llbracket \tau_1[\tau/\alpha] \rrbracket_{\mathcal{E}} \quad (\text{F-STP0})$$

Also from Definition 66 it suffices to prove that

$$\forall \tau', T'' < T . (p, T'', e) \in \llbracket \tau_2[\tau'/\alpha] \rrbracket_{\mathcal{E}}$$

This means given some $\tau', T'' < T$ and we need to prove

$$(p, T'', e) \in \llbracket \tau_2[\tau'/\alpha] \rrbracket_{\mathcal{E}} \quad (\text{F-STP1})$$

$$\underline{\text{IH}}: \llbracket (\tau_1) \ \sigma\iota \cup \{\alpha \mapsto \tau'\} \rrbracket \subseteq \llbracket (\tau_2) \ \sigma\iota \cup \{\alpha \mapsto \tau'\} \rrbracket$$

Instantiating (F-STP0) with τ', T'' we get

$$(p, T'', e) \in \llbracket \tau_1[\tau'/\alpha] \rrbracket_{\mathcal{E}}$$

and finally from IH we get the desired.

12. sub-indexPoly:

$$\frac{\Psi; \Theta, i; \Delta \vdash \tau_1 <: \tau_2}{\Psi; \Theta; \Delta \vdash \forall i. \tau_1 <: \forall i. \tau_2} \text{sub-indexPoly}$$

To prove: $\forall (p, T, \Lambda i.e) \in \llbracket (\forall i. \tau_1) \ \sigma\iota \rrbracket . (p, T, \Lambda i.e) \in \llbracket (\forall i. \tau_2) \ \sigma\iota \rrbracket$

This means given some $(p, T, \Lambda i.e) \in \llbracket (\forall i. \tau_1) \ \sigma\iota \rrbracket$ we need to prove

$$(p, T, \Lambda i.e) \in \llbracket (\forall i.\tau_2) \sigma\iota \rrbracket$$

From Definition 66 we are given that

$$\forall I, T' < T . (p, T', e) \in \llbracket \tau_1[I/i] \rrbracket_{\mathcal{E}} \quad (\text{F-SIP0})$$

Also from Definition 66 it suffices to prove that

$$\forall I', T'' < T . (p, T'', e) \in \llbracket \tau_2[I'/i] \rrbracket_{\mathcal{E}}$$

This means given some $I', T'' < T$ and we need to prove

$$(p, T'', e) \in \llbracket \tau_2[I'/i] \rrbracket_{\mathcal{E}} \quad (\text{F-SIP1})$$

$$\underline{\text{IH}}: \llbracket (\tau_1) \sigma\iota \cup \{i \mapsto I'\} \rrbracket \subseteq \llbracket (\tau_2) \sigma\iota \cup \{i \mapsto I'\} \rrbracket$$

Instantiating (F-SIP0) with I', T'' we get

$$(p, T'', e) \in \llbracket \tau_1[I'/i] \rrbracket_{\mathcal{E}}$$

and finally from IH we get the desired

13. sub-constraint:

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau_2 \quad \Theta; \Delta \models c_2 \implies c_1}{\Psi; \Theta; \Delta \vdash c_1 \implies \tau_1 <: c_2 \implies \tau_2} \text{ sub-constraint}$$

To prove: $\forall (p, T, \Lambda.e) \in \llbracket (c_1 \implies \tau_1) \sigma\iota \rrbracket . (p, T, \Lambda.e) \in \llbracket (c_2 \implies \tau_2) \sigma\iota \rrbracket$

This means given some $(p, T, \Lambda.e) \in \llbracket (c_1 \implies \tau_1) \sigma\iota \rrbracket$ we need to prove

$$(p, T, \Lambda.e) \in \llbracket (c_2 \implies \tau_2) \sigma\iota \rrbracket$$

From Definition 66 we are given that

$$. \models c_1\iota \implies (p, T, e) \in \llbracket \tau_1 \sigma\iota \rrbracket_{\mathcal{E}} \quad (\text{F-SC0})$$

Also from Definition 66 it suffices to prove that

$$. \models c_2\iota \implies (p, T, e) \in \llbracket \tau_2 \sigma\iota \rrbracket_{\mathcal{E}}$$

This means given some $. \models c_2\iota$ and we need to prove

$$(p, T, e) \in \llbracket \tau_2 \sigma\iota \rrbracket_{\mathcal{E}} \quad (\text{F-SC1})$$

Since we are given that $\Theta; \Delta \models c_2 \implies c_1$ therefore we know that $. \models c_1\iota$

Hence from (F-SC0) we have

$$(p, T, e) \in \llbracket \tau_1 \sigma\iota \rrbracket_{\mathcal{E}} \quad (\text{F-SC2})$$

$$\underline{\text{IH}}: \llbracket (\tau_1) \sigma\iota \rrbracket \subseteq \llbracket (\tau_2) \sigma\iota \rrbracket$$

Therefore we get the desired from IH and (F-SC2)

14. sub-CAnd:

$$\frac{\Psi; \Theta; \Delta \vdash \tau_1 <: \tau_2 \quad \Theta; \Delta \models c_1 \implies c_2}{\Psi; \Theta; \Delta \vdash c_1 \& \tau_1 <: c_2 \& \tau_2} \text{ sub-CAnd}$$

To prove: $\forall (p, v) \in \llbracket (c_1 \& \tau_1) \sigma \iota \rrbracket. (p, v) \in \llbracket (c_2 \& \tau_2) \sigma \iota \rrbracket$

This means given some $(p, v) \in \llbracket (c_1 \& \tau_1) \sigma \iota \rrbracket$ we need to prove $(p, v) \in \llbracket (c_2 \& \tau_2) \sigma \iota \rrbracket$

From Definition 66 we are given that

$$. \models c_1 \iota \wedge (p, e) \in \llbracket \tau_1 \sigma \iota \rrbracket_{\mathcal{E}} \quad (\text{F-SCA0})$$

Also from Definition 66 it suffices to prove that

$$. \models c_2 \iota \wedge (p, e) \in \llbracket \tau_2 \sigma \iota \rrbracket_{\mathcal{E}}$$

Since we are given that $\Theta; \Delta \models c_2 \implies c_1$ and $. \models c_1 \iota$ therefore we also know that $. \models c_2 \iota$

Also from (F-SCA0) we have $(p, e) \in \llbracket \tau_1 \sigma \iota \rrbracket_{\mathcal{E}}$ (F-SCA1)

$$\underline{\text{IH}}: \llbracket (\tau_1) \sigma \iota \rrbracket \subseteq \llbracket (\tau_2) \sigma \iota \rrbracket$$

Therefore we get the desired from IH and (F-SCA1)

15. sub-familyAbs:

$$\frac{\Psi; \Theta, i : S \vdash \tau <: \tau'}{\Psi; \Theta \vdash \lambda_t i : S . \tau <: \lambda_t i : S . \tau'} \text{ sub-familyAbs}$$

To prove:

$$\forall f \in \llbracket \lambda_t i : S . \tau \sigma \iota \rrbracket. f \in \llbracket \lambda_t i : S . \tau' \sigma \iota \rrbracket$$

This means given $f \in \llbracket \lambda_t i : S . \tau \sigma \iota \rrbracket$ and we need to prove

$$f \in \llbracket \lambda_t i : S . \tau' \sigma \iota \rrbracket$$

This means from Definition 66 we are given

$$\forall I. f I \in \llbracket \tau [I/i] \sigma \iota \rrbracket \quad (\text{F-SFAbs0})$$

This means from Definition 66 we need to prove

$$\forall I'. f I' \in \llbracket \tau' [I'/i] \sigma \iota \rrbracket$$

This further means that given some I' we need to prove

$$f I' \in \llbracket \tau' [I'/i] \sigma \iota \rrbracket \quad (\text{F-SFAbs1})$$

Instantiating (F-SFAbs0) with I' we get

$$f I' \in \llbracket \tau [I'/i] \sigma \iota \rrbracket$$

From IH we know that $\llbracket \tau \sigma \iota \cup \{i \mapsto I' \iota\} \rrbracket \subseteq \llbracket \tau' \sigma \iota \cup \{i \mapsto I' \iota\} \rrbracket$

And this completes the proof.

16. Sub-tfamilyApp1:

$$\frac{}{\Psi; \Theta; \Delta \vdash \lambda_{ti} : S . \tau I <: \tau[I/i]} \text{sub-familyApp1}$$

To prove:

$$\forall (p, T, v) \in \llbracket \lambda_{ti} : S . \tau I \sigma \iota \rrbracket . (p, T, v) \in \llbracket \tau[I/i] \sigma \iota \rrbracket$$

This means given $(p, T, v) \in \llbracket \lambda_{ti} : S . \tau I \sigma \iota \rrbracket$ and we need to prove $(p, T, v) \in \llbracket \tau[I/i] \sigma \iota \rrbracket$

This means from Definition 66 we are given

$$(p, T, v) \in \llbracket \lambda_{ti} : S . \tau \rrbracket I \sigma \iota$$

This further means that we have

$$(p, T, v) \in f I \sigma \iota \text{ where } f I \sigma \iota = \llbracket \tau[I/i] \sigma \iota \rrbracket$$

This means we have $(p, T, v) \in \llbracket \tau[I/i] \sigma \iota \rrbracket$

And this completes the proof.

17. Sub-tfamilyApp2:

$$\frac{}{\Psi; \Theta; \Delta \vdash \tau[I/i] <: \lambda_{ti} : S . \tau I} \text{sub-familyApp2}$$

To prove: $\forall (p, T, v) \in \llbracket \tau[I/i] \sigma \iota \rrbracket . (p, T, v) \in \llbracket \lambda_{ti} : S . \tau I \sigma \iota \rrbracket$

This means given $(p, T, v) \in \llbracket \tau[I/i] \sigma \iota \rrbracket$ (Sub-tF0)

And we need to prove

$$(p, T, v) \in \llbracket \lambda_{ti} : S . \tau I \sigma \iota \rrbracket$$

This means from Definition 66 it suffices to prove that

$$(p, T, v) \in \llbracket \lambda_{ti} : S . \tau \rrbracket I \sigma \iota$$

It further suffices to prove that

$$(p, T, v) \in f I \sigma \iota \text{ where } f I \sigma \iota = \llbracket \tau[I/i] \sigma \iota \rrbracket$$

which means we need to show that

$$(p, T, v) \in \llbracket \tau[I/i] \sigma \iota \rrbracket$$

We get this directly from (Sub-tF0)

□

Lemma 73 (Expression subtyping lemma). $\forall \Psi, \Theta, \tau, \tau'.$

$$\Psi; \Theta \vdash \tau <: \tau' \implies \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}} \subseteq \llbracket \tau' \sigma \iota \rrbracket_{\mathcal{E}}$$

Proof. To prove: $\forall(p, T, e) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}} \implies (p, T, e) \in \llbracket \tau' \sigma \iota \rrbracket_{\mathcal{E}}$

This means given some $(p, T, e) \in \llbracket \tau \sigma \iota \rrbracket_{\mathcal{E}}$ it suffices to prove that $(p, T, e) \in \llbracket \tau' \sigma \iota \rrbracket_{\mathcal{E}}$

This means from Definition 66 we are given

$$\forall t < T, v.e \downarrow_t v \implies (p, T - t, v) \in \llbracket \tau \sigma \iota \rrbracket \quad (\text{S-E0})$$

Similarly from Definition 66 it suffices to prove that

$$\forall t' < T, v'.e \downarrow_{t'} v' \implies (p, T - t', v') \in \llbracket \tau' \sigma \iota \rrbracket$$

This means given some $t' < T, v'$ s.t $e \downarrow_{t'} v'$ it suffices to prove that $(p, T - t', v') \in \llbracket \tau' \sigma \iota \rrbracket$

Instantiating (S-E0) with t', v' we get $(p, T - t', v') \in \llbracket \tau \sigma \iota \rrbracket$

And finally from Lemma 72 we get the desired. □

Lemma 74 (Γ subtyping lemma). $\forall \Psi, \Theta, \Gamma_1, \Gamma_2, \sigma, \iota.$

$$\Psi; \Theta \vdash \Gamma_1 <: \Gamma_2 \implies \llbracket \Gamma_1 \sigma \iota \rrbracket \subseteq \llbracket \Gamma_2 \sigma \iota \rrbracket$$

Proof. Proof by induction on $\Psi; \Theta \vdash \Gamma_1 <: \Gamma_2$

1. sub-lBase:

$$\frac{}{\Psi; \Theta \vdash \Gamma <: .} \text{ sub-lBase}$$

To prove: $\forall(p, T, \gamma) \in \llbracket \Gamma_1 \sigma \iota \rrbracket_{\mathcal{E}}. (p, T, \gamma) \in \llbracket . \rrbracket_{\mathcal{E}}$

This means given some $(p, T, \gamma) \in \llbracket \Gamma_1 \sigma \iota \rrbracket_{\mathcal{E}}$ it suffices to prove that $(p, T, \gamma) \in \llbracket . \rrbracket_{\mathcal{E}}$

From Definition 67 it suffices to prove that

$$\exists f : \mathcal{V}ars \rightarrow \mathcal{P}ots. (\forall x \in \text{dom}(.). (f(x), T, \gamma(x)) \in \llbracket \Gamma(x) \rrbracket_{\mathcal{E}}) \wedge (\sum_{x \in \text{dom}(.)} f(x) \leq p)$$

We choose f as a constant function $f' = 0$ and we get the desired

2. sub-lInd:

$$\frac{x : \tau' \in \Gamma_1 \quad \Psi; \Theta \vdash \tau' <: \tau \quad \Psi; \Theta \vdash \Gamma_1/x <: \Gamma_2}{\Psi; \Theta \vdash \Gamma_1 <: \Gamma_2, x : \tau} \text{ sub-lBase}$$

To prove: $\forall(p, T, \gamma) \in \llbracket \Gamma_1 \sigma \iota \rrbracket_{\mathcal{E}}. (p, T, \gamma) \in \llbracket \Gamma_2, x : \tau \rrbracket_{\mathcal{E}}$

This means given some $(p, T, \gamma) \in \llbracket \Gamma_1 \sigma \iota \rrbracket_{\mathcal{E}}$ it suffices to prove that $(p, T, \gamma) \in \llbracket \Gamma_2, x : \tau \rrbracket_{\mathcal{E}}$

This means from Definition 67 we are given that

$$\exists f : \mathcal{V}ars \rightarrow \mathcal{P}ots.$$

$$(\forall x \in \text{dom}(\Gamma_1). (f(x), T, \gamma(x)) \in \llbracket \Gamma(x) \rrbracket_{\mathcal{E}}) \quad (\text{L0})$$

$$(\sum_{x \in \text{dom}(\Gamma_1)} f(x) \leq p) \quad (\text{L1})$$

Similarly from Definition 67 it suffices to prove that

$$\exists f' : \mathcal{V}ars \rightarrow \mathcal{P}ots. (\forall y \in \text{dom}(\Gamma_2, x : \tau). (f'(y), T, \gamma(y)) \in \llbracket (\Gamma_2, x : \tau)(y) \rrbracket_{\mathcal{E}}) \wedge (\sum_{y \in \text{dom}(\Gamma_2, x : \tau)} f'(y) \leq p)$$

We choose f' as f and it suffices to prove that

(a) $\forall y \in \text{dom}(\Gamma_2, x : \tau). (f(y), T, \gamma(y)) \in \llbracket (\Gamma_2, x : \tau)(y) \rrbracket_{\mathcal{E}}$:

This means given some $y \in \text{dom}(\Gamma_2, x : \tau)$ it suffices to prove that $(f(y), T, \gamma(y)) \in \llbracket \tau_2 \rrbracket_{\mathcal{E}}$ where say $(\Gamma_2, x : \tau)(y) = \tau_2$

From Lemma 75 we know that

$$y : \tau_1 \in \Gamma_1 \wedge \Psi; \Theta \vdash \tau_1 <: \tau_2$$

By instantiating (L0) with the given y

$$(f(y), T, \gamma(y)) \in \llbracket \tau_1 \rrbracket_{\mathcal{E}}$$

Finally from Lemma 73 we also get $(f(y), T, \gamma(y)) \in \llbracket \tau_2 \rrbracket_{\mathcal{E}}$

And we are done

(b) $(\sum_{y \in \text{dom}(\Gamma_2, x : \tau)} f(y) \leq p)$:

From (L1) we know that $(\sum_{x \in \text{dom}(\Gamma_1)} f(x) \leq p)$ and since from Lemma 75 we know that $\text{dom}(\Gamma_2, x : \tau) \subseteq \text{dom}(\Gamma_1)$ therefore we also have

$$(\sum_{y \in \text{dom}(\Gamma_2, x : \tau)} f(y) \leq p)$$

□

Lemma 75 (Γ Subtyping: domain containment). $\forall p, \gamma, \Gamma_1, \Gamma_2.$

$$\Psi; \Theta \vdash \Gamma_1 <: \Gamma_2 \implies \forall x : \tau \in \Gamma_2. x : \tau' \in \Gamma_1 \wedge \Psi; \Theta \vdash \tau' <: \tau$$

Proof. Proof by induction on $\Psi; \Theta \vdash \Gamma_1 <: \Gamma_2$

1. sub-lBase:

$$\frac{}{\Psi; \Theta \vdash \Gamma_1 <: .} \text{sub-lBase}$$

To prove: $\forall x : \tau' \in (.). x : \tau \in \Gamma_1 \wedge \Psi; \Theta \vdash \tau' <: \tau$

Trivial

2. sub-lInd:

$$\frac{x : \tau' \in \Gamma_1 \quad \Psi; \Theta \vdash \tau' <: \tau \quad \Psi; \Theta \vdash \Gamma_1/x <: \Gamma_2}{\Psi; \Theta \vdash \Gamma_1 <: \Gamma_2, x : \tau_x} \text{sub-lBase}$$

To prove: $\forall y : \tau_1 \in (\Gamma_2, x : \tau_x). y : \tau \in \Gamma_1 \wedge \Psi; \Theta \vdash \tau' <: \tau$

This means given some $y : \tau \in (\Gamma_2, x : \tau_x)$ it suffices to prove that

$$y : \tau \in \Gamma_1 \wedge \Psi; \Theta \vdash \tau' <: \tau$$

The following cases arise:

- $y = x$:

In this case we are given that $x : \tau' \in \Gamma_1 \wedge \Psi; \Theta \vdash \tau' <: \tau$

Therefore we are done

- $y \neq x$:

Since we are given that $\Psi; \Theta \vdash \Gamma_1/x <: \Gamma_2$ therefore we get the desired from IH

□

Lemma 76 (Ω subtyping lemma). $\forall \Psi, \Theta, \Omega_1, \Omega_2, \sigma, \iota.$

$$\Psi; \Theta \vdash \Omega_1 <: \Omega_2 \implies \llbracket \Omega_1 \sigma \iota \rrbracket \subseteq \llbracket \Omega_2 \sigma \iota \rrbracket$$

Proof. Proof by induction on $\Psi; \Theta \vdash \Omega_1 <: \Omega_2$

1. sub-lBase:

$$\frac{}{\Psi; \Theta \vdash \Omega <: .} \text{sub-mBase}$$

To prove: $\forall (0, T, \delta) \in \llbracket \Omega_1 \sigma \iota \rrbracket_{\mathcal{E}}. (0, T, \delta) \in \llbracket . \rrbracket_{\mathcal{E}}$

This means given some $(0, T, \delta) \in \llbracket \Omega_1 \sigma \iota \rrbracket_{\mathcal{E}}$ it suffices to prove that $(0, T, \delta) \in \llbracket . \rrbracket_{\mathcal{E}}$

We get the desired directly from Definition 67

2. sub-lInd:

$$\frac{x : \tau' \in \Omega_1 \quad \Psi; \Theta \vdash \tau' <: \tau \quad \Psi; \Theta \vdash \Omega_1/x <: \Omega_2}{\Psi; \Theta \vdash \Omega_1 <: \Omega_2, x : \tau} \text{sub-mInd}$$

To prove: $\forall (0, T, \delta) \in \llbracket \Omega_1 \sigma \iota \rrbracket_{\mathcal{E}}. (0, T, \delta) \in \llbracket \Omega_2, x : \tau \rrbracket_{\mathcal{E}}$

This means given some $(0, T, \delta) \in \llbracket \Omega_1 \sigma \iota \rrbracket_{\mathcal{E}}$ it suffices to prove that $(0, T, \delta) \in \llbracket \Omega_2, x : \tau \rrbracket_{\mathcal{E}}$

This means from Definition 67 we are given that

$$(\forall x : \tau \in \Omega_1. (0, T, \delta(x)) \in \llbracket \tau \rrbracket_{\mathcal{E}}) \quad (\text{L0})$$

Similarly from Definition 67 it suffices to prove that

$$(\forall y : \tau_y \in (\Omega_2, x : \tau). (0, T, \delta(y)) \in \llbracket \tau_y \rrbracket_{\mathcal{E}})$$

This means given some $y : \tau_y \in (\Omega_2, x : \tau)$ it suffices to prove that

$$(0, T, \delta(y)) \in \llbracket \tau_y \rrbracket_{\mathcal{E}}$$

From Lemma 77 we know that $\exists \tau'. y : \tau' \in \text{dom}(\Omega_1) \wedge \Psi; \Theta \vdash \tau' <: \tau_y$

Instantiating (L0) with $y : \tau'$ we get $(0, T, \delta(y)) \in \llbracket \tau' \rrbracket_{\mathcal{E}}$

And finally from Lemma 73 we get the desired

□

Lemma 77 (Ω Subtyping: domain containment). $\forall \Psi, \Theta, \Omega_1, \Omega_2.$

$$\Psi; \Theta \vdash \Omega_1 <: \Omega_2 \implies \forall x : \tau \in \Omega_2. x : \tau' \in \Omega_1 \wedge \Psi; \Theta \vdash \tau' <: \tau$$

Proof. Proof by induction on $\Psi; \Theta \vdash \Omega_1 <: \Omega_2$

1. sub-lBase:

$$\frac{}{\Psi; \Theta \vdash \Omega <: .} \text{sub-mBase}$$

To prove: $\forall x : \tau \in (.). x : \tau' \in \Omega \wedge \Psi; \Theta \vdash \tau' <: \tau$

Trivial

2. sub-Ind:

$$\frac{x : \tau' \in \Omega_1 \quad \Psi; \Theta \vdash \tau' <: \tau \quad \Psi; \Theta \vdash \Omega_1/x <: \Omega_2}{\Psi; \Theta \vdash \Omega_1 <: \Omega_2, x : \tau} \text{sub-mInd}$$

To prove: $\forall y : \tau \in (\Omega_2, x : \tau_x). y : \tau' \in \Omega_1 \wedge \Psi; \Theta \vdash \tau' <: \tau$

This means given some $y : \tau \in (\Omega_2, x : \tau)$ it suffices to prove that

$y : \tau' \in \Omega_1 \wedge \Psi; \Theta \vdash \tau' <: \tau$

The following cases arise:

- $y = x$:

In this case we are given that

$x : \tau' \in \Omega_1 \wedge \Psi; \Theta \vdash \tau' <: \tau$

Therefore we are done

- $y \neq x$:

Since we are given that $\Psi; \Theta \vdash \Omega_1/x <: \Omega_2$ therefore we get the desired from IH

□

Theorem 78 (Soundness 1). $\forall e, n, n', \tau \in \text{Type}, t.$

$$\vdash e : \mathbb{M} n \tau \wedge e \Downarrow_t^{n'} v \implies n' \leq n$$

Proof. From Theorem 71 we know that $(0, t + 1, e) \in \llbracket \mathbb{M} n \tau \rrbracket_{\mathcal{E}}$

From Definition 66 this means we have

$$\forall t' < t + 1. e \Downarrow_{t'} v' \implies (0, t + 1 - t', v') \in \llbracket \mathbb{M} n \tau \rrbracket$$

From the evaluation relation we know that $e \Downarrow_0 e$ therefore we have

$$(0, t + 1, e) \in \llbracket \mathbb{M} n \tau \rrbracket$$

Again from Definition 66 it means we have

$$\forall t'' < t + 1. e \Downarrow_{t''}^{n'} v \implies \exists p'. n' + p' \leq 0 + n \wedge (p', t + 1 - t'', v) \in \llbracket \tau \rrbracket$$

Since we are given that $e \Downarrow_t^{n'} v$ therefore we have

$$\exists p'. n' + p' \leq n \wedge (p', 1, v) \in \llbracket \tau \rrbracket$$

Since $p' \geq 0$ therefore we get $n' \leq n$

□

Theorem 79 (Soundness 2). $\forall e, n, n', \tau \in \text{Type}.$

$$\vdash e : [n] \mathbf{1} \multimap \mathbb{M} 0 \tau \wedge e () \Downarrow_{t_1} - \Downarrow_{t_2}^{n'} v \implies n' \leq n$$

Proof. From Theorem 71 we know that $(0, t_1 + t_2 + 2, e) \in \llbracket [n] \mathbf{1} \multimap \mathbb{M} 0 \tau \rrbracket_{\mathcal{E}}$

Therefore from Definition 66 we know that

$$\forall t' < t_1 + t_2 + 2, v. e \Downarrow_{t'} v \implies (0, t_1 + t_2 + 2 - t', v) \in \llbracket [n] \mathbf{1} \multimap \mathbb{M} 0 \tau \rrbracket \quad (\text{S0})$$

Since we know that $e () \Downarrow_{t_1} -$ therefore from E-app we know that $\exists e'. e \Downarrow_{t_1} \lambda x. e'$

Instantiating (S0) with $t_1, \lambda x. e'$ we get $(0, t_2 + 2, \lambda x. e') \in \llbracket [n] \mathbf{1} \multimap \mathbb{M} 0 \tau \rrbracket$

This means from Definition 66 we have

$$\forall p', e', t'' < t_2 + 2. (p', t'', e') \in \llbracket [n] \mathbf{1} \rrbracket_{\mathcal{E}} \implies (0 + p', t'', e'[e''/x]) \in \llbracket \mathbb{M} 0 \tau \rrbracket_{\mathcal{E}} \quad (\text{S1})$$

Claim: $\forall t. (I, t, ()) \in \llbracket [I] \mathbf{1} \rrbracket_{\mathcal{E}}$

Proof:

From Definition 66 it suffices to prove that

$$() \Downarrow_0 v \implies (I, t, v) \in \llbracket [I] \mathbf{1} \rrbracket$$

Since we know that $v = ()$ therefore it suffices to prove that

$$(I, t, v) \in \llbracket [I] \mathbf{1} \rrbracket$$

From Definition 66 it suffices to prove that

$$\exists p'. p' + I \leq I \wedge (p', t, v) \in \llbracket \mathbf{1} \rrbracket\}$$

We choose p' as 0 and we get the desired

Instantiating (S1) with $n, (), t_2 + 1$ we get $(n, t_2 + 1, e'[(\)/x]) \in \llbracket \mathbb{M}0 \tau \rrbracket_{\mathcal{E}}$

This means again from Definition 66 we have

$$\forall t' < t_2 + 1. e'[(\)/x] \Downarrow_{t'} v' \implies (n, t_2 + 1 - t', v') \in \llbracket \mathbb{M}0 \tau \rrbracket$$

From E-val we know that $v' = e'[(\)/x]$ and $t' = 0$ therefore we have

$$(n, t_2 + 1, e'[(\)/x]) \in \llbracket \mathbb{M}0 \tau \rrbracket$$

Again from Definition 66 we have

$$\forall t' < t_2 + 1. e'[(\)/x] \Downarrow_{t'}^{n'} v'' \implies \exists p'. n' + p' \leq n + 0 \wedge (p', t_2 + 1 - t', v'') \in \llbracket \tau \rrbracket$$

Since we are given that $e \Downarrow_{t_1} v$ therefore we get

$$\exists p'. n' + p' \leq n \wedge (p', 1, v'') \in \llbracket \tau \rrbracket$$

Since $p' \geq 0$ therefore we have $n' \leq n$

□

Corollary 80 (Soundness). $\forall \Gamma, e, q, q', \tau, T, p_l.$

$$.; .; .; \Gamma \vdash e : [q] \mathbf{1} \multimap \mathbb{M}0 [q'] \tau \wedge$$

$$(p_l, T, \gamma) \in \llbracket \Gamma \rrbracket_{\mathcal{E}} \wedge$$

$$e (\) \gamma \Downarrow_{t_1} v_t \Downarrow_{t_2}^J v \wedge$$

$$t_1 + t_2 < T$$

\implies

$$\exists p_v. (p_v, T - t_1 - t_2, v) \in \llbracket \tau \rrbracket \wedge J \leq (q + p_l) - (q' + p_v)$$

Proof. From Theorem 71 we know that $(p_l, T, e) \in \llbracket [q] \mathbf{1} \multimap \mathbb{M}0 [q'] \tau \rrbracket_{\mathcal{E}}$

Therefore from Definition 66 we know that

$$\forall T' < T, v. e \gamma \Downarrow_{T'} v \implies (p_l, T - T', v) \in \llbracket [q] \mathbf{1} \multimap \mathbb{M}0 [q'] \tau \rrbracket \quad (\text{S0})$$

Since we know that $e (\) \gamma \Downarrow_{t_1} v_t$ therefore from E-app we know that

$$\exists e'. e \Downarrow_{t_1} \lambda x. e' \text{ and } e'[(\)/x] \Downarrow_{t_1'} v_t \text{ s.t. } t_1' + t_1'' + 1 = t_1$$

Instantiating (S0) with $t_1', \lambda x. e'$ we get $(p_l, T - t_1', \lambda x. e') \in \llbracket [q] \mathbf{1} \multimap \mathbb{M}0 [q'] \tau \rrbracket$

This means from Definition 66 we have

$$\forall p', T' < (T - t_1'), e'. (p', T', e'') \in \llbracket [q] \mathbf{1} \rrbracket_{\mathcal{E}} \implies (p_l + p', T', e'[e''/x]) \in \llbracket \mathbb{M}0 [q'] \tau \rrbracket_{\mathcal{E}} \quad (\text{S1})$$

Claim: $\forall T. (I, T, ()) \in \llbracket [I] \mathbf{1} \rrbracket_{\mathcal{E}}$

Proof:

From Definition 66 it suffices to prove that

$$\forall T'' < T, v. (\) \Downarrow_{T''} v \implies (I, T - T'', v) \in \llbracket [I] \mathbf{1} \rrbracket$$

From (E-val) we know that $T'' = 0$ and $v = ()$ therefore it suffices to prove that

$(I, T, ()) \in \llbracket [I] \mathbf{1} \rrbracket$

From Definition 66 it further suffices to prove that

$\exists p'. p' + I \leq I \wedge (p', T, ()) \in \llbracket \mathbf{1} \rrbracket$

We choose p' as 0 and we get the desired

□

Using the claim we know that we have $(q, T - t'_1 - 1, ()) \in \llbracket [q] \mathbf{1} \rrbracket_{\mathcal{E}}$

Instantiating (S1) with $q, T - t'_1 - 1, ()$ and using the claim proved above we get

$(p_l + q, T - t'_1 - 1, e'[(\)/x]) \in \llbracket \mathbb{M}0 [q'] \tau \rrbracket_{\mathcal{E}}$

This means again from Definition 66 we have

$\forall T_1 < T - t'_1 - 1. e'[(\)/x] \Downarrow v' \implies (p_l + q, T - t'_1 - 1 - T_1, v') \in \llbracket \mathbb{M}0 [q'] \tau \rrbracket$

Instantiating with t''_1, v_t and since $t_1 < T$, therefore we also have $t''_1 < T - t'_1$.

Also since we are given that $e(\) \gamma \Downarrow_{t_1} v_t$, therefore we know that $v' = v_t$. Thus, we have

$(p_l + q, T - t'_1 - 1 - t''_1, v_t) \in \llbracket \mathbb{M}0 [q'] \tau \rrbracket$

Again from Definition 66 we have

$\forall v'', t'_2 < T - t'_1 - t''_1 - 1. v_t \Downarrow_{t'_2}^J v'' \implies \exists p'. J + p' \leq p_l + q \wedge (p', T - t'_1 - t''_1 - 1 - t'_2, v'') \in \llbracket [q'] \tau \rrbracket$

Instantiating with v, t_2 and since $t_2 < T - t'_1 - t''_1 - 1$ and $e \Downarrow_{t_1} v_t \Downarrow_{t'_2}^J v$ therefore we get

$\exists p'. J + p' \leq p_l + q \wedge (p', T - t'_1 - t''_1 - 1 - t_2, v) \in \llbracket [q'] \tau \rrbracket$ (S2)

Since we have $(p', T - t'_1 - t''_1 - 1 - t_2, v) \in \llbracket [q'] \tau \rrbracket$ therefore from Definition 66 we have

$\exists p'_1. p'_1 + q' \leq p' \wedge (p'_1, T - t'_1 - t''_1 - 1 - t_2, v) \in \llbracket \tau \rrbracket$ (S3)

In order to prove $\exists p_v. (p_v, T - t_1 - t_2, v) \in \llbracket \tau \rrbracket \wedge J \leq (q + p_l) - (q' + p_v)$ we choose p_v as p'_1 and we need to prove

1. $(p'_1, T - t_1 - t_2, v) \in \llbracket \tau \rrbracket$:

Since from (S3) we have $(p'_1, T - t'_1 - t''_1 - 1 - t_2, v) \in \llbracket \tau \rrbracket$ and since $t'_1 + t''_1 + 1 = t_1$ therefore also have

$(p'_1, T - t_1 - t_2, v) \in \llbracket \tau \rrbracket$

2. $J \leq (q + p_l) - (q' + p_v)$:

From (S2) and (S3) we get

$J \leq (p_l + q) - (q' + p'_1)$

□

2.5 Embedding Univariate RAML

Univariate RAML's type syntax

$$\begin{aligned} \text{Types } \tau &::= \mathbf{b} \mid L^{\vec{q}} \tau \mid (\tau_1, \tau_2) \\ A &::= \tau \xrightarrow{q/q'} \tau \end{aligned}$$

Type translation

$$\begin{aligned} \langle \mathit{unit} \rangle &= \mathbf{1} \\ \langle \mathbf{b} \rangle &= !\mathbf{b} \\ \langle L^{\vec{q}} \tau \rangle &= \exists s. (\llbracket \phi(\vec{q}, s) \rrbracket \mathbf{1} \otimes L^s \langle \tau \rangle) \\ \langle (\tau_1, \tau_2) \rangle &= (\langle \tau_1 \rangle \otimes \langle \tau_2 \rangle) \\ \langle \tau_1 \xrightarrow{q/q'} \tau_2 \rangle &= ([q] \mathbf{1} \multimap \langle \tau_1 \rangle \multimap \mathbb{M}0 [q'] \langle \tau_2 \rangle) \end{aligned}$$

Type context translation

$$\begin{aligned} \langle \cdot \rangle &= \cdot \\ \langle \Gamma, x : \tau \rangle &= \langle \Gamma \rangle, x : \langle \tau \rangle \end{aligned}$$

Function context translation

$$\begin{aligned} \langle \cdot \rangle &= \cdot \\ \langle \Sigma, x : \tau \rangle &= \langle \Sigma \rangle, x : \langle \tau \rangle \end{aligned}$$

Judgment translation

$$\boxed{\Sigma; \Gamma \vdash_{q'}^q e_r : \tau \rightsquigarrow \cdot; \cdot; \langle \Sigma \rangle; \langle \Gamma \rangle \vdash e_a : [q] \mathbf{1} \multimap \mathbb{M} 0 ([q'] \langle \tau \rangle)}$$

Definition 81. $\phi(\vec{q}, n) \triangleq \sum_{1 \leq i \leq k} \binom{n}{i} q_i$ as defined in [2, 1]

Expression translation

$$\frac{}{\Sigma; \cdot \vdash_{q'}^{q+K^{unit}} () : unit \rightsquigarrow \lambda u. \text{release } - = u \text{ in bind } - = \uparrow^{K^{unit}} \text{ in bind } a = \text{store}() \text{ in ret}(a)} \text{unit}$$

$$\frac{}{\Sigma; \cdot \vdash_{q'}^{q+K^{base}} c : \mathbf{b} \rightsquigarrow \lambda u. \text{release } - = u \text{ in bind } - = \uparrow^{K^{base}} \text{ in bind } a = \text{store}(!c) \text{ in ret}(a)} \text{base}$$

$$\frac{}{\Sigma; x : \tau \vdash_{q'}^{q+K^{var}} x : \tau \rightsquigarrow \lambda u. \text{release } - = u \text{ in bind } - = \uparrow^{K^{var}} \text{ in bind } a = \text{store } x \text{ in ret}(a)} \text{var}$$

$$\frac{\tau_1 \xrightarrow{q/q'} \tau_2 \in \Sigma(f)}{\Sigma; x : \tau_1 \vdash_{q'-K_2^{app}}^{q+K_1^{app}} f x : \tau_2 \rightsquigarrow \lambda u. E_0} \text{app}$$

where

$$E_0 = \text{release } - = u \text{ in bind } - = \uparrow^{K_1^{app}} \text{ in bind } P = \text{store}() \text{ in } E_1$$

$$E_1 = \text{bind } f_1 = (f P x) \text{ in release } f_2 = f_1 \text{ in bind } - = \uparrow^{K_2^{app}} \text{ in bind } f_3 = \text{store } f_2 \text{ in ret } f_3$$

$$\frac{}{\Sigma; \emptyset \vdash_{q'}^{q+K^{nil}} nil : L^{\vec{p}} \tau \rightsquigarrow \lambda u. \text{release } - = u \text{ in bind } - = \uparrow^{K^{nil}} \text{ in bind } a = \text{store}() \text{ in bind } b = \text{store}\langle\langle a, nil \rangle\rangle \text{ in ret}(b)} \text{nil}$$

$$\frac{\vec{p} = (p_1, \dots, p_k)}{\Sigma; x_h : \tau, x_t : L^{(\prec \vec{p})} \tau \vdash_{q'}^{q+p_1+K^{cons}} \text{cons}(x_h, x_t) : L^{\vec{p}} \tau \rightsquigarrow \lambda u. \text{release } - = u \text{ in bind } - = \uparrow^{K^{cons}} \text{ in } E_0} \text{cons}$$

where

$$E_0 = x_t; x. \text{let}\langle\langle x_1, x_2 \rangle\rangle = x \text{ in } E_1$$

$$E_1 = \text{release } - = x_1 \text{ in bind } a = \text{store}() \text{ in bind } b = \text{store}\langle\langle a, x_h :: x_2 \rangle\rangle \text{ in ret}(b)$$

$$\frac{\begin{array}{l} \Sigma; \Gamma \vdash_{q'+K_2^{matN}}^{q-K_1^{matN}} e_1 : \tau' \rightsquigarrow e_{a1} \\ \vec{p} = (p_1, \dots, p_k) \quad \Sigma; \Gamma, h : \tau, t : L^{(\prec \vec{p})} \tau \vdash_{q'+K_2^{matN}}^{q+p_1-K_1^{matC}} e_2 : \tau' \rightsquigarrow e_{a2} \end{array}}{\Sigma; \Gamma; x : L^{\vec{p}} \tau \vdash_{q'}^q \text{match } x \text{ with } |nil \mapsto e_1 \mid h :: t \mapsto e_2 : \tau' \rightsquigarrow \lambda u. E_0} \text{match}$$

where

$$\begin{aligned}
E_0 &= \text{release } - = u \text{ in } E_{0,1} \\
E_{0,1} &= x; a. \text{let}\langle\langle x_1, x_2 \rangle\rangle = a \text{ in } E_1 \\
E_1 &= \text{match } x_2 \text{ with } | \text{nil} \mapsto E_2 \mid h :: l_t \mapsto E_3 \\
E_2 &= \text{bind } - = \uparrow^{K_1^{\text{mat}N}} \text{ in } E_{2,1} \\
E_{2,1} &= \text{bind } b = \text{store}() \text{ in } E'_2 \\
E'_2 &= \text{bind } c = (e_{a1} b) \text{ in } E'_{2,1} \\
E'_{2,1} &= \text{release } d = c \text{ in } E'_{2,2} \\
E'_{2,2} &= \text{bind } - = \uparrow^{K_2^{\text{mat}N}} \text{ in } E'_{2,3} \\
E'_{2,3} &= \text{release } - = x_1 \text{ in store } d \\
E_3 &= \text{bind } - = \uparrow^{K_1^{\text{mat}C}} \text{ in } E_{3,1} \\
E_{3,1} &= \text{release } - = x_1 \text{ in } E_{3,2} \\
E_{3,2} &= \text{bind } b = \text{store}() \text{ in } E_{3,3} \\
E_{3,3} &= \text{bind } t = \text{ret}\langle\langle b, l_t \rangle\rangle \text{ in } E_{3,4} \\
E_{3,4} &= \text{bind } d = \text{store}() \text{ in } E_{3,5} \\
E_{3,5} &= \text{bind } f = e_{a2} d \text{ in } E_{3,6} \\
E_{3,6} &= \text{release } g = f \text{ in } E_{3,7} \\
E_{3,7} &= \text{bind } - = \uparrow^{K_2^{\text{mat}C}} \text{ in store } g
\end{aligned}$$

$$\frac{\Sigma; \Gamma, x : \tau_1, y : \tau_2 \vdash_{q'}^q e : \tau' \rightsquigarrow e_a \quad \tau = \tau_1 \Downarrow \tau_2 \quad \tau = \tau_1 = \tau_2 = \mathbf{1}}{\Sigma; \Gamma, z : \tau \vdash_{q'}^q e[z/x, z/y] : \tau' \rightsquigarrow E_0} \text{ Share-unit}$$

$$\begin{aligned}
E_0 &= \lambda u. E_1 \\
E_1 &= \text{bind } a = \text{coerce}_{\mathbf{1}, \mathbf{1}, \mathbf{1}} z \text{ in let}\langle\langle x, y \rangle\rangle = a \text{ in } e_a u \\
\text{coerce}_{\mathbf{1}, \mathbf{1}, \mathbf{1}} &: (\mathbf{1}) \multimap \text{M}0((\mathbf{1}) \otimes (\mathbf{1})) \\
\text{coerce}_{\mathbf{1}, \mathbf{1}, \mathbf{1}} &\triangleq \lambda u. \text{ret}\langle\langle !(), !() \rangle\rangle
\end{aligned}$$

$$\frac{\Sigma; \Gamma, x : \tau_1, y : \tau_2 \vdash_{q'}^q e : \tau' \rightsquigarrow e_a \quad \tau = \tau_1 \Downarrow \tau_2 \quad \tau = \tau_1 = \tau_2 = \mathbf{b}}{\Sigma; \Gamma, z : \tau \vdash_{q'}^q e[z/x, z/y] : \tau' \rightsquigarrow E_0} \text{ Share-base}$$

$$\begin{aligned}
E_0 &= \lambda u. E_1 \\
E_1 &= \text{bind } a = \text{coerce}_{\mathbf{b}, \mathbf{b}, \mathbf{b}} z \text{ in let}\langle\langle x, y \rangle\rangle = a \text{ in } e_a u \\
\text{coerce}_{\mathbf{1}, \mathbf{1}, \mathbf{1}} &: (\mathbf{b}) \multimap \text{M}0((\mathbf{b}) \otimes (\mathbf{b})) \\
\text{coerce}_{\mathbf{b}, \mathbf{b}, \mathbf{b}} &\triangleq \lambda u. \text{let } !u' = u \text{ in ret}\langle\langle !u', !u' \rangle\rangle
\end{aligned}$$

$$\frac{\Sigma; \Gamma, x : \tau_1, y : \tau_2 \vdash_{q'}^q e : \tau' \rightsquigarrow e_a \quad \tau = L^{\vec{p}} \tau'' \quad \tau_1 = L^{\vec{p}_1} \tau_1'' \quad \tau_2 = L^{\vec{p}_2} \tau_2'' \quad \tau'' = \tau_1'' \Downarrow \tau_2'' \quad \vec{p} = \vec{p}_1 + \vec{p}_2}{\Sigma; \Gamma, z : \tau \vdash_{q'}^q e[z/x, z/y] : \tau' \rightsquigarrow E_0} \text{ Share-list}$$

$$\begin{aligned}
E_0 &= \lambda u. E_1 \\
E_1 &= \text{bind } a = \text{coerce}_{\tau, \tau_1, \tau_2} z \text{ in let}\langle\langle x, y \rangle\rangle = a \text{ in } e_a u \\
\text{coerce}_{L^{\vec{p}} \tau, L^{\vec{p}_1} \tau_1, L^{\vec{p}_2} \tau_2} &: !((\tau) \multimap \text{M}0((\tau_1) \otimes (\tau_2))) \multimap (L^{\vec{p}} \tau) \multimap \text{M}0(L^{\vec{p}_1} \tau_1) \otimes (L^{\vec{p}_2} \tau_2) \\
\text{coerce}_{L^{\vec{p}} \tau, L^{\vec{p}_1} \tau_1, L^{\vec{p}_2} \tau_2} &\triangleq \text{fix } f. \lambda g. \lambda e. \text{let } !g' = g \text{ in } e; x. \text{let}\langle\langle p, l \rangle\rangle = x \text{ in } E_0
\end{aligned}$$

where

$E_0 \triangleq \text{release } - = p \text{ in } E_1$
 $E_1 \triangleq \text{match } l \text{ with } | \text{nil} \mapsto E_{2.1} \mid h :: t \mapsto E_3$
 $E_{2.1} \triangleq \text{bind } z_1 = \text{store}() \text{ in } E_{2.2}$
 $E_{2.2} \triangleq \text{bind } z_2 = \text{store}() \text{ in } E_{2.3}$
 $E_{2.3} \triangleq \text{ret} \langle \langle \langle z_1, \text{nil} \rangle \rangle, \langle \langle z_2, \text{nil} \rangle \rangle \rangle$
 $E_3 \triangleq \text{bind } H = g' h \text{ in } E_{3.1}$
 $E_{3.1} \triangleq \text{bind } o_t = () \text{ in } E_{3.2}$
 $E_{3.2} \triangleq \text{bind } T = f g \langle \langle o_t, t \rangle \rangle \text{ in } E_4$
 $E_4 \triangleq \text{let} \langle \langle H_1, H_2 \rangle \rangle = H \text{ in } E_5$
 $E_5 \triangleq \text{let} \langle \langle T_1, T_2 \rangle \rangle = T \text{ in } E_6$
 $E_6 \triangleq T_1; tp_1. \text{let} \langle \langle p'_1, l'_1 \rangle \rangle = tp_1 \text{ in } E_{7.1}$
 $E_{7.1} \triangleq T_2; tp_2. \text{let} \langle \langle p'_2, l'_2 \rangle \rangle = tp_2 \text{ in } E_{7.2}$
 $E_{7.2} \triangleq \text{release } - = p'_1 \text{ in } E_{7.3}$
 $E_{7.3} \triangleq \text{release } - = p'_2 \text{ in } E_{7.4}$
 $E_{7.4} \triangleq \text{bind } o_1 = \text{store}() \text{ in } E_{7.5}$
 $E_{7.5} \triangleq \text{bind } o_2 = \text{store}() \text{ in } E_8$
 $E_8 \triangleq \text{ret} \langle \langle \langle o_1, H_1 :: T_1 \rangle \rangle, \langle \langle o_2, H_2 :: T_2 \rangle \rangle \rangle$

$$\frac{\Sigma; \Gamma, x : \tau_1, y : \tau_2 \vdash_{q'}^q e : \tau' \rightsquigarrow e_a \quad \tau = \tau_1 \vee \tau_2 \quad \tau = (\tau_a, \tau_b) \quad \tau_1 = (\tau'_a, \tau'_b) \quad \tau_2 = (\tau''_a, \tau''_b)}{\Sigma; \Gamma, z : \tau \vdash_{q'}^q e[z/x, z/y] : \tau' \rightsquigarrow E_0} \text{Share-pair}$$

$E_0 = \lambda u. E_1$

$E_1 = \text{bind } a = \text{coerce}_{(\tau_a, \tau_b), (\tau'_a, \tau'_b), (\tau''_a, \tau''_b)} z \text{ in } \text{let} \langle \langle x, y \rangle \rangle = a \text{ in } e_a u$

$\text{coerce}_{(\tau_a, \tau_b), (\tau'_a, \tau'_b), (\tau''_a, \tau''_b)} : !((\tau_a) \multimap \mathbb{M} 0 (\tau'_a) \otimes (\tau''_a)) \multimap !((\tau_b) \multimap \mathbb{M} 0 (\tau'_b) \otimes (\tau''_b)) \multimap ((\tau_a, \tau_b)) \multimap \mathbb{M} 0 ((\tau'_a, \tau'_b)) \otimes ((\tau''_a, \tau''_b))$

$\text{coerce}_{(\tau_a, \tau_b), (\tau'_a, \tau'_b), (\tau''_a, \tau''_b)} \triangleq \lambda -g_1. \lambda -g_2. \lambda p. \text{let} ! \langle \langle p_1, p_2 \rangle \rangle = p \text{ in } E_0$

where

$E_0 \triangleq \text{let} ! g'_1 = g_1 \text{ in } E_1$

$E_1 \triangleq \text{let} ! g'_2 = g_2 \text{ in } E_2$

$E_2 \triangleq \text{bind } P'_1 = g'_1 p_1 \text{ in } E_3$

$E_3 \triangleq \text{bind } P'_2 = g'_2 p_2 \text{ in } E_4$

$E_4 \triangleq \text{let} ! \langle \langle p'_{11}, p'_{12} \rangle \rangle = P'_1 \text{ in } E_5$

$E_5 \triangleq \text{let} ! \langle \langle p'_{21}, p'_{22} \rangle \rangle = P'_2 \text{ in } E_6$

$E_6 \triangleq \text{ret} \langle \langle p'_{11}, p'_{21} \rangle \rangle, \langle \langle p'_{12}, p'_{22} \rangle \rangle$

$$\frac{\Sigma; \Gamma \vdash_{q'}^q e : \tau \rightsquigarrow e_a \quad \tau <: \tau'}{\Sigma; \Gamma \vdash_{q'}^q e : \tau' \rightsquigarrow e_a} \text{Sub}$$

$$\frac{\Sigma; \Gamma, x : \tau_1 \vdash_{q'}^q e : \tau \rightsquigarrow e_a \quad \tau'_1 <: \tau_1}{\Sigma; \Gamma, x : \tau'_1 \vdash_{q'}^q e : \tau \rightsquigarrow e_a} \text{Super}$$

$$\frac{\Sigma; \Gamma \vdash_{p'}^p e : \tau \rightsquigarrow e_a \quad q \geq p \quad q - p \geq q' - p'}{\Sigma; \Gamma \vdash_{q'}^q e : \tau \rightsquigarrow \lambda o. E_0} \text{Relax}$$

where

$E_0 = \text{release } - = o \text{ in } E_1$

$E_1 = \text{bind } a = \text{store}() \text{ in } E_2$
 $E_2 = \text{bind } b = e_a \ a \text{ in } E_3$
 $E_3 = \text{release } c = b \text{ in store } c$

$$\frac{\Sigma; \Gamma_1 \vdash_p^{q-K_1^{let}} e_1 : \tau_1 \rightsquigarrow e_{a1} \quad \Sigma; \Gamma_2, x : \tau_1 \vdash_{q'+K_3^{let}}^{p-K_2^{let}} e_2 : \tau_1 \rightsquigarrow e_{a2}}{\Sigma; \Gamma_1, \Gamma_2 \vdash_{q'}^q \text{let } x = e_1 \text{ in } e_2 : \tau \rightsquigarrow E_t} \text{Let}$$

where

$E_t = \lambda u. E_0$
 $E_0 = \text{release } - = u \text{ in } E_1$
 $E_1 = \text{bind } - = \uparrow^{K_1^{let}} \text{ in } E_2$
 $E_2 = \text{bind } a = \text{store}() \text{ in } E_3$
 $E_3 = \text{bind } b = e_{a1} \ a \text{ in } E_4$
 $E_4 = \text{release } x = b \text{ in } E_5$
 $E_5 = \text{bind } - = \uparrow^{K_2^{let}} \text{ in } E_6$
 $E_6 = \text{bind } c = \text{store}() \text{ in } E_7$
 $E_7 = \text{bind } d = e_{a2} \ c \text{ in } E_8$
 $E_8 = \text{release } f = d \text{ in } E_9$
 $E_9 = \text{bind } - = \uparrow^{K_3^{let}} \text{ in } E_{10}$
 $E_{10} = \text{bind } g = \text{store } f \text{ in ret } g$

$$\frac{}{\Sigma; x_1 : \tau_1, x_2 : \tau_2 \vdash_q^{q+K^{pair}} (x_1, x_2) : (\tau_1, \tau_2) \rightsquigarrow E_t} \text{pair}$$

where

$E_t = \lambda u. E_0$
 $E_0 = \text{release } - = u \text{ in } E_1$
 $E_1 = \text{bind } - = \uparrow^{K^{pair}} \text{ in } E_2$
 $E_2 = \text{bind } a = \text{store}(x_1, x_2) \text{ in ret } a$

$$\frac{\tau = (\tau_1, \tau_2) \quad \Sigma, \Gamma, x_1 : \tau_1, x_2 : \tau_2 \vdash_{q'+K_2^{matP}}^{q-K_1^{matP}} e : \tau' \rightsquigarrow e_t}{\Sigma; \Gamma, x : \tau \vdash_q^q \text{match } x \text{ with } (x_1, x_2) \rightarrow e : \tau' \rightsquigarrow E_t} \text{matP}$$

where

$E_t = \lambda u. E_0$
 $E_0 = \text{release } - = u \text{ in } E_1$
 $E_1 = \text{bind } - = \uparrow^{K_1^{matP}} \text{ in } E_2$
 $E_2 = \text{let } \langle\langle x_1, x_2 \rangle\rangle = x \text{ in } E_3$
 $E_3 = \text{bind } a = \text{store}() \text{ in } E_4$
 $E_4 = \text{bind } b = e_t \ a \text{ in } E_5$
 $E_5 = \text{release } c = b \text{ in } E_6$
 $E_6 = \text{bind } - = \uparrow^{K_2^{matP}} \text{ in } E_7$
 $E_7 = \text{bind } d = \text{store } c \text{ in ret } d$

$$\frac{\Sigma; \Gamma \vdash_{q'}^q e : \tau \rightsquigarrow e_a}{\Sigma; \Gamma, x : \tau' \vdash_{q'}^q e : \tau \rightsquigarrow e_a} \text{Augment}$$

2.5.1 Type preservation

Theorem 82 (Type preservation: Univariate RAML to λ -Amor). If $\Sigma; \Gamma \vdash_{q'}^q e : \tau$ in Univariate RAML then there exists e' such that $\Sigma; \Gamma \vdash_{q'}^q e : \tau \rightsquigarrow e'$ such that there is a derivation of $.; ; ; (\Sigma), (\Gamma) \vdash e' : [q] \mathbf{1} \multimap \mathbb{M} 0 ([q'](\tau))$ in λ -Amor .

Proof. By induction on $\Sigma; \Gamma \vdash_{q'}^q e : \tau$

1. unit:

$$\frac{}{\Sigma; . \vdash_{q+K^{unit}}^q () : unit \rightsquigarrow \lambda u. \text{release } - = u \text{ in bind } - = \uparrow^{K^{unit}} \text{ in bind } a = \text{store}() \text{ in ret}(a)} \text{unit}$$

$$E_0 = \lambda u. \text{release } - = u \text{ in bind } - = \uparrow^{K^{unit}} \text{ in bind } a = \text{store}() \text{ in ret}(a)$$

$$E_1 = \text{release } - = u \text{ in bind } - = \uparrow^{K^{unit}} \text{ in bind } a = \text{store}() \text{ in ret}(a)$$

$$T_0 = [q + K^{unit}] \mathbf{1} \multimap \mathbb{M} 0 ([q](unit))$$

$$T_1 = [q + K^{unit}] \mathbf{1}$$

$$T_2 = \mathbb{M}(q + K^{unit}) ([q] \mathbf{1})$$

$$T_{2.1} = \mathbb{M}(q) ([q] \mathbf{1})$$

$$T_3 = \mathbb{M} K^{unit} \mathbf{1}$$

$$T_4 = \mathbb{M} 0 ([q] \mathbf{1})$$

$$T_5 = \mathbb{M} q ([q] \mathbf{1})$$

D1:

$$\frac{.; ; ; (\Sigma); . \vdash \text{store}() : T_5 \quad .; ; ; (\Sigma); a : [q] \mathbf{1} \vdash \text{ret}(a) : T_4}{.; ; ; (\Sigma); . \vdash \text{bind } a = \text{store}() \text{ in ret}(a) : T_5}$$

D0:

$$\frac{}{.; ; ; (\Sigma); . \vdash \uparrow^{K^{unit}} : T_3}$$

D0.0:

$$\frac{D0 \quad D1}{.; ; ; (\Sigma); . \vdash \text{bind } - = \uparrow^{K^{unit}} \text{ in bind } a = \text{store}() \text{ in ret}(a) : T_2} \text{T-bind}$$

Main derivation:

$$\frac{\frac{.; ; ; (\Sigma); u : T_1 \vdash u : T_1}{.; ; ; (\Sigma); u : T_1 \vdash E_1 : T_4} \text{T-var} \quad D0.0}{.; ; ; (\Sigma); . \vdash E_0 : T_0} \text{T-lam} \quad \text{T-release}$$

2. base:

$$\frac{}{\Sigma; . \vdash_{q+K^{base}}^q c : \mathbf{b} \rightsquigarrow \lambda u. \text{release } - = u \text{ in bind } - = \uparrow^{K^{base}} \text{ in bind } a = \text{store}(!c) \text{ in ret}(a)} \text{base}$$

$$E_0 = \lambda u. \text{release } - = u \text{ in bind } - = \uparrow^{K^{base}} \text{ in bind } a = \text{store}(!c) \text{ in ret}(a)$$

$$E_1 = \text{release } - = u \text{ in bind } - = \uparrow^{K^{base}} \text{ in bind } a = \text{store}(!c) \text{ in ret}(a)$$

$$T_0 = [q + K^{base}] \mathbf{1} \multimap \mathbb{M} 0 ([q](\mathbf{b}))$$

$$T_1 = [q + K^{base}] \mathbf{1}$$

$$T_2 = \mathbb{M}(q + K^{base}) ([q]!\mathbf{b})$$

$$T_{2.1} = \mathbb{M}(q) ([q]!\mathbf{b})$$

$$T_3 = \mathbb{M} K^{base} (\mathbf{1})$$

$$T_4 = \mathbb{M} 0 ([q]!\mathbf{b})$$

$$T_5 = \mathbb{M} q ([q]!\mathbf{b})$$

D1:

$$\frac{\frac{.; ; ; (\Sigma); \cdot \vdash \text{store}(!c) : T_5 \quad .; ; ; (\Sigma); a : [q]!\mathbf{b} \vdash \text{ret}(a) : T_4}{.; ; ; (\Sigma); \cdot \vdash \text{bind } a = \text{store}(!c) \text{ in ret}(a) : T_{2.1}}}{.; ; ; (\Sigma); \cdot \vdash \text{bind } - = \uparrow^{K^{base}} \text{ in bind } a = \text{store}(!c) \text{ in ret}(a) : T_2} \text{T-bind}$$

D0:

$$\frac{}{.; ; ; (\Sigma); \cdot \vdash \uparrow^{K^{base}} : T_3}$$

D0.0:

$$\frac{\frac{D0 \quad D1}{.; ; ; (\Sigma); \cdot \vdash \text{bind } - = \uparrow^{K^{base}} \text{ in bind } a = \text{store}(!c) \text{ in ret}(a) : T_2} \text{T-bind}}{.; ; ; (\Sigma); \cdot \vdash \text{bind } - = \uparrow^{K^{base}} \text{ in bind } a = \text{store}(!c) \text{ in ret}(a) : T_2} \text{T-bind}$$

Main derivation:

$$\frac{\frac{\frac{.; ; ; (\Sigma); u : T_1 \vdash u : T_1 \quad D0.0}{.; ; ; (\Sigma); u : T_1 \vdash E_1 : T_4} \text{T-release}}{.; ; ; (\Sigma); \cdot \vdash E_0 : T_0} \text{T-lam}}{.; ; ; (\Sigma); \cdot \vdash E_0 : T_0} \text{T-lam}$$

3. var:

$$\frac{}{\Sigma; x : \tau \vdash_q^{q+K^{var}} x : \tau \rightsquigarrow \lambda u. \text{bind } - = \uparrow^{K^{var}} \text{ in ret}(x) \text{ var}}$$

$$E_0 = \lambda u. \text{release } - = u \text{ in bind } - = \uparrow^{K^{var}} \text{ in bind } a = \text{store } x \text{ in ret}(a)$$

$$E_1 = \text{release } - = u \text{ in bind } - = \uparrow^{K^{var}} \text{ in bind } a = \text{store } x \text{ in ret}(a)$$

$$T_0 = [q + K^{var}] \mathbf{1} \multimap \mathbb{M} 0 ([q](\tau))$$

$$T_1 = [q + K^{var}] \mathbf{1}$$

$$T_2 = \mathbb{M} 0 ([q + K^{var}](\tau))$$

$$T_3 = \mathbb{M} K^{var} (\mathbf{1})$$

$$T_4 = \mathbb{M} 0 ([q](\tau))$$

$$T_5 = \mathbb{M} q ([q](\tau))$$

D1:

$$\frac{\frac{.; ; ; (\Sigma); x : (\tau) \vdash \text{store } x : T_5 \quad .; ; ; (\Sigma); a : [q](\tau) \vdash \text{ret}(a) : T_4}{.; ; ; (\Sigma); x : (\tau) \vdash \text{bind } a = \text{store } x \text{ in ret}(a) : T_5}}{.; ; ; (\Sigma); x : (\tau) \vdash \text{bind } a = \text{store } x \text{ in ret}(a) : T_5} \text{T-bind}$$

D0:

$$\frac{}{.; .; .; (\Sigma); x : (\tau) \vdash \uparrow^{K^{var}} : T_3}$$

D0.0:

$$\frac{\frac{D0 \quad D1}{.; .; .; (\Sigma); x : (\tau) \vdash \text{bind } - = \uparrow^{K^{var}} \text{ in bind } a = \text{store } x \text{ in ret}(a) : T_2}}{\text{T-bind}}$$

Main derivation:

$$\frac{\frac{\frac{.; .; .; (\Sigma); u : T_1 \vdash u : T_1}{\text{T-var}} \quad D0.0}{.; .; .; (\Sigma); x : (\tau), u : T_1 \vdash E_1 : T_4}{\text{T-release}}}{.; .; .; (\Sigma); x : (\tau) \vdash E_0 : T_0}{\text{T-lam}}$$

4. app:

$$\frac{\tau_1 \xrightarrow{q/q'} \tau_2 \in \Sigma(f)}{\Sigma; x : \tau_1 \vdash \frac{q+K_1^{app}}{q'-K_2^{app}} f \ x : \tau_2 \rightsquigarrow \lambda u. E_0}{\text{app}}}$$

where

$$E_0 = \text{release } - = u \text{ in bind } - = \uparrow^{K_1^{app}} \text{ in bind } P = \text{store}() \text{ in } E_1$$

$$E_1 = \text{bind } f_1 = (f \ P \ x) \text{ in release } f_2 = f_1 \text{ in bind } - = \uparrow^{K_2^{app}} \text{ in bind } f_3 = \text{store } f_2 \text{ in ret } f_3$$

$$E_{1.1} = \text{release } f_2 = f_1 \text{ in bind } - = \uparrow^{K_2^{app}} \text{ in bind } f_3 = \text{store } f_2 \text{ in ret } f_3$$

$$E_{1.2} = \text{bind } - = \uparrow^{K_2^{app}} \text{ in bind } f_3 = \text{store } f_2 \text{ in ret } f_3$$

$$E_{1.3} = \text{bind } f_3 = \text{store } f_2 \text{ in ret } f_3$$

$$E_{1.4} = \text{store } f_2$$

$$E_{1.5} = \text{ret } f_3$$

$$E_{0.1} = \text{bind } - = \uparrow^{K_1^{app}} \text{ in bind } F = f \text{ in } E_1$$

$$T_0 = [q + K_1^{app}] \mathbf{1} \multimap \mathbb{M} 0 ([q' - K_2^{app}] (\tau))$$

$$T_{0.1} = [q + K_1^{app}] \mathbf{1}$$

$$T_{0.2} = \mathbb{M} 0 ([q' - K_2^{app}] (\tau_2))$$

$$T_1 = \mathbb{M}(q + K_1^{app}) \mathbf{1}$$

$$T_{1.2} = \mathbb{M} 0 [q' - K_2^{app}] (\tau_2)$$

$$T_2 = \mathbb{M}(K_1^{app}) \mathbf{1}$$

$$T_3 = \mathbb{M}(q) (\tau_2)$$

$$T_4 = \mathbb{M} q ((\tau_1) \multimap \mathbb{M} 0 [q'] (\tau_2))$$

$$T_{4.1} = ((\tau_1) \multimap \mathbb{M} 0 [q'] (\tau_2))$$

$$T_{4.2} = \mathbb{M} 0 [q'] (\tau_2)$$

$$T_{4.3} = [q'] (\tau_2)$$

$$T_{4.4} = \mathbb{M}(q' - K_2^{app}) [q' - K_2^{app}] (\tau_2)$$

$$T_{4.41} = \mathbb{M}(q')[q' - K_2^{app}] (\tau_2)$$

$$T_{4.5} = [q' - K_2^{app}] (\tau_2)$$

$$T_{4.6} = \mathbb{M}0 [q' - K_2^{app}] (\tau_2)$$

D2.3:

$$\frac{\frac{.; ; ; (\Sigma); f_2 : (\tau_2) \vdash E_{1.4} : T_{4.4}}{\quad} \quad \frac{.; ; ; (\Sigma); f_3 : T_{4.5} \vdash E_{1.5} : T_{4.6}}{\quad}}{.; ; ; (\Sigma); f_2 : (\tau_2), f_3 : T_{4.5} \vdash E_{1.3} : T_{4.4}}$$

D2.2:

$$\frac{\frac{.; ; ; (\Sigma); \cdot \vdash \uparrow^{K_2^{app}} : \mathbb{M} K_2^{app} \mathbf{1}}{\quad} \quad D2.3}{.; ; ; (\Sigma); f_2 : (\tau_2) \vdash E_{1.2} : T_{4.41}}$$

D2.1:

$$\frac{\frac{.; ; ; (\Sigma); f_1 : T_{4.3} \vdash f_1 : T_{4.3}}{\quad} \quad D2.2}{.; ; ; (\Sigma); f_1 : T_{4.3} \vdash E_{1.1} : T_{1.2}}$$

D2:

$$\frac{\frac{.; ; ; (\Sigma); x : (\tau_1), P : [q] \mathbf{1} \vdash f P x : T_{4.2}}{\quad} \quad D2.1}{.; ; ; (\Sigma); x : (\tau_1), P : [q] \mathbf{1} \vdash E_1 : T_{1.2}}$$

D1:

$$\frac{\frac{.; ; ; (\Sigma); \cdot \vdash \text{store}() : \mathbb{M} q [q] \mathbf{1}}{\quad} \quad D2}{.; ; ; (\Sigma); x : (\tau_1) \vdash \text{bind } P = \text{store}() \text{ in } E_1 : T_{1.2}}$$

D0:

$$\frac{\frac{.; ; ; (\Sigma); x : (\tau_1) \vdash \uparrow^{K_1^{app}} : T_1}{\quad} \quad D1}{.; ; ; (\Sigma); x : (\tau_1) \vdash E_{0.1} : T_{1.2}}$$

Main derivation:

$$\frac{\frac{\frac{.; ; ; (\Sigma); u : T_{0.1} \vdash u : T_{0.1}}{\quad} \quad \text{T-var} \quad D0}{.; ; ; (\Sigma); x : (\tau_1), u : T_{0.1} \vdash E_0 : T_{0.2}}}{.; ; ; (\Sigma); x : (\tau_1) \vdash \lambda u. E_0 : T_0}$$

5. nil:

$$\frac{\Sigma; \emptyset \vdash_q^{q+K^{nil}} \text{nil} : L\bar{p}\tau \rightsquigarrow}{\lambda u. \text{release } - = u \text{ in bind } - = \uparrow^{K^{nil}} \text{ in bind } a = \text{store}() \text{ in bind } b = \text{store}\langle\langle a, \text{nil} \rangle\rangle \text{ in ret}(b)} \text{nil}$$

$$E_0 = \lambda u. \text{release } - = u \text{ in bind } - = \uparrow^{K^{nil}} \text{ in bind } a = \text{store}() \text{ in bind } b = \text{store}\langle\langle a, \text{nil} \rangle\rangle \text{ in ret}(b)$$

$$E_1 = \text{release } - = u \text{ in bind } - = \uparrow^{K^{nil}} \text{ in bind } a = \text{store}() \text{ in bind } b = \text{store}\langle\langle a, \text{nil} \rangle\rangle \text{ in ret}(b)$$

$$E_2 = \text{bind } - = \uparrow^{K^{nil}} \text{ in bind } a = \text{store}() \text{ in bind } b = \text{store}\langle\langle a, \text{nil} \rangle\rangle \text{ in ret}(b)$$

$$E_3 = \text{bind } a = \text{store}() \text{ in bind } b = \text{store}\langle\langle a, \text{nil} \rangle\rangle \text{ in ret}(b)$$

$$E_4 = \text{bind } b = \text{store}\langle\langle a, \text{nil} \rangle\rangle \text{ in ret}(b)$$

$$E_5 = \text{ret}(b)$$

$$T_0 = [q + K^{\text{nil}}] \mathbf{1} \multimap \mathbb{M} 0 ([q] \exists n. \phi(\vec{p}, n) \otimes \text{list}[n](\tau))$$

$$T_1 = [(q + K^{\text{nil}})] \mathbf{1}$$

$$T_2 = \mathbb{M} 0 ([q] \exists n. [\phi(\vec{p}, n)] \mathbf{1} \otimes \text{list}[n](\tau))$$

$$T_3 = \mathbb{M}(q + K^{\text{nil}}) ([q] \exists n. [\phi(\vec{p}, n)] \mathbf{1} \otimes \text{list}[n](\tau))$$

$$T_4 = \mathbb{M} K^{\text{nil}} \mathbf{1}$$

$$T_5 = \mathbb{M}(q) ([q] \exists n. [\phi(\vec{p}, n)] \mathbf{1} \otimes \text{list}[n](\tau))$$

$$T_{5.1} = ([q] \exists n. [\phi(\vec{p}, n)] \mathbf{1} \otimes \text{list}[n](\tau))$$

$$T_6 = \mathbb{M}(0) ([q] \exists n. [\phi(\vec{p}, n)] \mathbf{1} \otimes \text{list}[n](\tau))$$

D4:

$$\frac{\frac{\phi(\vec{p}, 0) = 0}{\frac{\frac{\cdot; \cdot; \cdot; (\Sigma); a : [0] \mathbf{1} \vdash a : [0] \mathbf{1}}{\cdot; \cdot; \cdot; (\Sigma); a : [0] \mathbf{1} \vdash \langle\langle a, \text{nil} \rangle\rangle : T_6[0/n]}{\cdot; \cdot; \cdot; (\Sigma); a : [0] \mathbf{1} \vdash \langle\langle a, \text{nil} \rangle\rangle : T_6}}{\cdot; \cdot; \cdot; (\Sigma); a : [0] \mathbf{1} \vdash a : [0] \mathbf{1}} \quad \frac{\cdot; \cdot; \cdot; (\Sigma); a : [0] \mathbf{1} \vdash \text{nil} : \text{list}[0](\tau)}{\cdot; \cdot; \cdot; (\Sigma); a : [0] \mathbf{1} \vdash \langle\langle a, \text{nil} \rangle\rangle : T_6[0/n]}}{\cdot; \cdot; \cdot; (\Sigma); a : [0] \mathbf{1} \vdash \langle\langle a, \text{nil} \rangle\rangle : T_6}}{\cdot; \cdot; \cdot; (\Sigma); a : [0] \mathbf{1} \vdash \langle\langle a, \text{nil} \rangle\rangle : T_6}}$$

D3:

$$\frac{\cdot; \cdot; \cdot; (\Sigma); b : T_{5.1} \vdash E_5 : T_6}{\cdot; \cdot; \cdot; (\Sigma); b : T_{5.1} \vdash E_5 : T_6}}$$

D2:

$$\frac{\frac{\frac{D4}{\cdot; \cdot; \cdot; (\Sigma); a : [0] \mathbf{1} \vdash \text{store}\langle\langle a, \text{nil} \rangle\rangle : T_5}}{\cdot; \cdot; \cdot; (\Sigma); a : [0] \mathbf{1} \vdash E_4 : T_5}}{\cdot; \cdot; \cdot; (\Sigma); a : [0] \mathbf{1} \vdash \text{store}\langle\langle a, \text{nil} \rangle\rangle : T_5}} \quad D3}{\cdot; \cdot; \cdot; (\Sigma); a : [0] \mathbf{1} \vdash E_4 : T_5}} \quad D2$$

D1:

$$\frac{\frac{\cdot; \cdot; \cdot; (\Sigma); \cdot \vdash \text{store}() : \mathbb{M} 0 [0] \mathbf{1}}{\cdot; \cdot; \cdot; (\Sigma); \cdot \vdash E_3 : T_5}}{\cdot; \cdot; \cdot; (\Sigma); \cdot \vdash \text{store}() : \mathbb{M} 0 [0] \mathbf{1}} \quad D2}{\cdot; \cdot; \cdot; (\Sigma); \cdot \vdash E_3 : T_5}} \quad D1$$

D0:

$$\frac{\frac{\cdot; \cdot; \cdot; (\Sigma); \cdot \vdash \uparrow^{K^{\text{nil}}} : T_4}{\cdot; \cdot; \cdot; (\Sigma); \cdot \vdash E_2 : T_3}}{\cdot; \cdot; \cdot; (\Sigma); \cdot \vdash \uparrow^{K^{\text{nil}}} : T_4}} \quad D1}{\cdot; \cdot; \cdot; (\Sigma); \cdot \vdash E_2 : T_3}} \quad D0$$

Main derivation:

$$\frac{\frac{\cdot; \cdot; \cdot; (\Sigma); u : T_1 \vdash u : T_1}{\cdot; \cdot; \cdot; (\Sigma); u : T_1 \vdash E_1 : T_2}}{\cdot; \cdot; \cdot; (\Sigma); \cdot \vdash E_0 : T_0}}{\cdot; \cdot; \cdot; (\Sigma); \cdot \vdash u : T_1 \vdash u : T_1}} \quad D0$$

6. cons:

$$\frac{\vec{p} = (p_1, \dots, p_k)}{\frac{\Sigma; x_h : \tau, x_t : L^{(\Leftarrow \vec{p})} \tau \vdash_q^{q+p_1+K^{\text{cons}}} \text{cons}(x_h, x_t) : L^p \tau \rightsquigarrow}{\lambda u. \text{release } - = u \text{ in bind } - = \uparrow^{K^{\text{cons}}} \text{ in } E_0} \text{ cons}}}$$

where

$$E_0 = x_t; x. \text{let}\langle\langle x_1, x_2 \rangle\rangle = x \text{ in } E_1$$

$$E_1 = \text{release} - = x_1 \text{ in bind } a = \text{store}() \text{ in store}\langle\langle a, x_h :: x_2 \rangle\rangle$$

$$T_0 = [q + p_1 + K^{\text{cons}}] \mathbf{1} \multimap \mathbb{M}0 ([q] \exists n'. [\phi(\vec{p}, n')] \mathbf{1} \otimes L^{n'}(\tau))$$

$$T_1 = [q + p_1 + K^{\text{cons}}] \mathbf{1}$$

$$T_2 = \mathbb{M}0 ([q] \exists n'. [\phi(\vec{p}, n')] \mathbf{1} \otimes L^{n'}(\tau))$$

$$T_{2.1} = \mathbb{M}(q + p_1) ([q] \exists n'. [\phi(\vec{p}, n')] \mathbf{1} \otimes L^{n'}(\tau))$$

$$T_{2.2} = \mathbb{M}(q + p_1 + \phi(\triangleleft \vec{p}, s)) ([q] \exists n'. [\phi(\vec{p}, n')] \mathbf{1} \otimes L^{n'}(\tau))$$

$$T_{2.3} = \mathbb{M}(q) ([q] \exists n'. [\phi(\vec{p}, n')] \mathbf{1} \otimes L^{n'}(\tau))$$

$$T_{2.4} = \exists n'. [\phi(\vec{p}, n')] \mathbf{1} \otimes L^{n'}(\tau)$$

$$T_{2.5} = [\phi(\vec{p}, n')] \mathbf{1} \otimes L^{n'}(\tau)$$

$$T_3 = [(p_1 + \phi(\triangleleft \vec{p}, s))] \mathbf{1}$$

$$T_l = \exists s. ([\phi(\triangleleft \vec{p}, s)] \mathbf{1} \otimes L^s(\tau))$$

$$T_{l1} = ([\phi(\triangleleft \vec{p}, s)] \mathbf{1} \otimes L^s(\tau))$$

$$T_{l2} = [\phi(\triangleleft \vec{p}, s)] \mathbf{1}$$

$$T_{l3} = L^s(\tau)$$

D1.4:

$$\frac{\frac{.; s : \mathbb{N}; .; (\Sigma); x_h : (\tau), x_2 : T_{l3}, a : T_3 \vdash \langle\langle a, x_h :: x_2 \rangle\rangle : T_{2.5}[(s+1)/n']}{s : \mathbb{N} \vdash s+1 : \mathbb{N}}}{.; s : \mathbb{N}; .; (\Sigma); x_h : (\tau), x_2 : T_{l3}, a : T_3 \vdash \langle\langle a, x_h :: x_2 \rangle\rangle : T_{2.4}}}{.; s : \mathbb{N}; .; (\Sigma); x_h : (\tau), x_2 : T_{l3}, a : T_3 \vdash \text{store}\langle\langle a, x_h :: x_2 \rangle\rangle : T_{2.3}}$$

D1.3:

$$\frac{\frac{.; s : \mathbb{N}; .; (\Sigma); . \vdash \text{store}() : \mathbb{M}(p_1 + \phi(\triangleleft \vec{p}, s)) [p_1 + \phi(\triangleleft \vec{p}, s)] \mathbf{1}}{.; s : \mathbb{N}; .; (\Sigma); x_h : (\tau), x_2 : T_{l3} \vdash \text{bind } a = \text{store}() \text{ in store}\langle\langle a, x_h :: x_2 \rangle\rangle : T_{2.2}}}{D1.4}$$

D1.2:

$$\frac{\frac{.; s : \mathbb{N}; .; (\Sigma); x_1 : T_{l2} \vdash x_1 : T_{l2}}{.; s : \mathbb{N}; .; (\Sigma); x_h : (\tau), x_1 : T_{l2}, x_2 : T_{l3} \vdash E_1 : T_{2.1}}}{D1.3}}$$

D1.1:

$$\frac{\frac{.; s : \mathbb{N}; .; (\Sigma); x : T_{l1} \vdash x : T_{l1}}{.; s : \mathbb{N}; .; (\Sigma); x_h : (\tau), x : T_{l1} \vdash \text{let}\langle\langle x_1, x_2 \rangle\rangle = x \text{ in } E_1 : T_{2.1}}}{D1.2}}$$

D1:

$$\frac{\frac{.; .; .; .; (\Sigma); x_t : T_l \vdash x_t : T_l}{.; .; .; .; (\Sigma); x_h : (\tau), x_t : T_l \vdash E_0 : T_{2.1}}}{D1.1}}$$

D0:

$$\frac{\frac{\cdot \vdash \uparrow^{K^{cons}}}{\cdot; \cdot; \cdot; \cdot; (\Sigma); \cdot \vdash \uparrow^{K^{cons}}} \quad D1}{\cdot; \cdot; \cdot; \cdot; (\Sigma); x_h : (\tau), x_t : T_l \vdash \text{bind } - = \uparrow^{K^{cons}} \text{ in } E_0 : T_{2.1}}}$$

Main derivation:

$$\frac{\frac{\frac{\cdot; \cdot; \cdot; \cdot; (\Sigma); u : T_1 \vdash u : T_1}{\cdot; \cdot; \cdot; \cdot; (\Sigma); x_h : (\tau), x_t : T_l, u : T_1 \vdash \text{release } - = u \text{ in bind } - = \uparrow^{K^{cons}} \text{ in } E_0 : T_2} \quad D0}{\cdot; \cdot; \cdot; \cdot; (\Sigma); x_h : (\tau), x_t : T_l \vdash \lambda u. \text{release } - = u \text{ in bind } - = \uparrow^{K^{cons}} \text{ in } E_0 : T_0}}}$$

7. match:

$$\frac{\frac{\Sigma; \Gamma \vdash_{q' + K_2^{matN}}^{q - K_1^{matN}} e_1 : \tau' \rightsquigarrow e_{a1} \quad \vec{p} = (p_1, \dots, p_k) \quad \Sigma; \Gamma, h : \tau, t : L^{(\triangleleft \vec{p})} \tau \vdash_{q' + K_2^{matC}}^{q + p_1 - K_1^{matC}} e_2 : \tau' \rightsquigarrow e_{a2}}{\Sigma; \Gamma, x : L^p \tau \vdash_q^q \text{ match } x \text{ with } |nil \mapsto e_1 \mid h :: t \mapsto e_2 : \tau' \rightsquigarrow \lambda u. E_0} \text{ match}}}$$

where

$$\begin{aligned} E_0 &= \text{release } - = u \text{ in } E_{0.1} \\ E_{0.1} &= x; a. \text{let} \langle\langle x_1, x_2 \rangle\rangle = a \text{ in } E_1 \\ E_1 &= \text{match } x_2 \text{ with } |nil \mapsto E_2 \mid h :: l_t \mapsto E_3 \\ E_2 &= \text{bind } - = \uparrow^{K_1^{matN}} \text{ in } E_{2.1} \\ E_{2.1} &= \text{bind } b = \text{store}() \text{ in } E'_2 \\ E'_2 &= \text{bind } c = (e_{a1} b) \text{ in } E'_{2.1} \\ E'_{2.1} &= \text{release } d = c \text{ in } E'_{2.2} \\ E'_{2.2} &= \text{bind } - = \uparrow^{K_2^{matN}} \text{ in } E'_{2.3} \\ E'_{2.3} &= \text{release } - = x_1 \text{ in store } d \\ E_3 &= \text{bind } - = \uparrow^{K_1^{matC}} \text{ in } E_{3.1} \\ E_{3.1} &= \text{release } - = x_1 \text{ in } E_{3.2} \\ E_{3.2} &= \text{bind } b = \text{store}() \text{ in } E_{3.3} \\ E_{3.3} &= \text{bind } t = \text{ret} \langle\langle b, l_t \rangle\rangle \text{ in } E_{3.4} \\ E_{3.4} &= \text{bind } d = \text{store}() \text{ in } E_{3.5} \\ E_{3.5} &= \text{bind } f = e_{a2} d \text{ in } E_{3.6} \\ E_{3.6} &= \text{release } g = f \text{ in } E_{3.7} \\ E_{3.7} &= \text{bind } - = \uparrow^{K_2^{matC}} \text{ in store } g \\ T_0 &= [q] \mathbf{1} \multimap \mathbb{M} 0 ([q'] (\tau')) \\ T_1 &= [q] \mathbf{1} \\ T_2 &= \mathbb{M} 0 ([q'] (\tau')) \\ T_{2.0} &= \mathbb{M} q' ([q'] (\tau')) \\ T_{2.1} &= \mathbb{M} q ([q'] (\tau')) \end{aligned}$$

$$\begin{aligned}
T_{2.10} &= \mathbb{M}(q - K_1^{matC}) ([q'] \langle \tau' \rangle) \\
T_{2.11} &= \mathbb{M}(q - K_1^{matN}) ([q'] \langle \tau' \rangle) \\
T_{2.12} &= \mathbb{M}(q - K_1^{matN}) ([q - K^{matN}] \mathbf{1}) \\
T_{2.13} &= ([q - K_1^{matN}] \mathbf{1}) \\
T_3 &= \mathbb{M}(q - K_1^{matC} + p_1 + \phi(\triangleleft \vec{p}, i)) [q'] \langle \tau' \rangle \\
T_{3.0} &= \mathbb{M}(q - K_1^{matC} + p_1) [q'] \langle \tau' \rangle \\
T_{3.1} &= \mathbb{M} 0 [q'] \langle \tau' \rangle \\
T_{3.2} &= \mathbb{M}(q' + K_2^{matC}) [q'] \langle \tau' \rangle \\
T_{4.0} &= \mathbb{M}(\phi(\triangleleft \vec{p}, i)) \mathbf{1} \\
T_{4.10} &= \mathbb{M} 0 T_{4.1} \\
T_{4.1} &= \exists s'. ([\phi(\triangleleft \vec{p}, s')]) \mathbf{1} \otimes L^{s'} \langle \tau \rangle \\
T_{4.11} &= ([\phi(\triangleleft \vec{p}, i)]) \mathbf{1} \otimes L^i \langle \tau \rangle \\
T_{4.12} &= ([\phi(\triangleleft \vec{p}, i)]) \mathbf{1} \\
T_{4.13} &= L^i \langle \tau \rangle \\
T_{4.2} &= \mathbb{M}(q - k_1^{matC} + p_1) [(q - k_1^{matC} + p_1)] \mathbf{1} \\
T_{4.3} &= \mathbb{M} 0 [(q' + K_2^{matC})] \langle \tau \rangle \\
T_{4.4} &= [(q' + K_2^{matC})] \langle \tau \rangle \\
T_b &= [\phi(\triangleleft \vec{p}, s')] \mathbf{1} \\
T_c &= \exists s'. ([\phi(\triangleleft \vec{p}, s')]) \mathbf{1} \otimes L^{s'} \langle \tau \rangle \\
T_d &= [q - K_1^{matC} + p_1] \mathbf{1} \\
T_f &= T_{4.4} \\
T_g &= \langle \tau \rangle \\
T_l &= \exists s. ([\phi(\vec{p}, s)]) \mathbf{1} \otimes L^s \langle \tau \rangle \\
T'_l &= ([\phi(\vec{p}, s)]) \mathbf{1} \otimes L^s \langle \tau \rangle \\
T_{l1} &= [\phi(\vec{p}, s)] \mathbf{1} \\
T_{l2} &= L^s \langle \tau \rangle \\
T_{l3} &= \exists s'. ([\phi(\triangleleft \vec{p}, s')]) \mathbf{1} \otimes L^{s'} \langle \tau \rangle \\
T_{l4} &= L^i \langle \tau \rangle \\
T_{ih1} &= [q - K_1^{matN}] \mathbf{1} \multimap \mathbb{M} 0 ([q' + K_2^{matN}] \langle \tau' \rangle) \\
T_{ih1.1} &= \mathbb{M} 0 ([q' + K_2^{matN}] \langle \tau' \rangle) \\
T_{ih1.2} &= ([q' + K_2^{matN}] \langle \tau' \rangle) \\
T_{ih2} &= [q + p_1 - K_1^{matC}] \mathbf{1} \multimap \mathbb{M} 0 ([q' + K_2^{matC}] \langle \tau' \rangle) \\
T_{ih2.1} &= \mathbb{M} 0 ([q' + K_2^{matC}] \langle \tau' \rangle)
\end{aligned}$$

D3.8:

$$.; s, i; s = i + 1; \langle \Sigma \rangle; g : T_g \vdash \text{store } g : \mathbb{M} q' [q'] \langle \tau \rangle$$

D3.7:

$$\frac{\frac{.; s, i; s = i + 1; (\Sigma); \cdot \vdash \uparrow^{K_2^{matC}} : \mathbb{M} K_2^{matC} \mathbf{1}}{.; s, i; s = i + 1; (\Sigma); g : T_g \vdash E_{3.7} : T_{3.3}}}{E_{3.8}}$$

D3.6:

$$\frac{\frac{.; s, i; s = i + 1; (\Sigma); f : T_f \vdash f : T_f}{.; s, i; s = i + 1; (\Sigma); f : T_f \vdash E_{3.6} : T_{3.2}}}{E_{3.7}}$$

D3.5:

$$\frac{\frac{.; s, i; s = i + 1; (\Sigma); (\Gamma), h : (\tau), t : T_c, d : T_d \vdash e_{a2} d : T_{4.3}}{.; s, i; s = i + 1; (\Sigma); (\Gamma), h : (\tau), t : T_c, d : T_d \vdash E_{3.5} : T_{3.2}}}{E_{3.6}}$$

D3.4:

$$\frac{\frac{.; s, i; s = i + 1; (\Sigma); \cdot \vdash \text{store}() : T_{4.2}}{.; s, i; s = i + 1; (\Sigma); (\Gamma), h : (\tau), t : T_c \vdash E_{3.4} : T_{3.1}}}{E_{3.5}}$$

D3.31:

$$\frac{\frac{.; s, i; s = i + 1; (\Sigma); l_t : T_{l4}, b : T_b \vdash \langle\langle b, l_t \rangle\rangle : T_{4.11}}{.; s, i; s = i + 1; (\Sigma); l_t : T_{l4}, b : T_b \vdash \langle\langle b, l_t \rangle\rangle : T_{4.1}}}{E_{3.6}}$$

D3.3:

$$\frac{\frac{D3.31}{.; s, i; s = i + 1; (\Sigma); l_t : T_{l4}, b : T_b \vdash \text{ret}\langle\langle b, l_t \rangle\rangle : T_{4.10}}{.; s, i; s = i + 1; (\Sigma); (\Gamma), h : (\tau), l_t : T_{l4}, b : T_b \vdash E_{3.3} : T_{3.1}}}{D_{3.4}}$$

D3.2:

$$\frac{\frac{.; s, i; s = i + 1; (\Sigma); \cdot \vdash \text{store}() : T_{4.0}}{.; s, i; s = i + 1; (\Sigma); (\Gamma); h : (\tau), l_t : T_{l4} \vdash E_{3.2} : T_3}}{D_{3.3}}$$

D3.1:

$$\frac{\frac{.; s, i; s = i + 1; (\Sigma); x_1 : T_{l1} \vdash x_1 : T_{l1}}{.; s, i; s = i + 1; (\Sigma); (\Gamma), x_1 : T_{l1}, h : (\tau), l_t : T_{l4} \vdash E_{3.1} : T_{2.10}}}{D_{3.2}}$$

D3:

$$\frac{\frac{.; s, i; s = i + 1; (\Sigma); \cdot \vdash \uparrow^{K_1^{matC}} : \mathbb{M} K_1^{matC} \mathbf{1}}{.; s, i; s = i + 1; (\Sigma); (\Gamma), x_1 : T_{l1}, h : (\tau), l_t : T_{l4} \vdash E_3 : T_{2.1}}}{D_{3.1}}$$

D2.32:

$$\frac{\frac{.; s; s = 0; (\Sigma); x_1 : T_{l1} \vdash x_1 : T_{l1}}{.; s; s = 0; (\Sigma); x_1 : T_{l1}, d : (\tau') \vdash \text{store } d : T_{2.0}}}{.; s; s = 0; (\Sigma); x_1 : T_{l1}, d : (\tau') \vdash E'_{2.3} : T_{2.0}}$$

D2.31:

$$\frac{\frac{.; s; s = 0; (\Sigma); \cdot \vdash \uparrow^{K_2^{matN}} : \mathbb{M} K_2^{matN} \mathbf{1}}{.; s; s = 0; (\Sigma); x_1 : T_{l1}, d : (\tau') \vdash E'_{2.2} : T_{3.2}}}{D_{2.32}}$$

D2.3:

$$\frac{\frac{\cdot; s; s = 0; (\Sigma); c : T_{ih1.2} \vdash c : T_{ih1.2}}{\cdot; s; s = 0; (\Sigma); x_1 : T_{l1}, c : T_{ih1.2} \vdash E'_{2.1} : T_2} \quad D2.31}{\cdot; s; s = 0; (\Sigma); x_1 : T_{l1}, c : T_{ih1.2} \vdash E'_{2.1} : T_2}$$

D2.22:

$$\frac{\cdot; s; s = 0; (\Sigma); b : T_{2.13} \vdash b : T_{2.13}}{\cdot; s; s = 0; (\Sigma); b : T_{2.13} \vdash b : T_{2.13}}$$

D2.21:

$$\frac{\cdot; s; s = 0; (\Sigma); (\Gamma) \vdash e_{a1} : T_{ih1}}{\cdot; s; s = 0; (\Sigma); (\Gamma) \vdash e_{a1} : T_{ih1}}$$

D2.2:

$$\frac{\frac{D2.21 \quad D2.22}{\cdot; s; s = 0; (\Sigma); (\Gamma), b : T_{2.13} \vdash e_{a1} b : T_{ih1.1}}}{\cdot; s; s = 0; (\Sigma); (\Gamma), b : T_{2.13} \vdash e_{a1} b : T_{ih1.1}}$$

D2.20:

$$\frac{\frac{D2.2 \quad D2.3}{\cdot; s; s = 0; (\Sigma); (\Gamma), x_1 : T_{l1}, b : T_{2.13} \vdash E'_2 : T_2}}{\cdot; s; s = 0; (\Sigma); (\Gamma), x_1 : T_{l1}, b : T_{2.13} \vdash E'_2 : T_2}}$$

D2.1:

$$\frac{\frac{\cdot; s; s = 0; (\Sigma); \cdot \vdash \text{store}() : T_{2.12}}{\cdot; s; s = 0; (\Sigma); (\Gamma), x_1 : T_{l1} \vdash E_{2.1} : T_{2.11}} \quad D2.20}{\cdot; s; s = 0; (\Sigma); (\Gamma), x_1 : T_{l1} \vdash E_{2.1} : T_{2.11}}$$

D2:

$$\frac{\cdot; s; s = 0; (\Sigma); \cdot \vdash \uparrow^{K_1^{matN}} : \mathbb{M} K_1^{matN} \mathbf{1}}{\cdot; s; s = 0; (\Sigma); (\Gamma), x_1 : T_{l1} \vdash E_2 : T_{2.1}} \quad D2.1$$

D1.1:

$$\frac{\frac{\cdot; s; \cdot; (\Sigma); x_2 : T_{l2} \vdash x_2 : T_{l2}}{\cdot; s; \cdot; (\Sigma); (\Gamma), x_1 : T_{l1}, x_2 : T_{l2} \vdash E_1 : T_{2.1}} \quad D2 \quad D3}{\cdot; s; \cdot; (\Sigma); (\Gamma), x_1 : T_{l1}, x_2 : T_{l2} \vdash E_1 : T_{2.1}}}$$

D1:

$$\frac{\frac{\cdot; s; \cdot; (\Sigma); a : T'_l \vdash a : T'_l}{\cdot; s; \cdot; (\Sigma); (\Gamma), a : T'_l \vdash \text{let}\langle\langle x_1, x_2 \rangle\rangle = a \text{ in } E_1 : T_{2.1}} \quad D1.1}{\cdot; s; \cdot; (\Sigma); (\Gamma), a : T'_l \vdash \text{let}\langle\langle x_1, x_2 \rangle\rangle = a \text{ in } E_1 : T_{2.1}}}$$

D0:

$$\frac{\frac{\cdot; \cdot; \cdot; (\Sigma); x : T_l \vdash x : T_l}{\cdot; \cdot; \cdot; (\Sigma); (\Gamma), x : T_l \vdash E_{0.1} : T_{2.1}} \quad D1}{\cdot; \cdot; \cdot; (\Sigma); (\Gamma), x : T_l \vdash E_{0.1} : T_{2.1}}}$$

Main derivation:

$$\frac{\frac{\cdot; \cdot; \cdot; (\Sigma); (\Gamma), x : T_l, u : T_1 \vdash u : T_1}{\cdot; \cdot; \cdot; (\Sigma); (\Gamma), x : T_l, u : T_1 \vdash E_0 : T_2} \quad D0}{\cdot; \cdot; \cdot; (\Sigma); (\Gamma), x : T_l \vdash \lambda u. E_0 : T_0}}$$

8. Share:

$$\frac{\Sigma; \Gamma, x : \tau_1, y : \tau_2 \vdash_{q'}^q e : \tau' \rightsquigarrow e_a \quad \tau = \tau_1 \vee \tau_2 \quad \tau = \tau_1 = \tau_2 = \mathbf{1}}{\Sigma; \Gamma, z : \tau \vdash_{q'}^q e[z/x, z/y] : \tau' \rightsquigarrow E_0} \text{Share-unit}$$

$$E_0 = \lambda u. E_1$$

$$E_1 = \text{bind } a = \text{coerce}_{\mathbf{1}, \mathbf{1}, \mathbf{1}} z \text{ in } \text{let} \langle\langle x, y \rangle\rangle = a \text{ in } e_a u$$

$$T_0 = [q] \mathbf{1} \multimap \mathbb{M} 0 ([q'] (\tau'))$$

D1:

$$\frac{\frac{.; .; .; (\Sigma); (\Gamma), u : [q] \mathbf{1}, x : (\tau_1), y : (\tau_2) \vdash e_a : T_0}{.; .; .; (\Sigma); (\Gamma), u : [q] \mathbf{1}, x : (\tau_1), y : (\tau_2) \vdash e_a u : \mathbb{M} 0 [q'] \mathbf{1}} \quad \frac{.; .; .; (\Sigma); u : [q] \mathbf{1} \vdash u : [q] \mathbf{1}}{}}{.; .; .; (\Sigma); (\Gamma), u : [q] \mathbf{1} \vdash u : [q] \mathbf{1}}$$

D0:

$$\frac{\frac{.; .; .; (\Sigma); a : ((\tau_1) \otimes (\tau_2)) \vdash a : ((\tau_1) \otimes (\tau_2))}{.; .; .; (\Sigma); (\Gamma), u : [q] \mathbf{1}, a : ((\tau_1) \otimes (\tau_2)) \vdash \text{let} \langle\langle x, y \rangle\rangle = a \text{ in } e_a u : [q] \multimap \mathbb{M} 0 [q] (\tau')} \quad D1}{.; .; .; (\Sigma); (\Gamma), u : [q] \mathbf{1}, a : ((\tau_1) \otimes (\tau_2)) \vdash \text{let} \langle\langle x, y \rangle\rangle = a \text{ in } e_a u : [q] \multimap \mathbb{M} 0 [q] (\tau')} D0$$

Main derivation:

$$\frac{\frac{\frac{Dc1 \quad \frac{.; .; .; (\Sigma); z : (\tau) \vdash z : (\tau)}{.; .; .; (\Sigma); z : (\tau) \vdash \text{coerce}_{\mathbf{1}, \mathbf{1}, \mathbf{1}} z : \mathbb{M} 0 ((\tau_1) \otimes (\tau_2))}{.; .; .; (\Sigma); (\Gamma), z : (\tau), u : [q] \mathbf{1} \vdash E_0 : \mathbb{M} 0 [q'] (\tau')}}{.; .; .; (\Sigma); (\Gamma), z : (\tau) \vdash \lambda u. E_0 : T_0}} \quad D0}{.; .; .; (\Sigma); (\Gamma), z : (\tau) \vdash \lambda u. E_0 : T_0} Dc1$$

$$\text{coerce}_{\mathbf{1}, \mathbf{1}, \mathbf{1}} : (\mathbf{1}) \multimap \mathbb{M} 0 ((\mathbf{1}) \otimes (\mathbf{1}))$$

$$\text{coerce}_{\mathbf{1}, \mathbf{1}, \mathbf{1}} \triangleq \lambda u. \text{ret} \langle\langle !(), !() \rangle\rangle$$

$$T_{c0} = (\mathbf{1}) \multimap \mathbb{M} 0 ((\mathbf{1}) \otimes (\mathbf{1}))$$

$$T_{c1} = \mathbb{M} 0 ((\mathbf{1}) \otimes (\mathbf{1}))$$

$$T_{c2} = (\mathbf{1}) \otimes (\mathbf{1})$$

Dc1:

$$\frac{\frac{\frac{.; .; .; .; \vdash \langle\langle !(), !() \rangle\rangle : T_{c2}}{.; .; .; .; u : (\mathbf{1}) \vdash \langle\langle !(), !() \rangle\rangle : T_{c2}}{.; .; .; .; u : (\mathbf{1}) \vdash \text{ret} \langle\langle !(), !() \rangle\rangle : T_{c1}}{.; .; .; .; \vdash \lambda u. \text{ret} \langle\langle !(), !() \rangle\rangle : T_{c0}}$$

$$\frac{\Sigma; \Gamma, x : \tau_1, y : \tau_2 \vdash_{q'}^q e : \tau' \rightsquigarrow e_a \quad \tau = \tau_1 \vee \tau_2 \quad \tau = \tau_1 = \tau_2 = \mathbf{b}}{\Sigma; \Gamma, z : \tau \vdash_{q'}^q e[z/x, z/y] : \tau' \rightsquigarrow E_0} \text{Share-base}$$

$$E_0 = \lambda u. E_1$$

$$E_1 = \text{bind } a = \text{coerce}_{\mathbf{b}, \mathbf{b}, \mathbf{b}} z \text{ in } \text{let} \langle\langle x, y \rangle\rangle = a \text{ in } e_a u$$

$$T_0 = [q] \mathbf{1} \multimap \mathbb{M} 0 [q'] (\tau')$$

D1:

$$\frac{\frac{.; .; .; (\Sigma); (\Gamma), u : [q] \mathbf{1}, x : (\tau_1), y : (\tau_2) \vdash e_a : T_0}{.; .; .; (\Sigma); (\Gamma), u : [q] \mathbf{1}, x : (\tau_1), y : (\tau_2) \vdash e_a u : \mathbb{M} 0 [q'] \mathbf{1}} \quad \frac{.; .; .; (\Sigma); u : [q] \mathbf{1} \vdash u : [q] \mathbf{1}}{D1}}{.; .; .; (\Sigma); (\Gamma), u : [q] \mathbf{1}, x : (\tau_1), y : (\tau_2) \vdash e_a u : \mathbb{M} 0 [q'] \mathbf{1}}$$

D0:

$$\frac{\frac{.; .; .; (\Sigma); a : ((\tau_1) \otimes (\tau_2)) \vdash a : ((\tau_1) \otimes (\tau_2))}{D1}}{.; .; .; (\Sigma); (\Gamma), u : [q] \mathbf{1}, a : ((\tau_1) \otimes (\tau_2)) \vdash \text{let} \langle\langle x, y \rangle\rangle = a \text{ in } e_a u : [q] \multimap \mathbb{M} 0 [q] (\tau')}$$

Main derivation:

$$\frac{\frac{\frac{Dc1 \quad \frac{.; .; .; (\Sigma); z : (\tau) \vdash z : (\tau)}{D0}}{.; .; .; (\Sigma); z : (\tau) \vdash \text{coerce}_{\mathbf{b}, \mathbf{b}, \mathbf{b}} z : \mathbb{M} 0 ((\tau_1) \otimes (\tau_2))}{.; .; .; (\Sigma); (\Gamma), z : (\tau), u : [q] \mathbf{1} \vdash E_0 : \mathbb{M} 0 [q'] (\tau')}}{.; .; .; (\Sigma); (\Gamma), z : (\tau) \vdash \lambda u. E_0 : T_0}}$$

$$\text{coerce}_{\mathbf{b}, \mathbf{b}, \mathbf{b}} : (\mathbf{b}) \multimap \mathbb{M} 0 ((\mathbf{b}) \otimes (\mathbf{b}))$$

$$\text{coerce}_{\mathbf{b}, \mathbf{b}, \mathbf{b}} \triangleq \lambda u. \text{let} !u' = u \text{ in } \text{ret} \langle\langle !u', !u' \rangle\rangle$$

$$T_{c0} = (\mathbf{b}) \multimap \mathbb{M} 0 ((\mathbf{b}) \otimes (\mathbf{b}))$$

$$T_{c1} = \mathbb{M} 0 ((\mathbf{b}) \otimes (\mathbf{b}))$$

$$T_{c2} = (\mathbf{b}) \otimes (\mathbf{b})$$

Dc2:

$$\frac{.; .; .; u' : \mathbf{b}; . \vdash \langle\langle !u', !u' \rangle\rangle : T_{c2}}{.; .; .; u' : \mathbf{b}; . \vdash \text{ret} \langle\langle !u', !u' \rangle\rangle : T_{c1}}$$

Dc1:

$$\frac{\frac{.; .; .; .; u : !\mathbf{b} \vdash u : !\mathbf{b}}{Dc2}}{.; .; .; .; u : (\mathbf{b}) \vdash \text{let} !u' = u \text{ in } \text{ret} \langle\langle !u', !u' \rangle\rangle : T_{c1}}{.; .; .; .; . \vdash \lambda u. \text{let} !u' = u \text{ in } \text{ret} \langle\langle !u', !u' \rangle\rangle : T_{c0}}$$

$$\frac{\tau = L^{\vec{p}} \tau'' \quad \tau_1 = L^{\vec{p}_1} \tau_1'' \quad \tau_2 = L^{\vec{p}_2} \tau_2'' \quad \tau'' = \tau_1'' \curlywedge \tau_2'' \quad \vec{p} = \vec{p}_1 + \vec{p}_2 \quad \text{Share-list}}{\Sigma; \Gamma, z : \tau \vdash_{q'}^q e[z/x, z/y] : \tau' \rightsquigarrow E_0}$$

$$E_0 = \lambda u. E_1$$

$$E_1 = \text{bind } a = \text{coerce}_{\tau, \tau_1, \tau_2} z \text{ in let} \langle\langle x, y \rangle\rangle = a \text{ in } e_a u$$

$$T_0 = [q] \mathbf{1} \multimap \mathbb{M} 0 ([q'] (\tau'))$$

D1:

$$\frac{\frac{.; .; .; (\Sigma); (\Gamma), u : [q] \mathbf{1}, x : (\tau_1), y : (\tau_2) \vdash e_a : T_0}{.; .; .; (\Sigma); (\Gamma), u : [q] \mathbf{1}, x : (\tau_1), y : (\tau_2) \vdash e_a u : \mathbb{M} 0 [q'] \mathbf{1}} \quad \frac{.; .; .; (\Sigma); u : [q] \mathbf{1} \vdash u : [q] \mathbf{1}}{D1}}{.; .; .; (\Sigma); (\Gamma), u : [q] \mathbf{1}, a : ((\tau_1) \otimes (\tau_2)) \vdash \text{let} \langle\langle x, y \rangle\rangle = a \text{ in } e_a u : [q] \multimap \mathbb{M} 0 [q] (\tau')}$$

D0:

$$\frac{\frac{.; .; .; (\Sigma); a : ((\tau_1) \otimes (\tau_2)) \vdash a : ((\tau_1) \otimes (\tau_2))}{D1}}{.; .; .; (\Sigma); (\Gamma), u : [q] \mathbf{1}, a : ((\tau_1) \otimes (\tau_2)) \vdash \text{let} \langle\langle x, y \rangle\rangle = a \text{ in } e_a u : [q] \multimap \mathbb{M} 0 [q] (\tau')}$$

Main derivation:

$$\frac{\frac{\frac{.; .; .; (\Sigma); z : (\tau) \vdash \text{coerce}_{\tau, \tau_1, \tau_2} z : \mathbb{M} 0 ((\tau_1) \otimes (\tau_2))}{D0}}{.; .; .; (\Sigma); (\Gamma), z : (\tau), u : [q] \mathbf{1} \vdash E_0 : \mathbb{M} 0 [q'] (\tau')}}{.; .; .; (\Sigma); (\Gamma), z : (\tau) \vdash \lambda u. E_0 : T_0}$$

$$\text{coerce}_{L^{\vec{p}}\tau, L^{\vec{p}_1}\tau_1, L^{\vec{p}_2}\tau_2} : !((\tau) \multimap \mathbb{M} 0 ((\tau_1) \otimes (\tau_2))) \multimap (L^{\vec{p}}\tau) \multimap \mathbb{M} 0 (L^{\vec{p}_1}\tau_1) \otimes (L^{\vec{p}_2}\tau_2)$$

$$\text{coerce}_{L^{\vec{p}}\tau, L^{\vec{p}_1}\tau_1, L^{\vec{p}_2}\tau_2} \triangleq \text{fix } f. \lambda_g. \lambda e. \text{let } !g' = g \text{ in } e; x. \text{let} \langle\langle p, l \rangle\rangle = x \text{ in } E_0$$

where

$$E_0 \triangleq \text{release } - = p \text{ in } E_1$$

$$E_1 \triangleq \text{match } l \text{ with } | \text{nil} \mapsto E_{2.1} \mid h :: t \mapsto E_3$$

$$E_{2.1} \triangleq \text{bind } z_1 = \text{store}() \text{ in } E_{2.2}$$

$$E_{2.2} \triangleq \text{bind } z_2 = \text{store}() \text{ in } E_{2.3}$$

$$E_{2.3} \triangleq \text{ret} \langle\langle\langle z_1, \text{nil} \rangle\rangle, \langle\langle z_2, \text{nil} \rangle\rangle\rangle$$

$$E_3 \triangleq \text{bind } H = g' h \text{ in } E_{3.1}$$

$$E_{3.1} \triangleq \text{bind } o_t = () \text{ in } E_{3.2}$$

$$E_{3.2} \triangleq \text{bind } T = f g \langle\langle o_t, t \rangle\rangle \text{ in } E_4$$

$$E_4 \triangleq \text{let} \langle\langle H_1, H_2 \rangle\rangle = H \text{ in } E_5$$

$$E_5 \triangleq \text{let}\langle\langle T_1, T_2 \rangle\rangle = T \text{ in } E_6$$

$$E_6 \triangleq T_1; tp_1. \text{let}\langle\langle p'_1, l'_1 \rangle\rangle = tp_1 \text{ in } E_{7.1}$$

$$E_{7.1} \triangleq T_2; tp_2. \text{let}\langle\langle p'_2, l'_2 \rangle\rangle = tp_2 \text{ in } E_{7.2}$$

$$E_{7.2} \triangleq \text{release } - = p'_1 \text{ in } E_{7.3}$$

$$E_{7.3} \triangleq \text{release } - = p'_2 \text{ in } E_{7.4}$$

$$E_{7.4} \triangleq \text{bind } o_1 = \text{store}() \text{ in } E_{7.5}$$

$$E_{7.5} \triangleq \text{bind } o_2 = \text{store}() \text{ in } E_8$$

$$E_8 \triangleq \text{ret}\langle\langle\langle o_1, H_1 :: T_1 \rangle\rangle, \langle\langle o_2, H_2 :: T_2 \rangle\rangle\rangle$$

$$T_0 = !((\tau) \multimap \mathbb{M}0((\tau_1) \otimes (\tau_2))) \multimap (L^{\vec{p}}\tau) \multimap \mathbb{M}0((L^{\vec{p}^1}\tau_1) \otimes (L^{\vec{p}^2}\tau_2))$$

$$T_1 = !((\tau) \multimap \mathbb{M}0((\tau_1) \otimes (\tau_2)))$$

$$T'_1 = ((\tau) \multimap \mathbb{M}0((\tau_1) \otimes (\tau_2)))$$

$$T_{1.0} = \exists s.([\phi(\vec{p}, s)] \mathbf{1} \otimes L^s(\tau))$$

$$T_{1.1} = ([\phi(\vec{p}, s)] \mathbf{1} \otimes L^s(\tau))$$

$$T_{1.2} = [\phi(\vec{p}, s)] \mathbf{1}$$

$$T_{1.3} = L^s(\tau)$$

$$T_2 = (L^{\vec{p}}\tau) \multimap \mathbb{M}0((L^{\vec{p}^1}\tau_1) \otimes (L^{\vec{p}^2}\tau_2))$$

$$T_3 = \mathbb{M}0((L^{\vec{p}^1}\tau_1) \otimes (L^{\vec{p}^2}\tau_2))$$

$$T_{3.1} = \mathbb{M}(\phi(\vec{p}, s))((L^{\vec{p}^1}\tau_1) \otimes (L^{\vec{p}^2}\tau_2))$$

$$T_{3.11} = \mathbb{M}(\phi(\triangleleft \vec{p}, s - 1))([\phi(\triangleleft \vec{p}, s - 1)]) \mathbf{1}$$

$$T_{3.12} = [(\phi(\triangleleft \vec{p}, s - 1))] \mathbf{1}$$

$$T_4 = \mathbb{M}0((\tau_1) \otimes (\tau_2))$$

$$T_{4.1} = ((\tau_1) \otimes (\tau_2))$$

$$T_5 = \mathbb{M}0((L^{\triangleleft \vec{p}^1}\tau_1) \otimes (L^{\triangleleft \vec{p}^2}\tau_2))$$

$$T_{5.1} = ((L^{\triangleleft \vec{p}^1}\tau_1) \otimes (L^{\triangleleft \vec{p}^2}\tau_2))$$

$$T_{5.2} = (L^{\triangleleft \vec{p}^1}\tau_1) = \exists s'_1.([\phi(\triangleleft \vec{p}^1, s'_1)] \mathbf{1} \otimes L^{s'_1}(\tau_1))$$

$$T_{5.21} = ([\phi(\triangleleft \vec{p}^1, s'_1)] \mathbf{1} \otimes L^{s'_1}(\tau_1))$$

$$T_{5.22} = [\phi(\triangleleft \vec{p}^1, s'_1)] \mathbf{1}$$

$$T_{5.23} = L^{s'_1}(\tau_1)$$

$$T_{5.3} = (L^{\triangleleft \vec{p}^2}\tau_2) = \exists s'_2.([\phi(\triangleleft \vec{p}^2, s'_2)] \mathbf{1} \otimes L^{s'_2}(\tau_2))$$

$$T_{5.31} = ([\phi(\triangleleft \vec{p}^2, s'_2)] \mathbf{1} \otimes L^{s'_2}(\tau_2))$$

$$T_{5.32} = [\phi(\triangleleft \vec{p}^2, s'_2)] \mathbf{1}$$

$$\begin{aligned}
T_{5.33} &= L^{s'_2}(\tau_2) \\
P_1 &= \vec{p}_1 \downarrow_1 + \phi(\langle \vec{p}_1, s'_1 \rangle) \\
P_2 &= \vec{p}_2 \downarrow_1 + \phi(\langle \vec{p}_2, s'_2 \rangle) \\
T_6 &= \mathbb{M} P_1 ([P_1] \mathbf{1}) \\
T_{6.1} &= [P_1] \mathbf{1} \\
T_7 &= \mathbb{M} P_2 ([P_2] \mathbf{1}) \\
T_{7.1} &= [P_2] \mathbf{1} \\
T_{8.0} &= \mathbb{M}(\vec{p} \downarrow_1) ((L^{\vec{p}_1} \tau_1) \otimes (L^{\vec{p}_2} \tau_2)) \\
T_{8.1} &= \mathbb{M}(\vec{p} \downarrow_1 + P_1) ((L^{\vec{p}_1} \tau_1) \otimes (L^{\vec{p}_2} \tau_2)) \\
T_{8.2} &= \mathbb{M}(\vec{p} \downarrow_1 + P_1 + P_2) ((L^{\vec{p}_1} \tau_1) \otimes (L^{\vec{p}_2} \tau_2)) \\
T_{8.3} &= \mathbb{M}(\vec{p}_2 \downarrow_1 + P_2) ((L^{\vec{p}_1} \tau_1) \otimes (L^{\vec{p}_2} \tau_2)) \\
T_{8.4} &= \mathbb{M} 0 ((L^{\vec{p}_1} \tau_1) \otimes (L^{\vec{p}_2} \tau_2)) \\
T_{8.41} &= (L^{\vec{p}_1} \tau_1) \otimes (L^{\vec{p}_2} \tau_2) \\
T_{8.5} &= (L^{\vec{p}_1} \tau_1) \\
T_{8.51} &= \exists s_1. ([\phi(\vec{p}_1, s_1)] \mathbf{1} \otimes L[s_1](\tau_1)) \\
T_{8.52} &= ([\phi(\vec{p}_1, s'_1)] \mathbf{1} \otimes L^{s'_1}(\tau_1)) \\
T_{8.6} &= (L^{\vec{p}_2} \tau_2) \\
T_{8.61} &= \exists s_2. ([\phi(\vec{p}_2, s_2)] \mathbf{1} \otimes L[s_2](\tau_2)) \\
T_{8.62} &= ([\phi(\vec{p}_2, s'_2)] \mathbf{1} \otimes L^{s'_2}(\tau_2))
\end{aligned}$$

D1.82:

$$\frac{\cdot; s'_2, s'_1, s, \cdot; g' : T'_1, f : T_0; H_2 : \langle \tau_2 \rangle, l'_2 : T_{5.33}, o_2 : T_{7.1} \vdash \langle\langle o_2, H_2 :: l'_2 \rangle\rangle : T_{8.62}}{\cdot; s'_2, s'_1, s, \cdot; g' : T'_1, f : T_0; H_2 : \langle \tau_2 \rangle, l'_2 : T_{5.33}, o_2 : T_{7.1} \vdash \langle\langle o_2, H_2 :: l'_2 \rangle\rangle : T_{8.61}}$$

D1.81:

$$\frac{\cdot; s'_2, s'_1, s, \cdot; g' : T'_1, f : T_0; H_1 : \langle \tau_1 \rangle, l'_1 : T_{5.23}, o_1 : T_{6.1} \vdash \langle\langle o_1, H_1 :: l'_1 \rangle\rangle : T_{8.52}}{\cdot; s'_2, s'_1, s, \cdot; g' : T'_1, f : T_0; H_1 : \langle \tau_1 \rangle, l'_1 : T_{5.23}, o_1 : T_{6.1} \vdash \langle\langle o_1, H_1 :: l'_1 \rangle\rangle : T_{8.51}}$$

D1.8:

$$\frac{\frac{\frac{\cdot; s'_2, s'_1, s, \cdot; g' : T'_1, f : T_0; H_1 : \langle \tau_1 \rangle, H_2 : \langle \tau_2 \rangle, l'_1 : T_{5.23}, l'_2 : T_{5.33}, o_1 : T_{6.1}, o_2 : T_{7.1} \vdash \langle\langle\langle o_1, H_1 :: l'_1 \rangle\rangle, \langle\langle o_2, H_2 :: l'_2 \rangle\rangle\rangle : T_{8.41}}{\cdot; s'_2, s'_1, s, \cdot; g' : T'_1, f : T_0; H_1 : \langle \tau_1 \rangle, H_2 : \langle \tau_2 \rangle, l'_1 : T_{5.23}, l'_2 : T_{5.33}, o_1 : T_{6.1}, o_2 : T_{7.1} \vdash \text{ret}\langle\langle\langle o_1, H_1 :: l'_1 \rangle\rangle, \langle\langle o_2, H_2 :: l'_2 \rangle\rangle\rangle : T_{8.4}}}{\cdot; s'_2, s'_1, s, \cdot; g' : T'_1, f : T_0; H_1 : \langle \tau_1 \rangle, H_2 : \langle \tau_2 \rangle, l'_1 : T_{5.23}, l'_2 : T_{5.33}, o_1 : T_{6.1}, o_2 : T_{7.1} \vdash E_8 : T_{8.4}}$$

D1.75:

$$\frac{\frac{\cdot; s'_2, s'_1, s, \cdot; g' : T'_1, f : T_0; \cdot \vdash \text{store}() : T_7}{\cdot; s'_2, s'_1, s, \cdot; g' : T'_1, f : T_0; H_1 : \langle \tau_1 \rangle, H_2 : \langle \tau_2 \rangle, l'_1 : T_{5.23}, l'_2 : T_{5.33}, o_1 : T_{6.1} \vdash \text{bind } o_2 = \text{store}() \text{ in } E_8 : T_{8.3}}}{\cdot; s'_2, s'_1, s, \cdot; g' : T'_1, f : T_0; H_1 : \langle \tau_1 \rangle, H_2 : \langle \tau_2 \rangle, l'_1 : T_{5.23}, l'_2 : T_{5.33}, o_1 : T_{6.1} \vdash E_{7.5} : T_{8.3}} \quad D1.8$$

D1.74:

$$\frac{\frac{\overline{.; s'_2, s'_1, s; ; g' : T'_1, f : T_0; . \vdash \text{store}() : T_6}}{D1.75}}{.; s'_2, s'_1, s; ; g' : T'_1, f : T_0; H_1 : (\tau_1), H_2 : (\tau_2), l'_1 : T_{5.23}, l'_2 : T_{5.33} \vdash \text{bind } o_1 = \text{store}() \text{ in } E_{7.5} : T_{8.2}}}{.; s'_2, s'_1, s; ; g' : T'_1, f : T_0; H_1 : (\tau_1), H_2 : (\tau_2), l'_1 : T_{5.23}, l'_2 : T_{5.33} \vdash E_{7.4} : T_{8.2}}$$

D1.73:

$$\frac{\frac{\overline{.; s'_2, s'_1, s; ; g' : T'_1, f : T_0; p'_2 : T_{5.32} \vdash p'_2 : T_{5.32}}}{D1.74}}{.; s'_2, s'_1, s; ; g' : T'_1, f : T_0; H_1 : (\tau_1), H_2 : (\tau_2), l'_1 : T_{5.23}, p'_2 : T_{5.32}, l'_2 : T_{5.33} \vdash \text{release } - = p'_2 \text{ in } E_{7.4} : T_{8.1}}}{.; s'_2, s'_1, s; ; g' : T'_1, f : T_0; H_1 : (\tau_1), H_2 : (\tau_2), l'_1 : T_{5.23}, p'_2 : T_{5.32}, l'_2 : T_{5.33} \vdash E_{7.3} : T_{8.1}}$$

D1.72:

$$\frac{\frac{\overline{.; s'_2, s'_1, s; ; g' : T'_1, f : T_0; p'_1 : T_{5.22} \vdash p'_1 : T_{5.22}}}{D1.73}}{.; s'_2, s'_1, s; ; g' : T'_1, f : T_0; H_1 : (\tau_1), H_2 : (\tau_2), p'_1 : T_{5.22}, l'_1 : T_{5.23}, p'_2 : T_{5.32}, l'_2 : T_{5.33} \vdash \text{release } - = p'_1 \text{ in } E_{7.3} : T_{8.0}}}{.; s'_2, s'_1, s; ; g' : T'_1, f : T_0; H_1 : (\tau_1), H_2 : (\tau_2), p'_1 : T_{5.22}, l'_1 : T_{5.23}, p'_2 : T_{5.32}, l'_2 : T_{5.33} \vdash E_{7.2} : T_{8.0}}$$

D1.711:

$$\frac{\frac{\overline{.; s'_2, s'_1, s; ; g' : T'_1, f : T_0; tp_2 : T_{5.31} \vdash tp_2 : T_{5.31}}}{D1.72}}{.; s'_2, s'_1, s; ; g' : T'_1, f : T_0; H_1 : (\tau_1), H_2 : (\tau_2), p'_1 : T_{5.22}, l'_1 : T_{5.23}, tp_2 : T_{5.31} \vdash \text{let}\langle\langle p'_2, l'_2 \rangle\rangle = tp_2 \text{ in } E_{7.2} : T_{8.0}}$$

D1.71:

$$\frac{\frac{\overline{.; s'_1, s; ; g' : T'_1, f : T_0; T_2 : T_{5.3} \vdash T_2 : T_{5.3}}}{D1.711}}{.; s'_1, s; ; g' : T'_1, f : T_0; H_1 : (\tau_1), H_2 : (\tau_2), T_2 : T_{5.3}, p'_1 : T_{5.22}, l'_1 : T_{5.23} \vdash T_2; tp_2. \text{let}\langle\langle p'_2, l'_2 \rangle\rangle = tp_2 \text{ in } E_{7.2} : T_{8.0}}}{.; s'_1, s; ; g' : T'_1, f : T_0; H_1 : (\tau_1), H_2 : (\tau_2), T_2 : T_{5.3}, p'_1 : T_{5.22}, l'_1 : T_{5.23} \vdash E_7 : T_{8.0}}$$

D1.61:

$$\frac{\frac{\overline{.; s'_1, s; ; g' : T'_1, f : T_0; tp_1 : T_{5.21} \vdash tp_1 : T_{5.21}}}{D1.71}}{.; s'_1, s; ; g' : T'_1, f : T_0; H_1 : (\tau_1), H_2 : (\tau_2), T_2 : T_{5.3}, tp_1 : T_{5.21} \vdash \text{let}\langle\langle p'_1, l'_1 \rangle\rangle = tp_1 \text{ in } E_7 : T_{8.0}}$$

D1.6:

$$\frac{\frac{\overline{.; s; ; g' : T'_1, f : T_0; T_1 : T_{5.2} \vdash T_1 : T_{5.2}}}{D1.61}}{.; s; ; g' : T'_1, f : T_0; p : T_{1.2}, H_1 : (\tau_1), H_2 : (\tau_2), T_1 : T_{5.2}, T_1 : T_{5.3} \vdash T_1; tp_1. \text{let}\langle\langle p'_1, l'_1 \rangle\rangle = tp_1 \text{ in } E_7 : T_{8.0}}}{.; s; ; g' : T'_1, f : T_0; H_1 : (\tau_1), H_2 : (\tau_2), T_1 : T_{5.2}, T_1 : T_{5.3} \vdash E_6 : T_{8.0}}$$

D1.5:

$$\frac{\frac{\overline{.; s; .; g' : T'_1, f : T_0; T : T_{5.1} \vdash T : T_{5.1}} \quad D1.6}{.; s; .; g' : T'_1, f : T_0; p : T_{1.2}, H_1 : (\tau_1), H_2 : (\tau_2), T : T_{5.1} \vdash \text{let}\langle\langle T_1, T_2 \rangle\rangle = T \text{ in } E_6 : T_{8.0}}}{.; s; .; g' : T'_1, f : T_0; p : T_{1.2}, H_1 : (\tau_1), H_2 : (\tau_2), T : T_{5.1} \vdash E_5 : T_{8.0}}$$

D1.4:

$$\frac{\frac{\overline{.; s; .; g' : T'_1, f : T_0; H : T_{4.1} \vdash H : T_{4.1}} \quad D1.5}{.; s; .; g' : T'_1, f : T_0; p : T_{1.2}, H : T_{4.1}, T : T_{5.1} \vdash \text{let}\langle\langle H_1, H_2 \rangle\rangle = H \text{ in } E_5 : T_{8.0}}}{.; s; .; g' : T'_1, f : T_0; p : T_{1.2}, H : T_{4.1}, T : T_{5.1} \vdash E_4 : T_{8.0}}$$

D1.3:

$$\frac{\frac{\overline{.; s; .; g' : T'_1, f : T_0; t : L^{s-1}(\tau), o_t : T_{3.12} \vdash f \langle\langle o_t, t \rangle\rangle : T_5} \quad D1.4}{.; s; .; g' : T'_1, f : T_0; p : T_{1.2}, H : T_{4.1}, t : L^{s-1}(\tau), o_t : T_{3.12} \vdash \text{bind } T = f \langle\langle o_t, t \rangle\rangle \text{ in } E_4 : T_{8.0}}$$

D1.21:

$$\frac{\frac{\overline{.; s; .; g' : T'_1, f : T_0; p : T_{1.2}, h : (\tau), t : L^{s-1}(\tau) \vdash \text{store}() : T_{3.11}} \quad D1.3}{.; s; .; g' : T'_1, f : T_0; p : T_{1.2}, h : (\tau), t : L^{s-1}(\tau) \vdash \text{bind } o_t = \text{store}() \text{ in } E_{3.2} : T_{3.1}}}{.; s; .; g' : T'_1, f : T_0; p : T_{1.2}, h : (\tau), t : L^{s-1}(\tau) \vdash E_{3.1} : T_{3.1}}$$

D1.2:

$$\frac{\frac{\overline{.; s; .; g' : T'_1, f : T_0; h : (\tau) \vdash g' h : T_4} \quad D1.3}{.; s; .; g' : T'_1, f : T_0; p : T_{1.2}, h : (\tau), t : L^{s-1}(\tau) \vdash \text{bind } H = g' h \text{ in } E_{3.1} : T_{3.1}}}{.; s; .; g' : T'_1, f : T_0; p : T_{1.2}, h : (\tau), t : L^{s-1}(\tau) \vdash E_3 : T_{3.1}}$$

D1.14:

$$\frac{\frac{\overline{.; s; .; g' : T'_1, f : T_0; z_2 : [0] \mathbf{1} \vdash z_2 : [0] \mathbf{1}} \quad \overline{.; s; .; g' : T'_1, f : T_0; z_2 : [0] \mathbf{1} \vdash \text{nil} : L^0(\tau_2)}}{.; s; .; g' : T'_1, f : T_0; z_2 : [0] \mathbf{1} \vdash \langle\langle z_2, \text{nil} \rangle\rangle : ([0] \mathbf{1} \otimes L^0(\tau_2))}}{.; s; .; g' : T'_1, f : T_0; z_2 : [0] \mathbf{1} \vdash \langle\langle z_2, \text{nil} \rangle\rangle : \exists s'. ([s'] \mathbf{1} \otimes L^{s'}(\tau_2))}$$

D1.13:

$$\frac{\frac{\overline{.; s; .; g' : T'_1, f : T_0; z_1 : [0] \mathbf{1} \vdash z_1 : [0] \mathbf{1}} \quad \overline{.; s; .; g' : T'_1, f : T_0; z_1 : [0] \mathbf{1} \vdash \text{nil} : L^0(\tau_1)}}{.; s; .; g' : T'_1, f : T_0; z_1 : [0] \mathbf{1} \vdash \langle\langle z_1, \text{nil} \rangle\rangle : ([0] \mathbf{1} \otimes L^0(\tau_1))}}{.; s; .; g' : T'_1, f : T_0; z_1 : [0] \mathbf{1} \vdash \langle\langle z_1, \text{nil} \rangle\rangle : \exists s'. ([s'] \mathbf{1} \otimes L^{s'}(\tau_1))}$$

D1.12:

$$\frac{\frac{\overline{.; s; .; g' : T'_1, f : T_0; z_1 : [0] \mathbf{1}, z_2 : [0] \mathbf{1} \vdash \langle\langle\langle z_1, \text{nil} \rangle\rangle, \langle\langle z_2, \text{nil} \rangle\rangle\rangle : T_{3.2}} \quad \overline{.; s; .; g' : T'_1, f : T_0; z_1 : [0] \mathbf{1}, z_2 : [0] \mathbf{1} \vdash \text{ret}\langle\langle\langle z_1, \text{nil} \rangle\rangle, \langle\langle z_2, \text{nil} \rangle\rangle\rangle : T_{3.1}}}{.; s; .; g' : T'_1, f : T_0; z_1 : [0] \mathbf{1}, z_2 : [0] \mathbf{1} \vdash E_{2.3} : T_{3.1}} \quad D1.13 \quad D1.14$$

D1.11:

$$\frac{\frac{\frac{}{.; s; .; g' : T'_1, f : T_0; \cdot \vdash \text{store}() : \mathbb{M}0 [0] \mathbf{1}}{D1.12}}{.; s; .; g' : T'_1, f : T_0; z_1 : [0] \mathbf{1} \vdash \text{bind } z_2 = \text{store}() \text{ in } E_{2.3} : T_{3.1}}}{.; s; .; g' : T'_1, f : T_0; z_1 : [0] \mathbf{1} \vdash E_{2.2} : T_{3.1}}$$

D1.10:

$$\frac{\frac{\frac{}{.; s; .; g' : T'_1, f : T_0; \cdot \vdash \text{store}() : \mathbb{M}0 [0] \mathbf{1}}{D1.11}}{.; s; .; g' : T'_1, f : T_0; \cdot \vdash \text{bind } z_1 = \text{store}() \text{ in } E_{2.2} : T_{3.1}}}{.; s; .; g' : T'_1, f : T_0; \cdot \vdash E_{2.1} : T_{3.1}}$$

D1:

$$\frac{\frac{\frac{}{.; s; .; g' : T'_1, f : T_0; l : T_{1.3} \vdash l : T_{1.3}}{D1.10 \quad D1.2}}{.; s; .; g' : T'_1, f : T_0; p : T_{1.2}, l : T_{1.3} \vdash \text{match } l \text{ with } | \text{nil} \mapsto E_2 \mid h :: t \mapsto E_3 : T_{3.1}}}$$

D0.3:

$$\frac{\frac{\frac{}{.; s; .; g' : T'_1, f : T_0; p : T_{1.2} \vdash p : T_{1.2}}{D1}}{.; s; .; g' : T'_1, f : T_0; p : T_{1.2}, l : T_{1.3} \vdash \text{release } - = p \text{ in } E_1 : T_3}}{.; s; .; g' : T'_1, f : T_0; p : T_{1.2}, l : T_{1.3} \vdash E_0 : T_3}$$

D0.2:

$$\frac{\frac{\frac{}{.; s; .; g' : T'_1, f : T_0; x : T_{1.1} \vdash x : T_{1.1}}{D0.3}}{.; s; .; g' : T'_1, f : T_0; x : T_{1.1} \vdash \text{let}\langle\langle p, l \rangle\rangle = x \text{ in } E_0 : T_3}$$

D0.1:

$$\frac{\frac{\frac{}{.; .; .; g' : T'_1, f : T_0; e : \langle L^p \tau \rangle \vdash e : \langle L^p \tau \rangle}}{D0.2}}{.; .; .; g' : T'_1, f : T_0; e : \langle L^p \tau \rangle \vdash e; x. \text{let}\langle\langle p, l \rangle\rangle = x \text{ in } E_0 : T_3}$$

D0:

$$\frac{\frac{\frac{\frac{}{.; .; .; f : T_0; g : T_1 \vdash g : T_1}}{D1.1}}{.; .; .; f : T_0; g : T_1, e : \langle L^p \tau \rangle \vdash \text{let}! g' = g \text{ in } e; x. \text{let}\langle\langle p, l \rangle\rangle = x \text{ in } E_0 : T_3}}{.; .; .; f : T_0; g : T_1 \vdash \lambda e. \text{let}! g' = g \text{ in } e; x. \text{let}\langle\langle p, l \rangle\rangle = x \text{ in } E_0 : T_2}}{.; .; .; f : T_0; \cdot \vdash \lambda g. \lambda e. \text{let}! g' = g \text{ in } e; x. \text{let}\langle\langle p, l \rangle\rangle = x \text{ in } E_0 : T_0}}{.; .; .; \cdot \vdash \text{fix} f. \lambda g. \lambda e. \text{let}! g' = g \text{ in } e; x. \text{let}\langle\langle p, l \rangle\rangle = x \text{ in } E_0 : T_0}$$

$$\frac{\Sigma; \Gamma, x : \tau_1, y : \tau_2 \vdash_{q'}^q e : \tau' \rightsquigarrow e_a \quad \tau = \tau_1 \curlyvee \tau_2 \quad \tau = (\tau_a, \tau_b) \quad \tau_1 = (\tau'_a, \tau'_b) \quad \tau_2 = (\tau''_a, \tau''_b)}{\Sigma; \Gamma, z : \tau \vdash_{q'}^q e[z/x, z/y] : \tau' \rightsquigarrow E_0} \text{Share-pair}$$

$$E_0 = \lambda u. E_1$$

$$E_1 = \text{bind } a = \text{coerce}_{(\tau_a, \tau_b), (\tau'_a, \tau'_b), (\tau''_a, \tau''_b)} z \text{ in } \text{let}\langle\langle x, y \rangle\rangle = a \text{ in } e_a u$$

$$T_0 = [q] \mathbf{1} \multimap \mathbb{M}0 ([q'] (\tau'))$$

D1:

$$\frac{\frac{.; .; .; (\Sigma); (\Gamma), u : [q] \mathbf{1}, x : (\tau_1), y : (\tau_2) \vdash e_a : T_0}{.; .; .; (\Sigma); (\Gamma), u : [q] \mathbf{1}, x : (\tau_1), y : (\tau_2) \vdash e_a u : \mathbb{M}0 [q'] \mathbf{1}} \quad \frac{.; .; .; (\Sigma); u : [q] \mathbf{1} \vdash u : [q] \mathbf{1}}{.; .; .; (\Sigma); (\Gamma), u : [q] \mathbf{1}, x : (\tau_1), y : (\tau_2) \vdash e_a u : \mathbb{M}0 [q'] \mathbf{1}}}{.; .; .; (\Sigma); (\Gamma), u : [q] \mathbf{1}, x : (\tau_1), y : (\tau_2) \vdash e_a u : \mathbb{M}0 [q'] \mathbf{1}}$$

D0:

$$\frac{\frac{.; .; .; (\Sigma); a : ((\tau_1) \otimes (\tau_2)) \vdash a : ((\tau_1) \otimes (\tau_2))}{.; .; .; (\Sigma); (\Gamma), u : [q] \mathbf{1}, a : ((\tau_1) \otimes (\tau_2)) \vdash \text{let}\langle\langle x, y \rangle\rangle = a \text{ in } e_a u : [q] \multimap \mathbb{M}0 [q] (\tau')} \quad D1}{.; .; .; (\Sigma); (\Gamma), u : [q] \mathbf{1}, a : ((\tau_1) \otimes (\tau_2)) \vdash \text{let}\langle\langle x, y \rangle\rangle = a \text{ in } e_a u : [q] \multimap \mathbb{M}0 [q] (\tau')}$$

Main derivation:

$$\frac{\frac{.; .; .; (\Sigma); z : (\tau) \vdash \text{coerce}_{(\tau_a, \tau_b), (\tau'_a, \tau'_b), (\tau''_a, \tau''_b)} z : \mathbb{M}0 ((\tau_1) \otimes (\tau_2))}{.; .; .; (\Sigma); (\Gamma), z : (\tau), u : [q] \mathbf{1} \vdash E_0 : \mathbb{M}0 [q'] (\tau')} \quad D0}{.; .; .; (\Sigma); (\Gamma), z : (\tau) \vdash \lambda u. E_0 : T_0}$$

$$\text{coerce}_{(\tau_a, \tau_b), (\tau'_a, \tau'_b), (\tau''_a, \tau''_b)} :!((\tau_a) \multimap \mathbb{M}0 ((\tau'_a) \otimes (\tau''_a))) \multimap !((\tau_b) \multimap \mathbb{M}0 ((\tau'_b) \otimes (\tau''_b))) \multimap ((\tau_a, \tau_b) \multimap \mathbb{M}0 ((\tau'_a, \tau'_b) \otimes ((\tau''_a, \tau''_b))))$$

$$\text{coerce}_{(\tau_a, \tau_b), (\tau'_a, \tau'_b), (\tau''_a, \tau''_b)} \triangleq \lambda g_1. \lambda g_2. \lambda p. \text{let} !\langle\langle p_1, p_2 \rangle\rangle = p \text{ in } E_0$$

where

$$E_0 \triangleq \text{let} ! g'_1 = g_1 \text{ in } E_1$$

$$E_1 \triangleq \text{let} ! g'_2 = g_2 \text{ in } E_2$$

$$E_2 \triangleq \text{bind } P'_1 = g'_1 p_1 \text{ in } E_3$$

$$E_3 \triangleq \text{bind } P'_2 = g'_2 p_2 \text{ in } E_4$$

$$E_4 \triangleq \text{let} !\langle\langle p'_{11}, p'_{12} \rangle\rangle = P'_1 \text{ in } E_5$$

$$E_5 \triangleq \text{let} !\langle\langle p'_{21}, p'_{22} \rangle\rangle = P'_2 \text{ in } E_6$$

$$E_6 \triangleq \text{ret}\langle\langle p'_{11}, p'_{21} \rangle\rangle, \langle\langle p'_{12}, p'_{22} \rangle\rangle$$

$$T_0 = !((\tau_a) \multimap \mathbb{M}0 ((\tau'_a) \otimes (\tau''_a))) \multimap !((\tau_b) \multimap \mathbb{M}0 ((\tau'_b) \otimes (\tau''_b))) \multimap ((\tau_a, \tau_b) \multimap \mathbb{M}0 ((\tau'_a, \tau'_b) \otimes ((\tau''_a, \tau''_b))))$$

$$T_{0.31} = !((\tau_a) \multimap \mathbb{M}0 ((\tau'_a) \otimes (\tau''_a)))$$

$$T_{0.32} = ((\tau_a) \multimap \mathbb{M}0 ((\tau'_a) \otimes (\tau''_a)))$$

$$T_{0.4} = !((\tau_b) \multimap \mathbb{M}0 ((\tau'_b) \otimes (\tau''_b))) \multimap ((\tau_a, \tau_b) \multimap \mathbb{M}0 ((\tau'_a, \tau'_b) \otimes ((\tau''_a, \tau''_b))))$$

$$T_{0.41} = !((\tau_b) \multimap \mathbb{M}0 ((\tau'_b) \otimes (\tau''_b)))$$

$$T_{0.42} = ((\tau_b) \multimap \mathbb{M}0 ((\tau'_b) \otimes (\tau''_b)))$$

$$T_{0.5} = ((\tau_a, \tau_b) \multimap \mathbb{M}0 ((\tau'_a, \tau'_b) \otimes ((\tau''_a, \tau''_b))))$$

$$T_{0.51} = ((\tau_a, \tau_b))$$

$$T_{0.6} = \mathbb{M}0 ((\tau'_a, \tau'_b) \otimes ((\tau''_a, \tau''_b)))$$

$$T_{0.61} = ((\tau'_a, \tau'_b) \otimes ((\tau''_a, \tau''_b)))$$

$$T_1 = \mathbb{M}0 ((\tau'_a) \otimes (\tau''_a))$$

$$T_{1.1} = ((\tau'_a) \otimes (\tau''_a))$$

$$T_{1.11} = (\tau'_a)$$

$$T_{1.12} = (\tau''_a)$$

$$T_2 = \mathbb{M}0((\tau'_b) \otimes (\tau''_b))$$

$$T_{2.1} = ((\tau'_b) \otimes (\tau''_b))$$

$$T_{2.11} = (\tau'_b)$$

$$T_{2.12} = (\tau''_b)$$

D6:

$$\frac{\frac{\frac{\cdot, \cdot, \cdot, f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; p'_{11} : T_{1.11}, p'_{12} : T_{1.12}, p'_{21} : T_{2.11}, p'_{22} : T_{2.12} \vdash \langle\langle p'_{11}, p'_{21} \rangle\rangle, \langle\langle p'_{12}, p'_{22} \rangle\rangle : T_{0.61}}{\cdot, \cdot, \cdot, f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; p'_{11} : T_{1.11}, p'_{12} : T_{1.12}, p'_{21} : T_{2.11}, p'_{22} : T_{2.12} \vdash \text{ret}\langle\langle p'_{11}, p'_{21} \rangle\rangle, \langle\langle p'_{12}, p'_{22} \rangle\rangle : T_{0.6}}}{\cdot, \cdot, \cdot, f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; p'_{11} : T_{1.11}, p'_{12} : T_{1.12}, p'_{21} : T_{2.11}, p'_{22} : T_{2.12} \vdash E_6 : T_{0.6}}$$

D5:

$$\frac{\frac{\frac{\cdot, \cdot, \cdot, f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; P'_2 : T_{2.1} \vdash P'_2 : T_{2.1}}{\cdot, \cdot, \cdot, f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; P'_2 : T_{2.1}, p'_{11} : T_{1.11}, p'_{12} : T_{1.12} \vdash \text{let}\langle\langle p'_{21}, p'_{22} \rangle\rangle = P'_2 \text{ in } E_6 : T_{0.6}}{\cdot, \cdot, \cdot, f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; P'_2 : T_{2.1}, p'_{11} : T_{1.11}, p'_{12} : T_{1.12} \vdash E_5 : T_{0.6}}}{D6}$$

D4:

$$\frac{\frac{\frac{\cdot, \cdot, \cdot, f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; P'_1 : T_{1.1} \vdash P'_1 : T_{1.1}}{\cdot, \cdot, \cdot, f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; P'_1 : T_{1.1}, P'_2 : T_{2.1} \vdash \text{let}\langle\langle p'_{11}, p'_{12} \rangle\rangle = P'_1 \text{ in } E_5 : T_{0.6}}{\cdot, \cdot, \cdot, f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; P'_1 : T_{1.1}, P'_2 : T_{2.1} \vdash E_4 : T_{0.6}}}{D5}$$

D3:

$$\frac{\frac{\frac{\cdot, \cdot, \cdot, f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; p_2 : (\tau_2) \vdash g'_2 p_2 : T_2}{\cdot, \cdot, \cdot, f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; p_2 : (\tau_2), P'_1 : T_{1.1} \vdash \text{bind } P'_2 = g'_2 p_2 \text{ in } E_4 : T_{0.6}}}{\cdot, \cdot, \cdot, f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; p_2 : (\tau_2), P'_1 : T_{1.1} \vdash E_3 : T_{0.6}}}{D4}$$

D2:

$$\frac{\frac{\frac{\cdot, \cdot, \cdot, f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; p_1 : (\tau_1) \vdash g'_1 p_1 : T_1}{\cdot, \cdot, \cdot, f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; p_1 : (\tau_1), p_2 : (\tau_2) \vdash \text{bind } P'_1 = g'_1 p_1 \text{ in } E_3 : T_{0.6}}}{\cdot, \cdot, \cdot, f : T_0; g'_1 : T_{0.32}, g'_2 : T_{0.42}; p_1 : (\tau_1), p_2 : (\tau_2) \vdash E_2 : T_{0.6}}}{D3}$$

D1:

$$\frac{\frac{\frac{\cdot, \cdot, \cdot, f : T_0; g'_1 : T_{0.32}; g_2 : T_{0.41} \vdash g_2 : T_{0.41}}{\cdot, \cdot, \cdot, f : T_0; g'_1 : T_{0.32}; g_2 : T_{0.41}, p_1 : (\tau_1), p_2 : (\tau_2) \vdash \text{let}\langle\langle g_2 \rangle\rangle = g_2 \text{ in } E_2 : T_{0.6}}}{\cdot, \cdot, \cdot, f : T_0; g'_1 : T_{0.32}; g_2 : T_{0.41}, p_1 : (\tau_1), p_2 : (\tau_2) \vdash E_1 : T_{0.6}}}{D2}$$

D0.1:

$$\frac{\frac{\frac{}{.; ; ; f : T_0; g_1 : T_{0.31} \vdash g_1 : T_{0.31}}{D1}}{.; ; ; f : T_0; g_1 : T_{0.31}, g_2 : T_{0.41}, p_1 : \langle \tau_1 \rangle, p_2 : \langle \tau_2 \rangle \vdash \text{let}! g'_1 = g_1 \text{ in } E_1 : T_{0.6}}{.; ; ; f : T_0; g_1 : T_{0.31}, g_2 : T_{0.41}, p_1 : \langle \tau_1 \rangle, p_2 : \langle \tau_2 \rangle \vdash E_0 : T_{0.6}}$$

D0:

$$\frac{\frac{\frac{\frac{\frac{}{.; ; ; f : T_0; p : T_{0.51} \vdash p : T_{0.51}}{D0.1}}{.; ; ; f : T_0; g_1 : T_{0.31}, g_2 : T_{0.41}, p : T_{0.51} \vdash \text{let}! \langle \langle p_1, p_2 \rangle \rangle = p \text{ in } E_0 : T_{0.6}}{.; ; ; f : T_0; g_1 : T_{0.31}, g_2 : T_{0.41} \vdash \lambda p. \text{let}! \langle \langle p_1, p_2 \rangle \rangle = p \text{ in } E_0 : T_{0.5}}{.; ; ; f : T_0; g_1 : T_{0.31} \vdash \lambda_{g_2}. \lambda p. \text{let}! \langle \langle p_1, p_2 \rangle \rangle = p \text{ in } E_0 : T_{0.4}}{.; ; ; f : T_0; \cdot \vdash \lambda_{g_1}. \lambda_{g_2}. \lambda p. \text{let}! \langle \langle p_1, p_2 \rangle \rangle = p \text{ in } E_0 : T_0}}{.; ; ; \cdot \vdash \text{fix} f. \lambda_{g_1}. \lambda_{g_2}. \lambda p. \text{let}! \langle \langle p_1, p_2 \rangle \rangle = p \text{ in } E_0 : T_0}$$

9. Sub:

$$\frac{\Sigma; \Gamma \vdash_{q'}^q e : \tau \rightsquigarrow e_a \quad \tau <: \tau'}{\Sigma; \Gamma \vdash_{q'}^q e : \tau' \rightsquigarrow e_a} \text{Sub}$$

Main derivation:

$$\frac{\frac{.; ; ; (\Sigma); (\Gamma) \vdash e_a : [q] \mathbf{1} \multimap \mathbb{M}0([q'](\tau)) \quad \tau <: \tau'}{.; ; ; (\Sigma); (\Gamma) \vdash e_a : [q] \mathbf{1} \multimap \mathbb{M}0([q'](\tau'))} \text{Lemma 83}}{.; ; ; (\Sigma); (\Gamma) \vdash e_a : [q] \mathbf{1} \multimap \mathbb{M}0([q'](\tau'))} \text{T-sub}$$

10. Super:

$$\frac{\Sigma; \Gamma, x : \tau_1 \vdash_{q'}^q e : \tau \rightsquigarrow e_a \quad \tau'_1 <: \tau_1}{\Sigma; \Gamma, x : \tau'_1 \vdash_{q'}^q e : \tau \rightsquigarrow e_a} \text{Super}$$

Main derivation:

$$\frac{.; ; ; (\Sigma); (\Gamma), x : \langle \tau_1 \rangle \vdash e_a : [q] \mathbf{1} \multimap \mathbb{M}0([q'](\tau)) \quad \frac{\tau'_1 <: \tau_1}{.; ; ; \vdash \langle \tau'_1 \rangle <: \langle \tau_1 \rangle} \text{Lemma 83}}{.; ; ; (\Sigma); (\Gamma), x : \langle \tau'_1 \rangle \vdash e_a : [q] \mathbf{1} \multimap \mathbb{M}0([q'](\tau))} \text{T-weaken}$$

11. Relax:

$$\frac{\Sigma; \Gamma \vdash_{p'}^p e : \tau \rightsquigarrow e_a \quad q \geq p \quad q - p \geq q' - p'}{\Sigma; \Gamma \vdash_{q'}^q e : \tau \rightsquigarrow \lambda o. E_0} \text{Relax}$$

where

$E_0 = \text{release } - = o \text{ in } E_1$

$E_1 = \text{bind } a = \text{store}() \text{ in } E_2$

$E_2 = \text{bind } b = e_a a \text{ in } E_3$

$E_3 = \text{release } c = b \text{ in store } c$

D2:

$$\frac{\overline{.; .; .; (\Sigma); b : [p'](\tau) \vdash b : [p'](\tau)}}{.; .; .; (\Sigma); c : (\tau) \vdash \text{store } c : \mathbb{M}(q - p + p') ([q - p + p'](\tau))} \frac{.; .; .; (\Sigma); b : [p'](\tau) \vdash E_3 : \mathbb{M}(q - p) ([q - p + p'](\tau))}{.}$$

D1.2:

$$\overline{.; .; .; (\Sigma); a : [p] \mathbf{1} \vdash a : [p] \mathbf{1}}$$

D1.1:

$$\overline{.; .; .; (\Sigma); (\Gamma) \vdash e_a : [p] \mathbf{1} \multimap \mathbb{M} 0 ([p'](\tau))} \text{IH}$$

D1:

$$\frac{\overline{.; .; .; (\Sigma); (\Gamma), a : [p] \mathbf{1} \vdash e_a a : \mathbb{M} 0 ([p'](\tau))} \quad \overline{.; .; .; (\Sigma); (\Gamma), a : [p] \mathbf{1} \vdash E_2 : \mathbb{M}(q - p) ([q - p + p'](\tau))} \quad D2}{.}$$

D0:

$$\frac{\overline{.; .; .; (\Sigma); \cdot \vdash \text{store}() : \mathbb{M} p ([p] \mathbf{1})} \quad D1}{.; .; .; (\Sigma); (\Gamma) \vdash E_1 : \mathbb{M}(q) ([q - p + p'](\tau))}$$

D0.0:

$$\frac{\overline{q' \leq q - p + p'} \text{ Given}}{.; .; \cdot \vdash ([q - p + p'](\tau)) <: ([q'](\tau))} \frac{.; .; \cdot \vdash \mathbb{M} 0 ([q - p + p'](\tau)) <: \mathbb{M} 0 ([q'](\tau))}{.}$$

Main derivation:

$$\frac{\overline{.; .; .; (\Sigma); o : [q] \mathbf{1} \vdash o : [q] \mathbf{1}} \quad D0}{.; .; .; (\Sigma); (\Gamma), o : [q] \mathbf{1} \vdash E_0 : \mathbb{M} 0 ([q - p + p'](\tau))} \quad \overline{.; .; .; (\Sigma); (\Gamma), o : [q] \mathbf{1} \vdash E_0 : \mathbb{M} 0 ([q'](\tau))} \quad D0.0}{.; .; .; (\Sigma); (\Gamma) \vdash \lambda o. E_0 : [q] \mathbf{1} \multimap \mathbb{M} 0 ([q'](\tau))} \text{T-sub}$$

12. Let:

$$\frac{\Sigma; \Gamma_1 \vdash_p^{q-K_1^{let}} e_1 : \tau_1 \rightsquigarrow e_{a1} \quad \Sigma; \Gamma_2, x : \tau_1 \vdash_{q'+K_3^{let}}^{p-K_2^{let}} e_2 : \tau_1 \rightsquigarrow e_{a2}}{\Sigma; \Gamma_1, \Gamma_2 \vdash_{q'}^q \text{let } x = e_1 \text{ in } e_2 : \tau \rightsquigarrow E_t} \text{Let}$$

where

$$E_t = \lambda u. E_0$$

$$E_0 = \text{release } - = u \text{ in } E_1$$

$$E_1 = \text{bind } - = \uparrow^{K_1^{let}} \text{ in } E_2$$

$$E_2 = \text{bind } a = \text{store}() \text{ in } E_3$$

$E_3 = \text{bind } b = e_{a1} \ a \text{ in } E_4$
 $E_4 = \text{release } x = b \text{ in } E_5$
 $E_5 = \text{bind } - = \uparrow^{K_2^{let}} \text{ in } E_6$
 $E_6 = \text{bind } c = \text{store}() \text{ in } E_7$
 $E_7 = \text{bind } d = e_{a2} \ c \text{ in } E_8$
 $E_8 = \text{release } f = d \text{ in } E_9$
 $E_9 = \text{bind } - = \uparrow^{K_3^{let}} \text{ in } E_{10}$
 $E_{10} = \text{bind } g = \text{store } f \text{ in ret } g$

$T_0 = [q] \mathbf{1} \multimap \mathbb{M} 0 ([q'](\tau))$
 $T_{0.1} = [q] \mathbf{1}$
 $T_{0.2} = \mathbb{M} 0 ([q'](\tau))$
 $T_{0.3} = \mathbb{M} q ([q'](\tau))$
 $T_{0.4} = \mathbb{M}(q - K_1^{let}) ([q'](\tau))$
 $T_{0.5} = \mathbb{M}(q - K_1^{let}) ([q - K_1^{let}] \mathbf{1})$
 $T_{0.51} = [q - K_1^{let}] \mathbf{1}$
 $T_{0.6} = \mathbb{M} 0 [p] (\tau_1)$
 $T_{0.61} = [p] (\tau_1)$
 $T_{0.7} = \mathbb{M} p ([q'](\tau))$
 $T_{0.8} = \mathbb{M}(p - K_2^{let}) ([q'](\tau))$
 $T_{0.9} = \mathbb{M}(p - K_2^{let}) ([p - K_2^{let}] \mathbf{1})$
 $T_{0.91} = [p - K_2^{let}] \mathbf{1}$
 $T_1 = \mathbb{M} 0 [(q' + K_3^{let})] (\tau)$
 $T_{1.1} = [(q' + K_3^{let})] (\tau)$
 $T_{1.2} = \mathbb{M}(q' + K_3^{let}) ([q'](\tau))$
 $T_{1.3} = \mathbb{M} q' ([q'](\tau))$

D10:

$$\frac{}{.; ; ; (\Sigma); g : [q'](\tau) \vdash \text{ret } g : \mathbb{M} 0 [q'](\tau)}$$

D9:

$$\frac{\frac{}{.; ; ; (\Sigma); f : (\tau) \vdash \text{store } f : T_{1.3}} \quad D10}{.; ; ; (\Sigma); f : (\tau) \vdash \text{bind } g = \text{store } f \text{ in ret } g : T_{1.3}}}{.; ; ; (\Sigma); f : (\tau) \vdash E_{10} : T_{1.3}}$$

D8:

$$\frac{\frac{}{.; ; ; . \vdash \uparrow^{K_3^{let}} : \mathbb{M} K_3^{let} \mathbf{1}} \quad D9}{.; ; ; (\Sigma); f : (\tau) \vdash \text{bind } - = \uparrow^{K_3^{let}} \text{ in } E_{10} : T_{1.2}}}{.; ; ; (\Sigma); f : (\tau) \vdash E_9 : T_{1.2}}$$

D7:

$$\frac{\frac{\frac{}{\cdot, \cdot, \cdot; (\Sigma); d : T_{1.1} \vdash d : T_{1.1}}{D8}}{\cdot, \cdot, \cdot; (\Sigma); d : T_{1.1} \vdash \text{release } f = d \text{ in } E_9 : T_{0.2}}}{\cdot, \cdot, \cdot; (\Sigma); d : T_{1.1} \vdash E_8 : T_{0.2}}$$

D6:

$$\frac{\frac{\frac{}{\cdot, \cdot, \cdot; (\Sigma); (\Gamma_2), c : T_{0.91} \vdash e_{a2} c : T_1}}{D7}}{\cdot, \cdot, \cdot; (\Sigma); (\Gamma_2), c : T_{0.91} \vdash \text{bind } d = e_{a2} c \text{ in } E_8 : T_{0.2}}}{\cdot, \cdot, \cdot; (\Sigma); (\Gamma_2), c : T_{0.91} \vdash E_7 : T_{0.2}}$$

D5:

$$\frac{\frac{\frac{}{\cdot, \cdot, \cdot; (\Sigma); \cdot \vdash \text{store}() : T_{0.9}}{D6}}{\cdot, \cdot, \cdot; (\Sigma); (\Gamma_2) \vdash \text{bind } c = \text{store}() \text{ in } E_7 : T_{0.8}}}{\cdot, \cdot, \cdot; (\Sigma); (\Gamma_2) \vdash E_6 : T_{0.8}}$$

D4:

$$\frac{\frac{\frac{}{\cdot, \cdot, \cdot; (\Sigma); \cdot \vdash \uparrow^{K_2^{let}} : \mathbb{M} K_2^{let} \mathbf{1}}{D5}}{\cdot, \cdot, \cdot; (\Sigma); (\Gamma_2) \vdash \text{bind } - = \uparrow^{K_2^{let}} \text{ in } E_6 : T_{0.7}}}{\cdot, \cdot, \cdot; (\Sigma); (\Gamma_2) \vdash E_5 : T_{0.7}}$$

D3:

$$\frac{\frac{\frac{}{\cdot, \cdot, \cdot; (\Sigma); b : T_{0.61} \vdash b : T_{0.61}}{D4}}{\cdot, \cdot, \cdot; (\Sigma); (\Gamma_2), b : T_{0.61} \vdash \text{release } x = b \text{ in } E_5 : T_{0.2}}}{\cdot, \cdot, \cdot; (\Sigma); (\Gamma_2), b : T_{0.61} \vdash E_4 : T_{0.2}}$$

D2:

$$\frac{\frac{\frac{}{\cdot, \cdot, \cdot; (\Sigma); (\Gamma_1), a : T_{0.51} \vdash e_{a1} a : T_{0.6}}{D3}}{\cdot, \cdot, \cdot; (\Sigma); (\Gamma_1), (\Gamma_2), a : T_{0.51} \vdash \text{bind } b = e_{a1} a \text{ in } E_4 : T_{0.2}}}{\cdot, \cdot, \cdot; (\Sigma); (\Gamma_1), (\Gamma_2), a : T_{0.51} \vdash E_3 : T_{0.2}}$$

D1:

$$\frac{\frac{\frac{}{\cdot, \cdot, \cdot; (\Sigma); (\Gamma_1), (\Gamma_2) \vdash \text{store}() : T_{0.5}}{D2}}{\cdot, \cdot, \cdot; (\Sigma); (\Gamma_1), (\Gamma_2) \vdash \text{bind } a = \text{store}() \text{ in } E_3 : T_{0.4}}}{\cdot, \cdot, \cdot; (\Sigma); (\Gamma_1), (\Gamma_2) \vdash E_2 : T_{0.4}}$$

D0:

$$\frac{\frac{\frac{}{\cdot, \cdot, \cdot; \cdot \vdash \uparrow^{K_1^{let}} : \mathbb{M} K_1^{let} \mathbf{1}}{D1}}{\cdot, \cdot, \cdot; (\Sigma); (\Gamma_1), (\Gamma_2) \vdash \text{bind } - = \uparrow^{K_1^{let}} \text{ in } E_2 : T_{0.3}}}{\cdot, \cdot, \cdot; (\Sigma); (\Gamma_1), (\Gamma_2) \vdash E_1 : T_{0.3}}$$

Main derivation:

$$\frac{\frac{\frac{\cdot; \cdot; \cdot; (\Sigma); (\Gamma_1), (\Gamma_2), u : T_{0.1} \vdash u : T_{0.1}}{D0}}{\cdot; \cdot; \cdot; (\Sigma); (\Gamma_1), (\Gamma_2), u : T_{0.1} \vdash \text{release } - = u \text{ in } E_1 : T_{0.2}}}{\cdot; \cdot; \cdot; (\Sigma); (\Gamma_1), (\Gamma_2), u : T_{0.1} \vdash E_0 : T_{0.2}}}{\cdot; \cdot; \cdot; (\Sigma); (\Gamma_1), (\Gamma_2) \vdash \lambda u. E_0 : T_0}$$

13. Pair:

$$\frac{}{\Sigma; x_1 : \tau_1, x_2 : \tau_2 \vdash_q^{q+K^{pair}} (x_1, x_2) : (\tau_1, \tau_2) \rightsquigarrow E_t} \text{ pair}$$

where

$$E_t = \lambda u. E_0$$

$$E_0 = \text{release } - = u \text{ in } E_1$$

$$E_1 = \text{bind } - = \uparrow^{K^{pair}} \text{ in } E_2$$

$$E_2 = \text{bind } a = \text{store}(x_1, x_2) \text{ in ret } a$$

$$T_0 = [(q + K^{pair})] \mathbf{1} \multimap \mathbb{M} 0 ([q] (\tau_1) \otimes (\tau_2))$$

$$T_{0.1} = [(q + K^{pair})] \mathbf{1}$$

$$T_{0.2} = \mathbb{M} 0 ([q] (\tau_1) \otimes (\tau_2))$$

$$T_{0.3} = \mathbb{M} (q + K^{pair}) ([q] (\tau_1) \otimes (\tau_2))$$

$$T_{0.4} = \mathbb{M} q ([q] (\tau_1) \otimes (\tau_2))$$

D2:

$$\frac{}{\cdot; \cdot; \cdot; (\Sigma); a : [q] (\tau_1) \otimes (\tau_2) \vdash \text{ret } a : \mathbb{M} 0 [q] (\tau_1) \otimes (\tau_2)}$$

D1:

$$\frac{\frac{\frac{\cdot; \cdot; \cdot; (\Sigma); x_1 : (\tau_1), x_2 : (\tau_2) \vdash \text{store}(x_1, x_2) : T_{0.4}}{D2}}{\cdot; \cdot; \cdot; (\Sigma); x_1 : (\tau_1), x_2 : (\tau_2) \vdash \text{bind } a = \text{store}(x_1, x_2) \text{ in ret } a : T_{0.4}}}{\cdot; \cdot; \cdot; (\Sigma); x_1 : (\tau_1), x_2 : (\tau_2) \vdash E_2 : T_{0.4}}}$$

D0:

$$\frac{\frac{\frac{\cdot; \cdot; \cdot; (\Sigma); \cdot \vdash \uparrow^{K^{pair}} : \mathbb{M} K^{pair} \mathbf{1}}{D1}}{\cdot; \cdot; \cdot; (\Sigma); x_1 : (\tau_1), x_2 : (\tau_2) \vdash \text{bind } - = \uparrow^{K^{pair}} \text{ in } E_2 : T_{0.3}}}{\cdot; \cdot; \cdot; (\Sigma); x_1 : (\tau_1), x_2 : (\tau_2) \vdash E_1 : T_{0.3}}}$$

Main derivation:

$$\frac{\frac{\frac{\cdot; \cdot; \cdot; (\Sigma); x_1 : (\tau_1), x_2 : (\tau_2), u : T_{0.1} \vdash u : T_{0.1}}{D0}}{\cdot; \cdot; \cdot; (\Sigma); x_1 : (\tau_1), x_2 : (\tau_2), u : T_{0.1} \vdash \text{release } - = u \text{ in } E_1 : T_{0.2}}}{\cdot; \cdot; \cdot; (\Sigma); x_1 : (\tau_1), x_2 : (\tau_2), u : T_{0.1} \vdash E_0 : T_{0.2}}}{\cdot; \cdot; \cdot; (\Sigma); x_1 : (\tau_1), x_2 : (\tau_2) \vdash \lambda u. E_0 : T_0}$$

14. MatP:

$$\frac{\tau = (\tau_1, \tau_2) \quad \Sigma, \Gamma, x_1 : \tau_1, x_2 : \tau_2 \vdash_{q - K_1^{matP} / q' + K_2^{matP}} e : \tau' \rightsquigarrow e_t}{\Sigma; \Gamma, x : \tau \vdash_{q'} \text{match } x \text{ with } (x_1, x_2) \rightarrow e : \tau' \rightsquigarrow E_t} \text{matP}$$

where

$$E_t = \lambda u. E_0$$

$$E_0 = \text{release } - = u \text{ in } E_1$$

$$E_1 = \text{bind } - = \uparrow^{K_1^{matP}} \text{ in } E_2$$

$$E_2 = \text{let} \langle \langle x_1, x_2 \rangle \rangle = x \text{ in } E_3$$

$$E_3 = \text{bind } a = \text{store}() \text{ in } E_4$$

$$E_4 = \text{bind } b = e_t \ a \text{ in } E_5$$

$$E_5 = \text{release } c = b \text{ in } E_6$$

$$E_6 = \text{bind } - = \uparrow^{K_2^{matP}} \text{ in } E_7$$

$$E_7 = \text{bind } d = \text{store } c \text{ in ret } d$$

$$T_0 = [q] \mathbf{1} \multimap \mathbb{M} 0 ([q'] (\tau'))$$

$$T_{0.1} = [q] \mathbf{1}$$

$$T_{0.2} = \mathbb{M} 0 ([q'] (\tau'))$$

$$T_{0.3} = \mathbb{M} q ([q'] (\tau'))$$

$$T_{0.4} = \mathbb{M}(q - K_1^{matP}) ([q'] (\tau'))$$

$$T_{0.5} = \mathbb{M}(q - K_1^{matP}) ([(q - K_1^{matP})] \mathbf{1})$$

$$T_{0.51} = [(q - K_1^{matP})] \mathbf{1}$$

$$T_{0.6} = \mathbb{M} 0 ([q' + k_2^{matP}] (\tau'))$$

$$T_{0.61} = [(q' + k_2^{matP})] (\tau')$$

$$T_{0.7} = \mathbb{M}(q' + K_2^{matP}) [q'] (\tau')$$

$$T_{0.71} = [q'] (\tau')$$

$$T_{0.8} = \mathbb{M} q' ([q'] (\tau'))$$

D7:

$$\frac{}{.; .; .; (\Sigma); d : [q'] (\tau') \vdash \text{ret } d : \mathbb{M} 0 [q'] (\tau')}$$

D6:

$$\frac{.; .; .; (\Sigma); c : (\tau') \vdash \text{store } c : \mathbb{M} q' [q'] (\tau') \quad D7}{.; .; .; (\Sigma); c : (\tau') \vdash \text{bind } d = \text{store } c \text{ in ret } d : T_{0.8}} \quad D6$$

$$\frac{.; .; .; (\Sigma); c : (\tau') \vdash E_7 : T_{0.8}}{.; .; .; (\Sigma); c : (\tau') \vdash E_6 : T_{0.7}}$$

D5:

$$\frac{.; .; .; (\Sigma); c : (\tau') \vdash \uparrow^{K_2^{matP}} : \mathbb{M} K_2^{matP} \mathbf{1} \quad D6}{.; .; .; (\Sigma); c : (\tau') \vdash \text{bind } - = \uparrow^{K_2^{matP}} \text{ in } E_7 : T_{0.7}} \quad D5$$

$$\frac{.; .; .; (\Sigma); c : (\tau') \vdash E_6 : T_{0.7}}{.; .; .; (\Sigma); c : (\tau') \vdash E_5 : T_{0.6}}$$

D4:

$$\frac{\frac{\frac{\cdot; \cdot; \cdot; (\Sigma); b : T_{0.61} \vdash b : T_{0.61}}{D5}}{\cdot; \cdot; \cdot; (\Sigma); b : T_{0.61} \vdash \text{release } c = b \text{ in } E_6 : T_{0.2}}}{\cdot; \cdot; \cdot; (\Sigma); b : T_{0.61} \vdash E_5 : T_{0.2}}$$

D3:

$$\frac{\frac{\frac{\cdot; \cdot; \cdot; (\Sigma); (\Gamma), x_1 : (\tau_1), x_2 : (\tau_2), a : T_{0.51} \vdash e_t a : T_{0.6}}{D4}}{\cdot; \cdot; \cdot; (\Sigma); (\Gamma), x_1 : (\tau_1), x_2 : (\tau_2), a : T_{0.51} \vdash \text{bind } b = e_t a \text{ in } E_5 : T_{0.2}}}{\cdot; \cdot; \cdot; (\Sigma); (\Gamma), x_1 : (\tau_1), x_2 : (\tau_2), a : T_{0.51} \vdash E_4 : T_{0.2}}}$$

D2:

$$\frac{\frac{\frac{\cdot; \cdot; \cdot; (\Sigma); \cdot \vdash \text{store}() : T_{0.5}}{D3}}{\cdot; \cdot; \cdot; (\Sigma); (\Gamma), x_1 : (\tau_1), x_2 : (\tau_2) \vdash \text{bind } a = \text{store}() \text{ in } E_4 : T_{0.4}}}{\cdot; \cdot; \cdot; (\Sigma); (\Gamma), x_1 : (\tau_1), x_2 : (\tau_2) \vdash E_3 : T_{0.4}}}$$

D1:

$$\frac{\frac{\frac{\cdot; \cdot; \cdot; (\Sigma); x : (\tau) \vdash x : (\tau)}{D2}}{\cdot; \cdot; \cdot; (\Sigma); (\Gamma), x : (\tau) \vdash \text{let}\langle\langle x_1, x_2 \rangle\rangle = x \text{ in } E_3 : T_{0.4}}}{\cdot; \cdot; \cdot; (\Sigma); (\Gamma), x : (\tau) \vdash E_2 : T_{0.4}}}$$

D0:

$$\frac{\frac{\frac{\cdot; \cdot; \cdot; (\Sigma); \cdot \vdash \uparrow^{K_1^{matP}} : \mathbb{M} K_1^{matP} \mathbf{1}}{D1}}{\cdot; \cdot; \cdot; (\Sigma); (\Gamma), x : (\tau) \vdash \text{bind } - = \uparrow^{K_1^{matP}} \text{ in } E_2 : T_{0.3}}}{\cdot; \cdot; \cdot; (\Sigma); (\Gamma), x : (\tau) \vdash E_1 : T_{0.3}}}$$

Main derivation:

$$\frac{\frac{\frac{\frac{\cdot; \cdot; \cdot; (\Sigma); (\Gamma), x : (\tau), u : T_{0.1} \vdash u : T_{0.1}}{D0}}{\cdot; \cdot; \cdot; (\Sigma); (\Gamma), x : (\tau), u : T_{0.1} \vdash \text{release } - = u \text{ in } E_1 : T_{0.2}}}{\cdot; \cdot; \cdot; (\Sigma); (\Gamma), x : (\tau), u : T_{0.1} \vdash E_0 : T_{0.2}}}{\cdot; \cdot; \cdot; (\Sigma); (\Gamma), x : (\tau) \vdash \lambda u. E_0 : T_0}}$$

15. Augment:

$$\frac{\Sigma; \Gamma \vdash_{q'}^q e : \tau \rightsquigarrow e_a}{\Sigma; \Gamma, x : \tau' \vdash_{q'}^q e : \tau \rightsquigarrow e_a} \text{Augment}$$

Main derivation:

$$\frac{\cdot; \cdot; \cdot; (\Sigma); (\Gamma) \vdash e_a : [q] \mathbf{1} \multimap \mathbb{M} 0 ([q'](\tau))}{\cdot; \cdot; \cdot; (\Sigma); (\Gamma), x : (\tau') \vdash e_a : [q] \mathbf{1} \multimap \mathbb{M} 0 ([q'](\tau))} \text{T-weaken}$$

□

Lemma 83 (Subtyping preservation). $\forall \tau, \tau'$.

$$\tau <: \tau' \implies (\tau) <: (\tau')$$

Proof. Proof by induction on the $\tau <: \tau'$ relation

1. Base:

$$\overline{\mathbf{b} <: \mathbf{b}}$$

Main derivation:

$$\overline{.; \cdot \vdash !\mathbf{b} <: !\mathbf{b}}$$

2. Pair:

$$\frac{\tau_1 <: \tau'_1 \quad \tau_2 <: \tau'_2}{(\tau_1, \tau_2) <: (\tau'_1, \tau'_2)}$$

Main derivation:

$$\frac{\overline{(\tau_1) <: (\tau'_1)} \text{ IH1} \quad \overline{(\tau_2) <: (\tau'_2)} \text{ IH2}}{\overline{((\tau_1) \otimes (\tau_2)) <: ((\tau'_1) \otimes (\tau'_2))}}$$

3. List:

$$\frac{\tau_1 <: \tau_2 \quad \vec{p} \geq \vec{q}}{L^{\vec{p}}\tau_1 <: L^{\vec{q}}\tau_2}$$

Main derivation:

$$\frac{\overline{\vec{q} \leq \vec{p}} \text{ Given} \quad \overline{.; s \vdash (\tau_1) <: (\tau_2)} \text{ IH}}{\overline{.; s \vdash [\phi(\vec{p}, s)] \mathbf{1} <: [\phi(\vec{q}, s)] \mathbf{1}} \quad \overline{.; s \vdash L^s(\tau_1) <: L^s(\tau_2)}}{\overline{.; s \vdash ([\phi(\vec{p}, s)] \mathbf{1} \otimes L^s(\tau_1)) <: ([\phi(\vec{q}, s)] \mathbf{1} \otimes L^s(\tau_2))}}{\overline{.; \cdot \vdash \exists s.([\phi(\vec{p}, s)] \mathbf{1} \otimes L^s(\tau_1)) <: \exists s.([\phi(\vec{q}, s)] \mathbf{1} \otimes L^s(\tau_2))}}$$

□

2.5.2 Cross-language model: RAMLU to λ -Amor

Definition 84 (Logical relation for RAMLU to λ -Amor).

$$\begin{aligned}
[unit]_{\mathcal{V}}^H &\triangleq \{(T, {}^s v, {}^t v) \mid {}^s v \in \llbracket unit \rrbracket \wedge {}^t v \in \llbracket \mathbf{1} \rrbracket \wedge {}^s v = {}^t v\} \\
[\mathbf{b}]_{\mathcal{V}}^H &\triangleq \{(T, {}^s v, !{}^t v) \mid {}^s v \in \llbracket \mathbf{b} \rrbracket \wedge {}^t v \in \llbracket \mathbf{b} \rrbracket \wedge {}^s v = {}^t v\} \\
[(\tau_1, \tau_2)]_{\mathcal{V}}^H &\triangleq \{(T, \ell, \langle\langle {}^t v_1, {}^t v_2 \rangle\rangle) \mid H(\ell) = ({}^s v_1, {}^s v_2) \wedge (T, {}^s v_1, {}^t v_1) \in [\tau_1]_{\mathcal{V}} \wedge (T, {}^s v_2, {}^t v_2) \in [\tau_2]_{\mathcal{V}}\} \\
[L^{\vec{q}}\tau]_{\mathcal{V}}^H &\triangleq \{(T, \ell_s, \langle\langle (), {}^t t \rangle\rangle) \mid (T, \ell_s, {}^t t) \in [L \tau]_{\mathcal{V}}^H\} \\
\text{where} \\
[L \tau]_{\mathcal{V}}^H &\triangleq \{(T, NULL, nil)\} \cup \\
&\quad \{(T, \ell, {}^t v :: {}^t l_t) \mid H(\ell) = ({}^s v, \ell_s) \wedge (T, {}^s v, {}^t v) \in [\tau]_{\mathcal{V}} \wedge (T, \ell_s, {}^t l_t) \in [L \tau]_{\mathcal{V}}\} \\
[\tau_1 \xrightarrow{q/q'} \tau_2]_{\mathcal{V}}^H &\triangleq \{(T, f(x) = e_s, \text{fix}f.\lambda u.\lambda x.e_t) \mid \forall {}^s v', {}^t v', T' < T . \\
&\quad (T', {}^s v', {}^t v') \in [\tau_1]_{\mathcal{V}} \implies (T', e_s, e_t[() / u][{}^t v' / x][\text{fix}f.\lambda u.\lambda x.e_t / f]) \in [\tau_2]_{\mathcal{E}}^{\{x \mapsto {}^s v'\}, H}\} \\
[\tau]_{\mathcal{E}}^{V, H} &\triangleq \{(T, e_s, e_t) \mid \forall H', {}^s v, p, p', t < T . V, H \vdash_{p'}^p e_s \Downarrow_t {}^s v, H' \implies \\
&\quad \exists {}^t v_t, {}^t v_f, J.e_t \Downarrow_- {}^t v_t \Downarrow_-^J {}^t v_f \wedge (T -t, {}^s v, {}^t v_f) \in [\tau]_{\mathcal{V}}^{H'} \wedge p - p' \leq J\}
\end{aligned}$$

Definition 85 (Interpretation of typing context).

$$[\Gamma]_{\mathcal{V}}^H = \{(T, V, \delta_t) \mid \forall x : \tau \in \text{dom}(\Gamma).(T, V(x), \delta_t(x)) \in [\tau]_{\mathcal{V}}^H\}$$

Definition 86 (Interpretation of function context).

$$[\Sigma]_{\mathcal{V}}^H = \{(T, \delta_{sf}, \delta_{tf}) \mid (\forall f : (\tau_1 \xrightarrow{q/q'} \tau_2) \in \text{dom}(\Sigma).(T, \delta_{sf}(f) \delta_{sf}, \delta_{tf}(f) \delta_{tf}) \in [(\tau_1 \xrightarrow{q/q'} \tau_2)]_{\mathcal{V}}^H)\}$$

Lemma 87 (Monotonicity for values). $\forall {}^s v, {}^t v, T, \tau, H.$

$$(T, {}^s v, {}^t v) \in [\tau]_{\mathcal{V}}^H \implies \forall T' \leq T . (T', {}^s v, {}^t v) \in [\tau]_{\mathcal{V}}^H$$

Proof. Given: $(T, {}^s v, {}^t v) \in [\tau]_{\mathcal{V}}^H$

To prove: $\forall T' \leq T . (T', {}^s v, {}^t v) \in [\tau]_{\mathcal{V}}^H$

This means given some $T' \leq T$ it suffices to prove that

$$(T', {}^s v, {}^t v) \in [\tau]_{\mathcal{V}}^H$$

By induction on τ

1. $\tau = unit$:

In this case we are given that $(T, {}^s v, {}^t v) \in [unit]_{\mathcal{V}}^H$

and we need to prove $(T', {}^s v, {}^t v) \in [unit]_{\mathcal{V}}^H$

We get the desired trivially from Definition 84

2. $\tau = \mathbf{b}$:

In this case we are given that $(T, {}^s v, !{}^t v') \in [\mathbf{b}]_{\mathcal{V}}^H$

and we need to prove $(T', {}^s v, !{}^t v') \in [\mathbf{b}]_{\mathcal{V}}^H$

We get the desired trivially from Definition 84

3. $\tau = L^{\vec{p}}\tau'$:

In this case we are given that $(T, {}^s v, {}^t v) \in [L^{\vec{p}}\tau']_{\mathcal{V}}^H$

Here let ${}^s v = \ell_s$ and ${}^t v = \langle\langle (), {}^t v_h :: {}^t l_t \rangle\rangle$

and we have $(T, \ell_s, {}^t v_h :: l_t) \in [L\tau']_{\mathcal{V}}^H$ (MV-L1)

And we need to prove $(T', \ell_s, {}^t v_h :: l_t) \in [L\vec{\tau}']_{\mathcal{V}}^H$

Therefore it suffices to prove that $(T', \ell_s, {}^t v_h :: l_t) \in [L\tau']_{\mathcal{V}}^H$

We induct on $(T, \ell_s, {}^t v_h :: l_t) \in [L\tau']_{\mathcal{V}}^H$

- $(T, \text{NULL}, \text{nil}) \in [L\vec{\tau}']_{\mathcal{V}}^H$:

In this case we need to prove that $(T', \text{NULL}, \text{nil}) \in [L\tau']_{\mathcal{V}}^H$

We get this directly from Definition 84

- $(T, \ell_s, {}^t v_h :: l_t) \in [L\tau']_{\mathcal{V}}^H$:

Since from (MV-L1) we are given that $(T, \ell_s, {}^t v_h :: l_t) \in [L\tau']_{\mathcal{V}}^H$

therefore from Definition 84 we have

$$H(\ell_s) = ({}^s v_h, \ell_{st}) \wedge (T, {}^s v_h, {}^t v_h) \in [\tau']_{\mathcal{V}} \wedge (T, \ell_{st}, l_t) \in [L\tau']_{\mathcal{V}} \quad (\text{MV-L2})$$

In this case we need to prove that $(T', \ell_s, {}^t v_h :: l_t) \in [L\tau']_{\mathcal{V}}^H$

From Definition 84 it further it suffices to prove that

- $H(\ell_s) = ({}^s v_h, \ell_{st})$:
Directly from (MV-L2)
- $(T', {}^s v_h, {}^t v_h) \in [\tau']_{\mathcal{V}}$:
From (MV-L2) and outer induction
- $(T', \ell_{st}, l_t) \in [L\tau']_{\mathcal{V}}$:
From (MV-L2) and inner induction

4. $\tau = (\tau_1, \tau_2)$:

In this case we are given that $(T, \ell, ({}^t v_1, {}^t v_2)) \in [(\tau_1, \tau_2)]_{\mathcal{V}}^H$

This means from Definition 84 we have

$$H(\ell) = ({}^s v_1, {}^s v_2) \wedge (T, {}^s v_1, {}^t v_1) \in [\tau_1]_{\mathcal{V}} \wedge (T, {}^s v_2, {}^t v_2) \in [\tau_2]_{\mathcal{V}} \quad (\text{MV-P0})$$

and we need to prove $(T', \ell, ({}^t v_1, {}^t v_2)) \in [(\tau_1, \tau_2)]_{\mathcal{V}}^H$

Similarly from Definition 84 it suffices to prove that

$$H(\ell) = ({}^s v_1, {}^s v_2) \wedge (T', {}^s v_1, {}^t v_1) \in [\tau_1]_{\mathcal{V}} \wedge (T', {}^s v_2, {}^t v_2) \in [\tau_2]_{\mathcal{V}}$$

We get this directly from (MV-P0), IH1 and IH2

□

Lemma 88 (Monotonicity for functions). $\forall {}^s v, {}^t v, T, \tau, H.$

$$(T, f(x) = e_s, \text{fix}f.\lambda u.\lambda x.e_t) \in [\tau_1 \xrightarrow{q/q'} \tau_2]_{\mathcal{V}}^H \implies \forall T' \leq T. (T', f(x) = e_s, \text{fix}f.\lambda u.\lambda x.e_t) \in [\tau_1 \xrightarrow{q/q'} \tau_2]_{\mathcal{V}}^H$$

Proof. We need to prove that $(T', f(x) = e_s, \text{fix}f.\lambda u.\lambda x.e_t) \in [\tau_1 \xrightarrow{q/q'} \tau_2]_{\mathcal{V}}^H$

This means from Definition 84 it suffices to prove that

$$\forall {}^s v', {}^t v', T'' < T'. (T'', {}^s v', {}^t v') \in [\tau_1]_{\mathcal{V}} \implies (T'', e_s, e_t[(\)/u][{}^t v'/x][\text{fix}f.\lambda u.\lambda x.e_t/f]) \in [\tau_2]_{\mathcal{E}}^{\{x \mapsto {}^s v'\}, H}$$

This means given some ${}^s v', {}^t v', T'' < T'$ s.t. $(T'', {}^s v', {}^t v') \in [\tau_1]_{\mathcal{V}}$ it suffices to prove that

$$(T'', e_s, e_t[(\)/u][{}^t v'/x][\text{fix}f.\lambda u.\lambda x.e_t/f]) \in [\tau_2]_{\mathcal{E}}^{\{x \mapsto {}^s v'\}, H} \quad (\text{MF0})$$

Since we are given that $(T, f(x) = e_s, \text{fix}f.\lambda u.\lambda x.e_t) \in [\tau_1 \xrightarrow{q/q'} \tau_2]_{\mathcal{V}'}^H$, therefore from Definition 84 we have

$$\forall {}^s v'_1, {}^t v'_1, T'_1 < T . (T'_1, {}^s v'_1, {}^t v'_1) \in [\tau_1]_{\mathcal{V}} \implies (T'_1, e_s, e_t[(\)/u][{}^t v'_1/x][\text{fix}f.\lambda u.\lambda x.e_t/f]) \in [\tau_2]_{\mathcal{E}}^{\{x \mapsto {}^s v'_1\}, H}$$

Instantiating with the given ${}^s v', {}^t v', T''$ we get the desired □

Lemma 89 (Monotonicity for expressions). $\forall e_s, e_t, T, \tau, H.$

$$(T, e_s, e_t) \in [\tau]_{\mathcal{E}}^H \implies \forall T' \leq T . (T', e_s, e_t) \in [\tau]_{\mathcal{E}}^H$$

Proof. To prove: $(T', e_s, e_t) \in [\tau]_{\mathcal{E}}^H$

This means from Definition 84 it suffices to prove that

$$\forall H', {}^s v, p, p', t < T' . V, H \vdash_{p'}^p e_s \Downarrow_t {}^s v, H' \implies \exists {}^t v_t, {}^t v_f, J.e_t \Downarrow_- {}^t v_t \Downarrow_-^J {}^t v_f \wedge (T' - t, {}^s v, {}^t v_f) \in [\tau]_{\mathcal{V}'}^H \wedge p - p' \leq J$$

This means given some $H', {}^s v, p, p', t < T'$ s.t $V, H \vdash_{p'}^p e_s \Downarrow_t {}^s v, H'$ it suffices to prove that

$$\exists {}^t v_t, {}^t v_f, J.e_t \Downarrow_- {}^t v_t \Downarrow_-^J {}^t v_f \wedge (T' - t, {}^s v, {}^t v_f) \in [\tau]_{\mathcal{V}'}^H \wedge p - p' \leq J \quad (\text{ME0})$$

Since we are given that $(T, e_s, e_t) \in [\tau]_{\mathcal{E}}^H$ therefore again from Definition 84 we know that

$$\forall H', {}^s v, p, p', t < T . V, H \vdash_{p'}^p e_s \Downarrow_t {}^s v, H' \implies \exists {}^t v_t, {}^t v_f, J.e_t \Downarrow_- {}^t v_t \Downarrow_-^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in [\tau]_{\mathcal{V}'}^H \wedge p - p' \leq J$$

Instantiating with the given $H', {}^s v, p, p', t$ and using Lemma 87 we get the desired □

Lemma 90 (Monotonicity for Γ). $\forall {}^s v, {}^t v, T, \tau, H.$

$$(T, V, \delta_t) \in [\Gamma]_{\mathcal{V}}^H \implies \forall T' \leq T . (T', V, \delta_t) \in [\Gamma]_{\mathcal{V}}^H$$

Proof. To prove: $(T', V, \delta_t) \in [\Gamma]_{\mathcal{V}}^H$

From Definition 85 it suffices to prove that

$$\forall x : \tau \in \text{dom}(\Gamma). (T', V(x), \delta_t(x)) \in [\tau]_{\mathcal{V}}^H$$

This means given some $x : \tau \in \text{dom}(\Gamma)$ it suffices to prove that

$$(T', V(x), \delta_t(x)) \in [\tau]_{\mathcal{V}}^H$$

Since we are given that $(T, V, \delta_t) \in [\Gamma]_{\mathcal{V}}^H$

therefore from Definition 85 we have

$$\forall x : \tau \in \text{dom}(\Gamma). (T, V(x), \delta_t(x)) \in [\tau]_{\mathcal{V}}^H$$

Instantiating it with the given x and using Lemma 87 we get the desired □

Lemma 91 (Monotonicity for Σ). $\forall {}^s v, {}^t v, T, \tau, H.$

$$(T, \delta_{sf}, \delta_{tf}) \in [\Sigma]_{\mathcal{V}}^H \implies \forall T' \leq T . (T', \delta_{sf}, \delta_{tf}) \in [\Sigma]_{\mathcal{V}}^H$$

Proof. To prove: $(T', \delta_{sf}, \delta_{tf}) \in [\Sigma]_{\mathcal{V}}^H$

From Definition 86 it suffices to prove that

$$(\forall f : (\tau_1 \xrightarrow{q/q'} \tau_2) \in \text{dom}(\Sigma). (T', \delta_{sf}(f) \delta_{sf}, \delta_{tf}(f) \delta_{tf}) \in [(\tau_1 \xrightarrow{q/q'} \tau_2)]_{\mathcal{V}'}^H)$$

This means given some $f : (\tau_1 \xrightarrow{q/q'} \tau_2) \in \text{dom}(\Sigma)$ it suffices to prove that

$$(T', \delta_{sf}(f) \delta_{sf}, \delta_{tf}(f) \delta_{tf}) \in [(\tau_1 \xrightarrow{q/q'} \tau_2)]_{\mathcal{V}'}^H$$

Since we are given that $(T, \delta_{sf}, \delta_{tf}) \in [\Sigma]_{\mathcal{V}}^H$

therefore from Definition 85 we have

$$(\forall f : (\tau_1 \xrightarrow{q/q'} \tau_2) \in \text{dom}(\Sigma). (T, \delta_{sf}(f) \delta_{sf}, \delta_{tf}(f) \delta_{tf}) \in [(\tau_1 \xrightarrow{q/q'} \tau_2)]_{\mathcal{V}'}^H)$$

Instantiating it with the given f and using Lemma 88 we get the desired □

Theorem 92 (Fundamental theorem). $\forall \Sigma, \Gamma, q, q', \tau, e_s, e_t, I, V, H, \delta_t, \delta_{sf}, \delta_{tf}, T.$

$$\begin{aligned} & \Sigma; \Gamma \vdash_{q'}^q e_s : \tau \rightsquigarrow e_t \wedge \\ & (T, V, \delta_t) \in [\Gamma]_{\mathcal{V}}^H \wedge (T, \delta_{sf}, \delta_{tf}) \in [\Sigma]_{\mathcal{V}}^H \\ & \implies \\ & (T, e_s \delta_{sf}, e_t \delta_{tf}) \in [\tau]_{\mathcal{E}}^{V, H} \end{aligned}$$

Proof. Proof by induction on $\Sigma; \Gamma \vdash_{q'}^q e_s : \tau \rightsquigarrow e_t$

1. unit:

$$\frac{}{\Sigma; \cdot \vdash_q^{q+K^{unit}} () : unit \rightsquigarrow E_t} \text{ unit}$$

where

$$E_t = \lambda u. \text{release } - = u \text{ in bind } - = \uparrow^{K^{unit}} \text{ in bind } a = \text{store}() \text{ in ret}(a)$$

$$E'_t = \text{release } - = u \text{ in bind } - = \uparrow^{K^{unit}} \text{ in bind } a = \text{store}() \text{ in ret}(a)$$

$$\text{To prove: } (T, x \delta_{sf}, E_t \delta_{tf}) \in [\tau]_{\mathcal{E}}^{V, H}$$

This means from Definition 84 we are given some

$${}^s v, H', {}^s v, r, r', t \text{ s.t. } V, H \vdash_{r'}^r () \Downarrow_t (), H. \text{ From (E:Unit) we know that } t = 1$$

Therefore it suffices to prove that

$$(a) \exists {}^t v_t, {}^t v_f, J. E_t \delta_{tf} \Downarrow_{-} {}^t v_t \Downarrow_{-}^J {}^t v_f \wedge (T - 1, (), {}^t v_f) \in [unit]_{\mathcal{V}}:$$

We choose ${}^t v_t, {}^t v_f, J$ as $E'_t, (), K^{unit}$ respectively

Since from E-app we know that $E_t \Downarrow E'_t$, also since $E'_t \Downarrow^{K^{unit}} ()$ (from E-release, E-bind, E-store, E-return)

Therefore we get the desired from Definition 85

$$(b) r - r' \leq J:$$

From (E:Unit) we know that $\exists p. r = p + K^{unit}, r' = p$ and since we know that $J = K^{unit}$, therefore we are done

2. base:

$$\frac{}{\Sigma; \cdot \vdash_q^{q+K^{base}} c : \mathbf{b} \rightsquigarrow E_t} \text{ unit}$$

where

$$E_t = \lambda u. \text{release } - = u \text{ in bind } - = \uparrow^{K^{base}} \text{ in bind } a = \text{store}(!c) \text{ in ret}(a)$$

$$E'_t = \text{release } - = u \text{ in bind } - = \uparrow^{K^{base}} \text{ in bind } a = \text{store}(!c) \text{ in ret}(a)$$

$$\text{To prove: } (T, x \delta_{sf}, E_t \delta_{tf}) \in [\mathbf{b}]_{\mathcal{E}}^{V, H}$$

This means from Definition 84 we are given some

$${}^s v, H', {}^s v, r, r', t \text{ s.t. } V, H \vdash_{r'}^r c \Downarrow_t c, H. \text{ From (E:base) we know that } t = 1$$

Therefore it suffices to prove that

(a) $\exists^t v_t, {}^t v_f, J. E_t () \Downarrow_- {}^t v_t \Downarrow_-^J {}^t v_f \wedge (T - 1, (), {}^t v_f) \in [\mathbf{b}]_{\mathcal{V}}$:

We choose ${}^t v_t, {}^t v_f, J$ as $E'_t, !c, K^{base}$ respectively

Since from E-app we know that $E_t \Downarrow E'_t$, also since $E'_t \Downarrow^{K^{base}} !c$ (from E-release, E-bind, E-store, E-return)

Therefore we get the desired from Definition 85

(b) $r - r' \leq J$:

From (E:base) we know that $\exists p. r = p + K^{base}$, $r' = p$ and since we know that $J = K^{base}$, therefore we are done

3. var:

$$\frac{}{\Sigma; x : \tau \vdash_q^{q+K^{var}} x : \tau \rightsquigarrow E_t} \text{ var}$$

where

$E_t = \lambda u. \text{release } - = u \text{ in bind } - = \uparrow^{K^{var}}$ in bind $a = \text{store } x \text{ in ret}(a)$

$E'_t = \text{release } - = () \text{ in bind } - = \uparrow^{K^{var}}$ in bind $a = \text{store } x \text{ in ret}(a)$

To prove: $(T, x\delta_{sf}, E_t () \delta_t \delta_{tf}) \in [\tau]_{\mathcal{E}}^{V,H}$

This means from Definition 84 we are given some

${}^s v, H', {}^s v, r, r', t$ s.t $V, H \vdash_{r'}^r x \Downarrow_t V(x), H$. From (E:Var) we know that $t = 1$

Therefore it suffices to prove that

(a) $\exists^t v_t, {}^t v_f, J. E_t () \Downarrow_- {}^t v_t \Downarrow_-^J {}^t v_f \wedge (T - 1, V(x), {}^t v_f) \in [\tau]_{\mathcal{V}}$:

We choose ${}^t v_t, {}^t v_f, J$ as $E'_t, \delta_t(x)$ respectively

Since from E-app we know that $E_t \Downarrow E'_t$, also since $E_t \Downarrow^{K^{var}} \delta_t(x)$ (from E-release, E-bind, E-store, E-return)

Therefore we get the desired from Definition 85 and Lemma 91

(b) $r - r' \leq J$:

From (E:VAR) we know that $\exists p. r = p + K^{var}$, $r' = p$ and $J = K^{var}$, so we are done

4. app:

$$\frac{\tau_1 \xrightarrow{q/q'} \tau_2 \in \Sigma(f)}{\Sigma; x : \tau_1 \vdash_{q'-K_2}^{q+K_1^{app}} f x : \tau_2 \rightsquigarrow E_t} \text{ app}$$

where

$E_t = \lambda u. E_0$

$E_0 = \text{release } - = u \text{ in bind } - = \uparrow^{K_1^{app}}$ in bind $P = \text{store}() \text{ in } E_1$

$E_1 = \text{bind } f_1 = (f P x) \text{ in release } f_2 = f_1 \text{ in bind } - = \uparrow^{K_2^{app}}$ in bind $f_3 = \text{store } f_2 \text{ in ret } f_3$

To prove: $(T, f x, E_t () \delta_t \delta_{tf}) \in [\tau_2]_{\mathcal{E}}^{V,H}$

This means from Definition 84 we are given some

${}^s v, H', {}^s v, r, r', t < T$ s.t $V, H \vdash_{r'}^r f x \delta_{sf} \Downarrow_t {}^s v, H'$

and it suffices to prove that

$$\exists^t v_t, {}^t v_f, J.E_t () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in [\tau_2]_{\mathcal{V}}^{H'} \wedge r - r' \leq J \quad (\text{F-A0})$$

Since we are given that $(T, \delta_{sf}, \delta_{tf}) \in [\Sigma]_{\mathcal{V}}^H$ therefore from Definition 86 we know that

$$(T, \delta_{sf}(f) \delta_{sf}, \delta_{tf}(f) \delta_{tf}) \in [(\tau_1 \xrightarrow{q/q'} \tau_2)]_{\mathcal{V}}^H$$

From Definition 84 we know that $\delta_{sf}(f) = (f(x) = e_s)$ and $\delta_{tf}(f) = \text{fix}f.\lambda u.\lambda x.e_t$ and we have

$$\forall^s v', {}^t v', T' < T. (T', {}^s v', {}^t v') \in [\tau_1]_{\mathcal{V}}^H \implies (T', e_s, e_t[()/u][{}^t v'/x][\text{fix}f.\lambda u.\lambda x.e_t/f]) \in [\tau_2]_{\mathcal{E}}^{\{x \mapsto {}^s v'\}, H} \quad (\text{F-A1})$$

Since we are given that $(T, V, \delta_t) \in [\Gamma]_{\mathcal{V}}^H$ therefore we have

$$(T, V(x), \delta_t(x)) \in [\tau_1]_{\mathcal{V}}^H$$

This means from Lemma 87 we also have $(T - 1, V(x), \delta_t(x)) \in [\tau_1]_{\mathcal{V}}^H$

Instantiating (F-A1) with $T - 1, V(x), \delta_t(x)$ we get

$$(T - 1, e_s, e_t[()/u][\delta_t(x)/x][\text{fix}f.\lambda u.\lambda x.e_t/f]) \in [\tau_2]_{\mathcal{E}}^{\{x \mapsto V(x)\}, H}$$

This means from Definition 84 we have

$$\begin{aligned} \forall H'_1, {}^s v_1, r_1, r'_1, t' < T - 1. V, H \vdash_{r'_1}^{r_1} e_s \Downarrow_{t'} {}^s v_1, H'_1 \implies \\ \exists^t v_t, {}^t v_f, J_1.e_t[()/u][\delta_t(x)/x][\text{fix}f.\lambda u.\lambda x.e_t/f] \Downarrow {}^t v_t \Downarrow^{J_1} {}^t v_f \wedge (T - 1 - t', {}^s v_1, {}^t v_f) \in [\tau_2]_{\mathcal{V}}^{H'_1} \wedge \\ r_1 - r'_1 \leq J_1 \quad (\text{F-A2}) \end{aligned}$$

Since we know that $V, H \vdash_{r'}^r f x \delta_{sf} \Downarrow_t {}^s v, H'$ where $t < T$ therefore from (E:FunApp) we know that

$V, H \vdash_{r'+K_2^{app}}^{r-K_1^{app}} e_s \Downarrow_{t-1} {}^s v, H'$ therefore instantiating (F-A2) with $H', {}^s v, r - K_1^{app}, r' + K_2^{app}, t - 1$ we get

$$\exists^t v_t, {}^t v_f, J_1.e_t[()/u][\delta_t(x)/x][\text{fix}f.\lambda u.\lambda x.e_t/f] \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in [\tau_2]_{\mathcal{V}}^{H'} \wedge (r - K_1^{app}) - (r' + K_2^{app}) \leq J_1 \quad (\text{F-A3})$$

From E-release, E-bind, E-store we know that $J = J_1 + K_1^{app} + K_2^{app}$ therefore we get the desired from (F-A3)

5. nil:

$$\frac{}{\Sigma; \emptyset \vdash_q^{q+K^{nil}} \text{nil} : L^{\vec{p}}\tau \rightsquigarrow E_t} \text{nil}$$

where

$E_t = \lambda u.\text{release} - = u$ in $\text{bind} - = \uparrow^{K^{nil}}$ in $\text{bind} a = \text{store}()$ in $\text{bind} b = \text{store}\langle\langle a, \text{nil} \rangle\rangle$ in $\text{ret}(b)$

To prove: $(T, \text{nil}, E_t ()) \delta_t \delta_{tf} \in [L^{\vec{p}}\tau]_{\mathcal{E}}^{V, H}$

This means from Definition 84 we are given some

$${}^s v, H', {}^s v, t < T \text{ s.t } \emptyset, \emptyset \vdash_p^p \text{nil} \Downarrow_t {}^s v, H'$$

From (E:NIL) we know that ${}^s v = \text{NULL}$, $H' = H$ and $t = 1$ and it suffices to prove that

(a) $\exists^t v_t, {}^t v_f, J.e_t \Downarrow^J {}^t v_t \Downarrow^J {}^t v_f \wedge (T - 1, nil, {}^t v_f) \in [L^{\vec{p}}\tau]_{\mathcal{V}}$:

From E-bind, E-release, E-return we know that ${}^t v = \langle\langle(), nil\rangle\rangle$ therefore from Definition 84 we get the desired

(b) $p - p' \leq J$:

Here $p = q + K^{nil}$, $p' = q$ and $J = K^{nil}$, so we are done

6. cons:

$$\frac{\vec{p} = (p_1, \dots, p_k)}{\Sigma; x_h : \tau, x_t : L^{\langle \vec{p} \rangle} \tau \vdash_q^{q+p_1+K^{cons}} cons(x_h, x_t) : L^p \tau \rightsquigarrow E_t} \text{ cons}$$

where

$$E_t = \lambda u. \text{release} - = u \text{ in bind} - = \uparrow^{K^{cons}} \text{ in } E_0$$

$$E_0 = x_t; x. \text{let} \langle\langle x_1, x_2 \rangle\rangle = x \text{ in } E_1$$

$$E_1 = \text{release} - = x_1 \text{ in bind } a = \text{store}() \text{ in store} \langle\langle a, x_h :: x_2 \rangle\rangle$$

$$E'_t = \text{release} - = () \text{ in bind} - = \uparrow^{K^{cons}} \text{ in } E_0$$

To prove: $(T, cons(x_h, x_t), E_t () \delta_t \delta_{tf}) \in [L^{\vec{p}}\tau]_{\mathcal{E}}^{V,H}$

This means from Definition 84 we are given some

$${}^s v, H', {}^s v, p, p', t < T \text{ s.t. } \emptyset, \emptyset \vdash_{p'}^p cons(x_h, x_t) \delta_{sf} \Downarrow_t {}^s v, H'$$

and it suffices to prove that

(a) $\exists^t v_t, {}^t v_f, J.E_t () \Downarrow^J {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, H'(\ell), {}^t v_f) \in [L^{\vec{p}}\tau]_{\mathcal{V}}^H$:

From (E-app) of λ -Amor we know that $E_t () \Downarrow E'_t$

Also from E-release, E-bind, E-store we know that ${}^t v_f = \langle\langle(), \delta_t(x_h) :: \delta_t(x_t) \downarrow_2\rangle\rangle$

Therefore it suffices to prove that $(T - t, \ell, \langle\langle(), \delta_t(x_h) :: \delta_t(x_t) \downarrow_2\rangle\rangle) \in [L^{\vec{p}}\tau]_{\mathcal{V}}^{H'}$

From Definition 84 it further suffices to prove that

$$(T - t, \ell, \delta_t(x_h) :: \delta_t(x_t) \downarrow_2) \in [L \tau]_{\mathcal{V}}^{H'}$$

Since from (E:CONS) rule of univariate RAML we know that $H' = H[\ell \mapsto v]$ where $v = (V(x_h), V(x_t))$

Therefore it further suffices to prove that

$$(T - t, V(x_h), \delta_t(x_h)) \in [\tau]_{\mathcal{V}}^{H'} \text{ and } (T - t, V(x_t), \delta_t(x_t) \downarrow_2) \in [L \tau]_{\mathcal{V}}^{H'}$$

Since we are given that $(T, V, \delta_{tf}) \in [\Sigma]_{\mathcal{V}'}^{V,H}$ therefore from Definition 85 and Lemma 87 it means we have

$$(T - t, V(x_h), \delta_t(x_h)) \in [\tau]_{\mathcal{V}}^H \quad (\text{F-C1})$$

and

$$(T - t, V(x_t), \delta_t(x_t)) \in [L^{\langle \vec{p} \rangle} \tau]_{\mathcal{V}}^H$$

$$\text{This means we also have } (T - t, V(x_t), \delta_t(x_t) \downarrow_2) \in [L \tau]_{\mathcal{V}}^H \quad (\text{F-C2})$$

Since $H' = H[\ell \mapsto v]$ where $v = (V(x_h), V(x_t))$ therefore we also have

We get the desired from (F-C1), (F-C2) and Definition 84

(b) $p - p' \leq J$:

From (E:CONS) we know that $p = q' + K^{cons}$ and $p' = q'$ for some q' . Also we know that $J = K^{cons}$. Therefore we are done.

7. match:

$$\frac{\begin{array}{c} \Sigma; \Gamma \vdash_{q'+K_2^{matN}}^{q-K_1^{matN}} e_1 : \tau' \rightsquigarrow e_{a1} \\ \vec{p} = (p_1, \dots, p_k) \quad \Sigma; \Gamma, h : \tau, t : L^{(< \vec{p})} \tau \vdash_{q'+K_2^{matN}}^{q+p_1-K_1^{matC}} e_2 : \tau' \rightsquigarrow e_{a2} \end{array}}{\Sigma; \Gamma; x : L^p \tau \vdash_q^q \text{ match } x \text{ with } |nil \mapsto e_1 | h :: t \mapsto e_2 : \tau' \rightsquigarrow \lambda u. E_0} \text{ match}$$

where

$$E_0 = \text{release } - = u \text{ in } E_{0.1}$$

$$E_{0.1} = x; a. \text{let} \langle\langle x_1, x_2 \rangle\rangle = a \text{ in } E_1$$

$$E_1 = \text{match } x_2 \text{ with } |nil \mapsto E_2 | h :: l_t \mapsto E_3$$

$$E_2 = \text{bind } - = \uparrow^{K_1^{matN}} \text{ in } E_{2.1}$$

$$E_{2.1} = \text{bind } b = \text{store}() \text{ in } E'_2$$

$$E'_2 = \text{bind } c = (e_{a1} b) \text{ in } E'_{2.1}$$

$$E'_{2.1} = \text{release } d = c \text{ in } E'_{2.2}$$

$$E'_{2.2} = \text{bind } - = \uparrow^{K_2^{matN}} \text{ in } E'_{2.3}$$

$$E'_{2.3} = \text{release } - = x_1 \text{ in store } d$$

$$E_3 = \text{bind } - = \uparrow^{K_1^{matC}} \text{ in } E_{3.1}$$

$$E_{3.1} = \text{release } - = x_1 \text{ in } E_{3.2}$$

$$E_{3.2} = \text{bind } b = \text{store}() \text{ in } E_{3.3}$$

$$E_{3.3} = \text{bind } t = \text{ret} \langle\langle b, l_t \rangle\rangle \text{ in } E_{3.4}$$

$$E_{3.4} = \text{bind } d = \text{store}() \text{ in } E_{3.5}$$

$$E_{3.5} = \text{bind } f = e_{a2} d \text{ in } E_{3.6}$$

$$E_{3.6} = \text{release } g = f \text{ in } E_{3.7}$$

$$E_{3.7} = \text{bind } - = \uparrow^{K_2^{matC}} \text{ in store } g$$

To prove: $(T, \text{match } x \text{ with } |nil \mapsto e_1 | h :: t \mapsto e_2, \lambda u. E_0) \delta_t \delta_{tf} \in [\tau']_{\mathcal{E}}^{V,H}$

This means from Definition 84 we are given some

$${}^s v, H', {}^s v, p, p', t < T \text{ s.t. } V, H \vdash_{p'}^p (\text{match } x \text{ with } |nil \mapsto e_1 | h :: t \mapsto e_2) \delta_{sf} \Downarrow_t {}^s v, H'$$

2 cases arise:

(a) $V(x) = \text{NULL}$:

Since $(T, V, \delta_t) \in [\Gamma]_{\mathcal{V}}^{V,H}$ therefore from Definition 85 and Definition 84 we have $\delta_t(x) = \langle\langle (), nil \rangle\rangle$

IH: $(T - 1, e_1 \delta_{sf}, e_{a1} () \delta_t \delta_{tf}) \in [\tau']_{\mathcal{E}}^{V,H}$

This means from Definition 84 we have

$$\forall H'_1, {}^s v_1, p_1, p'_1, t_1. V, H \vdash_{p'_1}^{p_1} e_1 \Downarrow_{t_1} {}^s v_1, H'_1 \implies \exists {}^t v_{t_1}, {}^t v_{f_1}, J_1. e_{a1} \Downarrow {}^t v_{t_1} \Downarrow^{J_1} {}^t v_{f_1} \wedge (T - 1 - t_1, {}^s v_1, {}^t v_{f_1}) \in [\tau']_{\mathcal{V}}^{H'_1} \wedge p_1 - p'_1 \leq J_1 \quad (\text{F-RUA-M0})$$

Since we are given that $V, H \vdash_{p'}^p$ (match x with $|nil \mapsto e_1 \mid h :: t \mapsto e_2\rangle \delta_{sf} \Downarrow_t {}^s v, H'$ therefore from (E:MatvhN) we know that $V, H \vdash_{p'+K_2^{matN}}^{p-K_1^{matN}} e_1 \Downarrow_{t-1} {}^s v, H'$ therefore instantiating (F-RUA-M0) with $H', {}^s v, p - K_1^{matN}, p' + K_2^{matN}$ we get

$$\exists {}^t v_{t_1}, {}^t v_{f_1}, J_1. e_{a1} \Downarrow {}^t v_{t_1} \Downarrow^{J_1} {}^t v_{f_1} \wedge (T - t, {}^s v, {}^t v_{f_1}) \in [\tau']_{\mathcal{V}}^{H'} \wedge p - K_1^{matN} - p' - K_2^{matN} \leq J_1 \quad (\text{F-RUA-M1})$$

It suffices to prove that

$$\exists {}^t v_t, {}^t v_f, J. \lambda u. E_0 () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in [\tau']_{\mathcal{V}}^{H'} \wedge p - p' \leq J$$

We choose ${}^t v_t$ as ${}^t v_{t_1}$, ${}^t v_f$ as ${}^t v_{f_1}$ and J as $J_1 + K_1^{matN} + K_2^{matN}$ and we get the desired from E-bind, E-release, E-store and (F-RUA-M1)

(b) $V(x) = \ell_s$:

Since $(T, V, \delta_t) \in [\Gamma]_{\mathcal{V}}^{V, H}$ therefore from Definition 85 and Definition 84 we have

$$\delta_t(x) = \langle\langle (), {}^t v_h :: l_t \rangle\rangle \text{ s.t}$$

$$H(\ell_s) = ({}^s v_h, \ell_{ts}), ({}^s v, {}^t v) \in [\tau']_{\mathcal{V}} \text{ and } (\ell_s, l_t) \in [L \tau']_{\mathcal{V}} \text{ and}$$

$$\text{Let } V' = V \cup \{h \mapsto {}^s v_h\} \cup \{t \mapsto \ell_{ts}\} \text{ and } \delta'_t = \delta_t \cup \{h \mapsto {}^t v_h\} \cup \{t \mapsto \ell_{ts}\}$$

From Definition 85 and Lemma 87 we have $(T - 1, V', \delta'_t) \in [\Gamma, h : \tau, t : L^{\leq \vec{p}} \tau]_{\mathcal{V}}^{V', H}$

Therefore from IH we have

$$(T - 1, e_2 \delta_{sf}, e_{a2} () \delta'_t \delta_{tf}) \in [\tau']_{\mathcal{E}}^{V', H}$$

This means from Definition 84 we have

$$\forall H'_2, {}^s v_2, p_2, p'_2, t_1. V, H \vdash_{p'_2}^{p_2} e_2 \Downarrow_{t_1} {}^s v_2, H'_2 \implies \exists {}^t v_{t_2}, {}^t v_{f_2}, J_2. e_{a2} \Downarrow {}^t v_{t_2} \Downarrow^{J_2} {}^t v_{f_2} \wedge (T - 1 - t_1, {}^s v_2, {}^t v_{f_2}) \in [\tau']_{\mathcal{V}}^{H'_2} \wedge p_2 - p'_2 \leq J_2 \quad (\text{F-RUA-M0.0})$$

Since we are given that $V, H \vdash_{p'}^p$ (match x with $|nil \mapsto e_1 \mid h :: t \mapsto e_2\rangle \delta_{sf} \Downarrow_t {}^s v, H'$ therefore from (E:MatvhC) we know that $V, H \vdash_{p'+K_2^{matC}}^{p-K_1^{matC}} e_2 \Downarrow_{t-1} {}^s v, H'$ therefore instantiating (F-RUA-M0.0) with $H', {}^s v, p - K_1^{matC}, p' + K_2^{matC}, t - 1$ we get

$$\exists {}^t v_{t_2}, {}^t v_{f_2}, J_2. e_{a2} \Downarrow {}^t v_{t_2} \Downarrow^{J_2} {}^t v_{f_2} \wedge (T - t, {}^s v_2, {}^t v_{f_2}) \in [\tau']_{\mathcal{V}}^{H'_2} \wedge p_2 - p'_2 \leq J_2 \quad (\text{F-RUA-M2})$$

It suffices to prove that

$$\exists {}^t v_t, {}^t v_f, J. \lambda u. E_0 () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in [\tau']_{\mathcal{V}}^{H'} \wedge p - p' \leq J$$

We choose ${}^t v_t$ as ${}^t v_{t_2}$, ${}^t v_f$ as ${}^t v_{f_2}$ and J as $J_2 + K_1^{matC} + K_2^{matC}$ and we get the desired from E-bind, E-release, E-store and (F-RUA-M2)

8. Share:

$$\frac{\Sigma; \Gamma, x : \tau_1, y : \tau_2 \vdash_q^q e : \tau' \rightsquigarrow e_a \quad \tau = \tau_1 \vee \tau_2 \quad \tau = \tau_1 = \tau_2 = \mathbf{1}}{\Sigma; \Gamma, z : \tau \vdash_q^q e[z/x, z/y] : \tau' \rightsquigarrow E_0} \text{ Share-unit}$$

$$E_0 = \lambda u. E_1$$

$$E_1 = \text{bind } a = \text{coerce}_{\mathbf{1}, \mathbf{1}, \mathbf{1}} z \text{ in let } \langle\langle x, y \rangle\rangle = a \text{ in } e_a u$$

$coerce_{1,1,1} \triangleq \lambda u. \text{ret}\langle\langle !(), !() \rangle\rangle$

To prove: $(T, e[z/x, z/y], E_0) \delta_t \delta_{tf} \in [\tau']_{\mathcal{E}}^{V, H}$

This means from Definition 84 we are given some

${}^s v, H', {}^s v, p, p', t$ s.t $V, H \vdash_{p'}^p e[z/x, z/y] \delta_{sf} \Downarrow_t {}^s v, H'$

And we need to prove

$\exists {}^t v_t, {}^t v_f, J. E_0() \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in [\tau]_{\mathcal{V}}^{H'} \wedge p - p' \leq J$

Let

$V' = V \cup \{x \mapsto V(z)\} \cup \{y \mapsto V(z)\}$

$\delta'_t = \delta_t \cup \{x \mapsto \delta_t(z)\} \cup \{y \mapsto \delta_t(z)\}$

Since we are given that $(T, V, \delta_t) \in [\Gamma, z : \mathbf{1}]_{\mathcal{V}}^{V, H}$ therefore from Definition 85 we also have

$(T, V', \delta'_t) \in [\Gamma, x : \mathbf{1}, y : \mathbf{1}]_{\mathcal{V}}^{V', H}$

IH

$(T, e, e_a) \delta'_t \delta_{tf} \in [\tau']_{\mathcal{E}}^{V', H}$

This means from Definition 84 we have

$\forall H'_1, {}^s v_1, p_1, p'_1, t_1. V', H \vdash_{p'_1}^{p_1} e \Downarrow_{t_1} {}^s v_1, H'_1 \implies \exists {}^t v_t, {}^t v_f, J. e_a() \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t_1, {}^s v_1, {}^t v_f) \in [\tau']_{\mathcal{V}}^{H'} \wedge p_1 - p'_1 \leq J$

Instantiating it with the given $H', {}^s v, p, p', t$ we get the desired

$$\frac{\Sigma; \Gamma, x : \tau_1, y : \tau_2 \vdash_{q'}^q e : \tau' \rightsquigarrow e_a \quad \tau = \tau_1 \checkmark \tau_2 \quad \tau = \tau_1 = \tau_2 = \mathbf{b}}{\Sigma; \Gamma, z : \tau \vdash_{q'}^q e[z/x, z/y] : \tau' \rightsquigarrow E_0} \text{Share-base}$$

$E_0 = \lambda u. E_1$

$E_1 = \text{bind } a = coerce_{\mathbf{b}, \mathbf{b}, \mathbf{b}} z \text{ in } \text{let}\langle\langle x, y \rangle\rangle = a \text{ in } e_a u$

$coerce_{\mathbf{b}, \mathbf{b}, \mathbf{b}} \triangleq \lambda u. \text{let } !u' = u \text{ in } \text{ret}\langle\langle u', u' \rangle\rangle$

Similar reasonign as in the unit case above

$$\frac{\tau = L^{\vec{p}} \tau'' \quad \Sigma; \Gamma, x : \tau_1, y : \tau_2 \vdash_{q'}^q e : \tau' \rightsquigarrow e_a \quad \tau = \tau_1 \checkmark \tau_2 \quad \tau_1 = L^{\vec{p}_1} \tau_1'' \quad \tau_2 = L^{\vec{p}_2} \tau_2'' \quad \tau'' = \tau_1'' \oplus \tau_2'' \quad \vec{p} = \vec{p}_1 \oplus \vec{p}_2}{\Sigma; \Gamma, z : \tau \vdash_{q'}^q e[z/x, z/y] : \tau' \rightsquigarrow E_0} \text{Share-list}$$

$E_0 = \lambda u. E_1$

$E_1 = \text{bind } a = coerce_{\tau, \tau_1, \tau_2} z \text{ in } \text{let}\langle\langle x, y \rangle\rangle = a \text{ in } e_a u$

$coerce_{L^{\vec{p}} \tau, L^{\vec{p}_1} \tau_1, L^{\vec{p}_2} \tau_2} \triangleq \text{fix } f. \lambda g. \lambda e. \text{let } !g' = g \text{ in } e; x. \text{let}\langle\langle p, l \rangle\rangle = x \text{ in } E_0$

where

$E_0 \triangleq \text{release } - = p \text{ in } E_1$
 $E_1 \triangleq \text{match } l \text{ with } | \text{nil} \mapsto E_{2.1} \mid h :: t \mapsto E_3$
 $E_{2.1} \triangleq \text{bind } z_1 = \text{store}() \text{ in } E_{2.2}$
 $E_{2.2} \triangleq \text{bind } z_2 = \text{store}() \text{ in } E_{2.3}$
 $E_{2.3} \triangleq \text{ret} \langle \langle \langle z_1, \text{nil} \rangle \rangle, \langle \langle z_2, \text{nil} \rangle \rangle \rangle$
 $E_3 \triangleq \text{bind } H = g' h \text{ in } E_{3.1}$
 $E_{3.1} \triangleq \text{bind } o_t = () \text{ in } E_{3.2}$
 $E_{3.2} \triangleq \text{bind } T = f g \langle \langle o_t, t \rangle \rangle \text{ in } E_4$
 $E_4 \triangleq \text{let} \langle \langle H_1, H_2 \rangle \rangle = H \text{ in } E_5$
 $E_5 \triangleq \text{let} \langle \langle T_1, T_2 \rangle \rangle = T \text{ in } E_6$
 $E_6 \triangleq T_1; tp_1. \text{let} \langle \langle p'_1, l'_1 \rangle \rangle = tp_1 \text{ in } E_{7.1}$
 $E_{7.1} \triangleq T_2; tp_2. \text{let} \langle \langle p'_2, l'_2 \rangle \rangle = tp_2 \text{ in } E_{7.2}$
 $E_{7.2} \triangleq \text{release } - = p'_1 \text{ in } E_{7.3}$
 $E_{7.3} \triangleq \text{release } - = p'_2 \text{ in } E_{7.4}$
 $E_{7.4} \triangleq \text{bind } o_1 = \text{store}() \text{ in } E_{7.5}$
 $E_{7.5} \triangleq \text{bind } o_2 = \text{store}() \text{ in } E_8$
 $E_8 \triangleq \text{ret} \langle \langle \langle \langle o_1, H_1 :: T_1 \rangle \rangle, \langle \langle o_2, H_2 :: T_2 \rangle \rangle \rangle$

To prove: $(T, e[z/x, z/y], E_0 ()) \delta_t \delta_{tf} \in [\tau']_{\mathcal{E}}^{V, H}$

This means from Definition 84 we are given some

$${}^s v, H', {}^s v, p, p', t < T \text{ s.t. } V, H \vdash_{p'}^p e[z/x, z/y] \delta_{sf} \Downarrow_t {}^s v, H'$$

And we need to prove

$$\exists {}^t v_t, {}^t v_f, J. E_0 () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T -t, {}^s v, {}^t v_f) \in [\tau]_{\mathcal{V}}^{H'} \wedge p - p' \leq J$$

Let

$$V' = V \cup \{x \mapsto V(z)\} \cup \{y \mapsto V(z)\}$$

$$\delta'_t = \delta_t \cup \{x \mapsto \delta_t(z)\} \cup \{y \mapsto \delta_t(z)\}$$

Since we are given that $(T, V, \delta_t) \in [\Gamma, z : \tau]_{\mathcal{V}}^{V, H}$ therefore from Definition 85 we also have

$$(T, V', \delta'_t) \in [\Gamma, x : \tau_1, y : \tau_2]_{\mathcal{V}}^{V', H}$$

III

$$(T, e, e_a ()) \delta'_t \delta_{tf} \in [\tau']_{\mathcal{E}}^{V', H}$$

This means from Definition 84 we have

$$\forall H'_1, {}^s v_1, p_1, p'_1, t_1. V', H \vdash_{p'_1}^{p_1} e \Downarrow_{t_1} {}^s v_1, H'_1 \implies \exists {}^t v_t, {}^t v_f, J. e_a () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T -t_1, {}^s v_1, {}^t v_f) \in [\tau']_{\mathcal{V}}^{H'} \wedge p_1 - p'_1 \leq J$$

Instantiating it with the given $H', {}^s v, p, p', t$ we get the desired

$$\frac{\tau = \tau_1 \curlywedge \tau_2 \quad \tau = (\tau_a, \tau_b) \quad \tau_1 = (\tau'_a, \tau'_b) \quad \tau_2 = (\tau''_a, \tau''_b) \quad \tau = \tau_1 \oplus \tau_2 \quad \text{Share-pair}}{\Sigma; \Gamma, x : \tau_1, y : \tau_2 \vdash_{q'}^q e : \tau' \rightsquigarrow e_a} \quad \Sigma; \Gamma, z : \tau \vdash_{q'}^q e[z/x, z/y] : \tau' \rightsquigarrow E_0$$

$$E_0 = \lambda u. E_1$$

$$E_1 = \text{bind } a = \text{coerce}_{(\tau_a, \tau_b), (\tau'_a, \tau'_b), (\tau''_a, \tau''_b)} z \text{ in let } \langle\langle x, y \rangle\rangle = a \text{ in } e_a u$$

$$\text{coerce}_{(\tau_a, \tau_b), (\tau'_a, \tau'_b), (\tau''_a, \tau''_b)} \triangleq \lambda -g_1. \lambda -g_2. \lambda p. \text{let } \langle\langle p_1, p_2 \rangle\rangle = p \text{ in } E_0$$

where

$$E_0 \triangleq \text{let } !g'_1 = g_1 \text{ in } E_1$$

$$E_1 \triangleq \text{let } !g'_2 = g_2 \text{ in } E_2$$

$$E_2 \triangleq \text{bind } P'_1 = g'_1 p_1 \text{ in } E_3$$

$$E_3 \triangleq \text{bind } P'_2 = g'_2 p_2 \text{ in } E_4$$

$$E_4 \triangleq \text{let } \langle\langle p'_{11}, p'_{12} \rangle\rangle = P'_1 \text{ in } E_5$$

$$E_5 \triangleq \text{let } \langle\langle p'_{21}, p'_{22} \rangle\rangle = P'_2 \text{ in } E_6$$

$$E_6 \triangleq \text{ret } \langle\langle p'_{11}, p'_{21} \rangle\rangle, \langle\langle p'_{12}, p'_{22} \rangle\rangle$$

Same reasoning as in the list subcase above

9. Sub:

$$\frac{\Sigma; \Gamma \vdash_{q'}^q e : \tau \rightsquigarrow e_a \quad \tau <: \tau'}{\Sigma; \Gamma \vdash_{q'}^q e : \tau' \rightsquigarrow e_a}$$

To prove: $(T, e, e_a () \delta_t \delta_{tf}) \in [\tau']_{\mathcal{E}}^{V, H}$

IH: $(T, e, e_a () \delta_t \delta_{tf}) \in [\tau]_{\mathcal{E}}^{V, H}$

We get the desired from IH and Lemma 94

10. Relax:

$$\frac{\Sigma; \Gamma \vdash_{p'}^p e : \tau \rightsquigarrow e_a \quad q \geq p \quad q - p \geq q' - p'}{\Sigma; \Gamma \vdash_{q'}^q e : \tau \rightsquigarrow E_t}$$

where

$$E_t = \lambda o. E_0$$

$$E_0 = \text{release } - = o \text{ in } E_1$$

$$E_1 = \text{bind } a = \text{store}() \text{ in } E_2$$

$$E_2 = \text{bind } b = e_a a \text{ in } E_3$$

$$E_3 = \text{release } c = b \text{ in store } c$$

To prove: $(T, e, E_t () \delta_t \delta_{tf}) \in [\tau]_{\mathcal{E}}^{V, H}$

This means from Definition 84 we are given some

${}^s v, H', {}^s v, r, r', t < T$ s.t $\emptyset, \emptyset \vdash_{r'}^r e \Downarrow_t {}^s v, H'$

And it suffices to prove that

$$\exists {}^t v_t, {}^t v_f, J. E_t () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T -t, {}^s v, {}^t v_f) \in [\tau]_{\mathcal{V}} \wedge r - r' \leq J \quad (\text{F-R0})$$

IH: $(T, e, e_a () \delta_t \delta_{tf}) \in [\tau]_{\mathcal{E}}^{V, H}$

This means from Definition 84 we have

$$\forall {}^s v_1, H'_1, r_1, r'_1, t_1 < T. V, H \vdash_{r'_1}^{r_1} e \Downarrow_{t_1} {}^s v_1, H' \implies \exists {}^t v_t, {}^t v_f, J. e_a () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T -t_1, {}^s v, {}^t v_f) \in [\tau]_{\mathcal{V}} \wedge r - r' \leq J$$

Instantiating it with the given ${}^s v, H', r, r', t$ we get

$$\exists {}^t v'_t, {}^t v'_f, J'. e_a () \Downarrow {}^t v'_t \Downarrow^{J'} {}^t v'_f \wedge (T -t, {}^s v, {}^t v'_f) \in [\tau]_{\mathcal{V}} \wedge r - r' \leq J' \quad (\text{F-R1})$$

In order to prove (F-R0) we choose ${}^t v_t, {}^t v_f, J$ as ${}^t v'_t, {}^t v'_f, J'$ and we get the desired from E-app, E-release, E-bind, E-store and (F-R1)

11. Super:

$$\frac{\Sigma; \Gamma, x : \tau_1 \vdash_{q'}^q e : \tau \rightsquigarrow e_a \quad \tau'_1 <: \tau_1}{\Sigma; \Gamma, x : \tau'_1 \vdash_{q'}^q e : \tau \rightsquigarrow e_a} \text{ Super}$$

Given: $(T, V, \delta_t) \in [\Gamma, x : \tau'_1]_{\mathcal{V}}^H$

To prove: $(T, e, e_a () \delta_t \delta_{tf}) \in [\tau]_{\mathcal{E}}^{V, H}$

This means from Definition 84 it suffices to prove that

$$\forall H', {}^s v, p, p', t < T. V, H \vdash_{p'}^p e_s \Downarrow_t {}^s v, H' \implies \exists {}^t v_t, {}^t v_f, J. e_a () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T -t, {}^s v, {}^t v_f) \in [\tau]_{\mathcal{V}}^{H'} \wedge p - p' \leq J$$

This means given some $H', {}^s v, p, p', t < T$ s.t $V, H \vdash_{p'}^p e_s \Downarrow_t {}^s v, H'$ it suffices to prove that

$$\exists {}^t v_t, {}^t v_f, J. e_a () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T -t, {}^s v, {}^t v_f) \in [\tau]_{\mathcal{V}}^{H'} \wedge p - p' \leq J \quad (\text{F-Su0})$$

Since we are given that $(T, V, \delta_t) \in [\Gamma, x : \tau'_1]_{\mathcal{V}}^H$ therefore from Definition 85 we know that $(T, V(x), \delta_t(x)) \in [\tau'_1]_{\mathcal{V}}^H$

Therefore from Lemma 93 we know that $(T, V(x), \delta_t(x)) \in [\tau_1]_{\mathcal{V}}^H$

IH: $(T, e, e_a () \delta_t \delta_{tf}) \in [\tau]_{\mathcal{E}}^{V, H}$

This means from Definition 84 we have

$$\forall H'_i, {}^s v_i, p_i, p'_i, t_1. V, H \vdash_{p'_i}^{p_i} e \Downarrow_{t_1} {}^s v, H' \implies \exists {}^t v_t, {}^t v_f, J. e_a () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T -t_1, {}^s v_i, {}^t v_f) \in [\tau]_{\mathcal{V}}^{H'} \wedge p_i - p'_i \leq J$$

Instantiating it with the given $H', {}^s v, p, p', t$ we get the desired

12. Let:

$$\frac{\Sigma; \Gamma_1 \vdash_p^{q-K_1^{let}} e_1 : \tau_1 \rightsquigarrow e_{a1} \quad \Sigma; \Gamma_2, x : \tau_1 \vdash_{q'+K_3^{let}}^{p-K_2^{let}} e_2 : \tau_1 \rightsquigarrow e_{a2}}{\Sigma; \Gamma_1, \Gamma_2 \vdash_{q'}^q \text{let } x = e_1 \text{ in } e_2 : \tau \rightsquigarrow E_t} \text{ Let}$$

where

$$E_t = \lambda u. E_0$$

$$E_0 = \text{release } - = u \text{ in } E_1$$

$$E_1 = \text{bind } - = \uparrow^{K_1^{let}} \text{ in } E_2$$

$$E_2 = \text{bind } a = \text{store}() \text{ in } E_3$$

$$E_3 = \text{bind } b = e_{a1} a \text{ in } E_4$$

$$E_4 = \text{release } x = b \text{ in } E_5$$

$$E_5 = \text{bind } - = \uparrow^{K_2^{let}} \text{ in } E_6$$

$$E_6 = \text{bind } c = \text{store}() \text{ in } E_7$$

$$E_7 = \text{bind } d = e_{a2} c \text{ in } E_8$$

$$E_8 = \text{release } f = d \text{ in } E_9$$

$$E_9 = \text{bind } - = \uparrow^{K_3^{let}} \text{ in } E_{10}$$

$$E_{10} = \text{bind } g = \text{store } f \text{ in ret } g$$

To prove: $(T, \text{let } x = e_1 \text{ in } e_2, E_t () \delta_t \delta_{tf}) \in [\tau]_{\mathcal{E}}^{V, H}$

This means from Definition 84 we are given some

$${}^s v, H', {}^s v, r, r', t < T \text{ s.t. } V, H \vdash_{r'}^r (\text{let } x = e_1 \text{ in } e_2) \delta_{sf} \Downarrow_t {}^s v, H'$$

it suffices to prove that

$$\exists {}^t v_t, {}^t v_f, J, e_t \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in [\tau]_{\mathcal{V}}^{H'} \wedge r - r' \leq J \quad (\text{F-L0})$$

Since we are given that $(T, V, \delta_t) \text{ in } [\Gamma_1, \Gamma_2]_{\mathcal{V}}^H$ therefore we know that

$$\exists V_1, V_2, \delta_t^1, \delta_t^2 \text{ s.t. } V = V_1, V_2, \delta_t = \delta_t^1, \delta_t^2 \text{ and}$$

$$(T, V_1, \delta_t^1) \in [\Gamma_1]_{\mathcal{V}}^H \text{ and } (T, V_2, \delta_t^2) \in [\Gamma_2]_{\mathcal{V}}^H$$

IH1

$$(T, e_1, e_{a1} () \delta_t^1 \delta_{tf}) \in [\tau_1]_{\mathcal{E}}^{V_1, H'}$$

This means from Definition 84 we have

$$\forall H'_1, {}^s v_1, p_1, p'_1, t_1. V, H \vdash_{p'_1}^{p_1} e_1 \Downarrow_{t_1} {}^s v_1, H' \implies \exists {}^t v_{t1}, {}^t v_{f1}, J_1. e_{a1} () \Downarrow {}^t v_{t1} \Downarrow^{J_1} {}^t v_{f1} \wedge (T - t_1, {}^s v_1, {}^t v_{f1}) \in [\tau_1]_{\mathcal{V}}^{H'_1} \wedge p_1 - p'_1 \leq J_1 \quad (\text{F-L1})$$

Since we know that $V, H \vdash_{r'}^r (\text{let } x = e_1 \text{ in } e_2) \delta_{sf} \Downarrow_t {}^s v, H'$ therefore from (E:Let) we know that $\exists H'_1, {}^s v_1, r_1, t_1 \text{ s.t. } V, H \vdash_{r_1}^{r - K_1^{let}} e_1 \delta_{sf} \Downarrow_{t_1} {}^s v_1, H'_1$

Instantiating (F-L1) with $H'_1, {}^s v_1, r - K_1^{let}, r_1, t_1$ we get

$$\exists {}^t v_{t1}, {}^t v_{f1}, J_1. e_{a1} () \Downarrow {}^t v_{t1} \Downarrow^{J_1} {}^t v_{f1} \wedge (T - t_1, {}^s v_1, {}^t v_{f1}) \in [\tau_1]_{\mathcal{V}}^{H'_1} \wedge r - K_1^{let} - r_1 \leq J_1 \quad (\text{F-L1.1})$$

IH2

$$(T - t_1, e_2, e_{a2} () \delta_t^2 \cup \{x \mapsto {}^t v_{f1}\} \delta_{tf}) \in [\tau]_{\mathcal{E}}^{V_2 \cup \{x \mapsto {}^s v_1\}, H'_1}$$

This means from Definition 84 we have

$$\forall H'_2, {}^s v_2, p_2, p'_2, t_2 < T - t_1. V, H \vdash_{p'_2}^{p_2} e_2 \Downarrow_{t_2} {}^s v_2, H' \implies \exists {}^t v_{t_2}, {}^t v_{f_2}, J_2.e_{a_2} () \Downarrow {}^t v_{t_2} \Downarrow^{J_2} {}^t v_{f_2} \wedge (T - t_1 - t_2, {}^s v_2, {}^t v_{f_2}) \in [\tau]_{\mathcal{V}}^{H'_2} \wedge p_2 - p'_2 \leq J_2 \quad (\text{F-L2})$$

Since we know that $V, H \vdash_{r'}^r (\text{let } x = e_1 \text{ in } e_2)\delta_{sf} \Downarrow_t {}^s v, H'$ therefore from (E:Let) we know that $\exists H'_2, {}^s v_2, t_2 < t - t_1$ s.t $V, H \vdash_{r'+K_3^{let}}^{r_1-K_2^{let}} e_2\delta_{sf} \Downarrow_{t_2} {}^s v, H'_2$

Instantiating (F-L2) with $H'_2, {}^s v, r_1 - K_2^{let}, r' + K_3^{let}, t_2$ we get

$$\exists {}^t v_{t_2}, {}^t v_{f_2}, J_2.e_{a_2} () \Downarrow {}^t v_{t_2} \Downarrow^{J_2} {}^t v_{f_2} \wedge (T - t_1 - t_2, {}^s v, {}^t v_{f_2}) \in [\tau]_{\mathcal{V}}^{H'_2} \wedge r_1 - K_2^{let} - (r' + K_3^{let}) \leq J_2 \quad (\text{F-L2.1})$$

In order to prove (F-L0) we choose ${}^t v_t$ as ${}^t v_{t_2}$, ${}^t v_f$ as ${}^t v_{f_2}$, J as $J_1 + J_2 + K_1^{let} + K_2^{let} + K_3^{let}$, t as $t_1 + t_2 + 1$ and we get the desired from (F-L1.1) and (F-L2.1) and Lemma 87

13. Pair:

$$\frac{}{\Sigma; x_1 : \tau_1, x_2 : \tau_2 \vdash_q^{q+K^{pair}} (x_1, x_2) : (\tau_1, \tau_2) \rightsquigarrow E_t} \text{pair}$$

where

$$E_t = \lambda u. E_0$$

$$E_0 = \text{release } - = u \text{ in } E_1$$

$$E_1 = \text{bind } - = \uparrow^{K^{pair}} \text{ in } E_2$$

$$E_2 = \text{bind } a = \text{store}(x_1, x_2) \text{ in ret } a$$

$$\text{Given: } (T, V, \delta_t) \in [x_1 : \tau_1, x_2 : \tau_2]_{\mathcal{V}}^H$$

$$\text{To prove: } (T, (x_1, x_2), E_t ()) \delta_t \delta_{tf} \in [(\tau_1, \tau_2)]_{\mathcal{E}}^{V, H}$$

This means from Definition 84 it suffices to prove that

$$\forall H', {}^s v, r, r', t < T. V, H \vdash_{r'}^r (x_1, x_2) \Downarrow_t {}^s v, H' \implies \exists {}^t v_t, {}^t v_f, J.E_t () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in [(\tau_1, \tau_2)]_{\mathcal{V}}^{H'} \wedge r - r' \leq J$$

This means given some $H', {}^s v, r, r', t < T$ s.t $V, H \vdash_{r'}^r (x_1, x_2) \Downarrow_t {}^s v, H'$ it suffices to prove that

$$\exists {}^t v_t, {}^t v_f, J.E_t () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in [(\tau_1, \tau_2)]_{\mathcal{V}}^{H'} \wedge r - r' \leq J \quad (\text{F-P0})$$

This means we need to prove that $\exists {}^t v_t, {}^t v_f, J$

- $E_t () \Downarrow {}^t v_t \Downarrow^J {}^t v_f$:

From E-app, E-release, E-bind, E-tick, E-store and E-return we know that ${}^t v_t = E_0$, ${}^t v_f = (\delta_t(x_1), \delta_t(x_2))$ and $J = K^{pair}$

- $(T - t, {}^s v, {}^t v_f) \in [(\tau_1, \tau_2)]_{\mathcal{V}}^{H'}$:

Since we are given that $V, H \vdash_{r'}^r (x_1, x_2) \Downarrow_t {}^s v, H'$, therefore from (E:Pair) we know that ${}^s v = \ell$ where $\ell \notin \text{dom}(H)$ and $H' = H[\ell \mapsto (V(x_1), V(x_2))]$

Since we are given that $(T, V, \delta_t) \in [x_1 : \tau_1, x_2 : \tau_2]_{\mathcal{V}}^H$ therefore from Definition 85, Definition 84 and Lemma 87 we get the desired.

- $r - r' \leq J$:

From (E:Pair) we know that $\exists p.r = p + K^{pair}$ and $r' = p$. Since we know that $J = K^{pair}$, therefore we are done.

14. MatP:

$$\frac{\tau = (\tau_1, \tau_2) \quad \Sigma, \Gamma, x_1 : \tau_1, x_2 : \tau_2 \vdash_{q-K_1^{matP}}^{q'+K_2^{matP}} e : \tau' \rightsquigarrow e_t}{\Sigma; \Gamma, x : \tau \vdash_q^{matP} \text{match } x \text{ with } (x_1, x_2) \rightarrow e : \tau' \rightsquigarrow E_t} \text{ matP}$$

where

$$E_t = \lambda u. E_0$$

$$E_0 = \text{release } - = u \text{ in } E_1$$

$$E_1 = \text{bind } - = \uparrow^{K_1^{matP}} \text{ in } E_2$$

$$E_2 = \text{let} \langle \langle x_1, x_2 \rangle \rangle = x \text{ in } E_3$$

$$E_3 = \text{bind } a = \text{store}() \text{ in } E_4$$

$$E_4 = \text{bind } b = e_t a \text{ in } E_5$$

$$E_5 = \text{release } c = b \text{ in } E_6$$

$$E_6 = \text{bind } - = \uparrow^{K_2^{matP}} \text{ in } E_7$$

$$E_7 = \text{bind } d = \text{store } c \text{ in ret } d$$

$$\text{Given: } (T, V, \delta_t) \in [\Gamma, x : \tau]_{\mathcal{V}}^H$$

$$\text{To prove: } (T, (\text{match } x \text{ with } (x_1, x_2) \rightarrow e), E_t ()) \delta_t \delta_{tf} \in [\tau]_{\mathcal{E}}^{V, H}$$

This means from Definition 84 it suffices to prove that

$$\forall H', {}^s v, p, p', t < T. V, H \vdash_{p'}^p (\text{match } x \text{ with } (x_1, x_2) \rightarrow e) \Downarrow_t {}^s v, H' \implies \exists {}^t v_t, {}^t v_f, J. E_t () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in [\tau']_{\mathcal{V}}^{H'} \wedge p - p' \leq J$$

This means given some $H', {}^s v, p, p', t < T$ s.t $V, H \vdash_{p'}^p (\text{match } x \text{ with } (x_1, x_2) \rightarrow e) \Downarrow_t {}^s v, H'$ it suffices to prove that

$$\exists {}^t v_t, {}^t v_f, J. E_t () \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in [\tau']_{\mathcal{V}}^{H'} \wedge p - p' \leq J \quad (\text{F-MP0})$$

Since we are given that $(T, V, \delta_t) \in [\Gamma, x : \tau]_{\mathcal{V}}^H$ therefore from Definition 85 and since $\tau = (\tau_1, \tau_2)$ therefore we know that $(T, V(x), \delta_t(x)) \in [(\tau_1, \tau_2)]_{\mathcal{V}}^H$

This means from Definition 84 that $\exists \ell$ s.t $H(\ell) = ({}^s v_1, {}^s v_2) \wedge (T, {}^s v_1, {}^t v_1) \in [\tau_1]_{\mathcal{V}} \wedge (T, {}^s v_2, {}^t v_2) \in [\tau_2]_{\mathcal{V}}$

$$\underline{\text{IH:}} (T, e, e_t ()) \delta_t \cup \{x_1 \mapsto {}^t v_1\} \cup \{x_2 \mapsto {}^t v_2\} \delta_{tf} \in [\tau']_{\mathcal{E}}^{V \cup \{x_1 \mapsto {}^s v_1\} \cup \{x_2 \mapsto {}^s v_2\}, H}$$

This means from Definition 84 we have

$$\forall H'_i, {}^s v_i, p_i, p'_i, t_1 < T - 1. V, H \vdash_{p'_i}^{p_i} e \Downarrow_{t_1} {}^s v, H'_i \implies \exists {}^t v_{t1}, {}^t v_{f1}, J_1. e () \Downarrow {}^t v_{t1} \Downarrow^J {}^t v_{f1} \wedge (T - t_1, {}^s v_i, {}^t v_{f1}) \in [\tau']_{\mathcal{V}}^{H'_i} \wedge p_i - p'_i \leq J_1$$

Since we are given that $V, H \vdash_{p'}^p (\text{match } x \text{ with } (x_1, x_2) \rightarrow e) \Downarrow {}^s v, H'$ therefore from (E:MatP) we know that

$$V \cup \{x_1 \mapsto {}^s v_1\} \cup \{x_2 \mapsto {}^s v_2\}, H \vdash_{p'+K_2^{matP}}^{p-K_1^{matP}} e \Downarrow_{t-1} {}^s v, H'$$

Instantiating it with the given $H', {}^s v, p - K_1^{matP}, p' + K_2^{matP}, t - 1$ we get

$$\exists {}^t v_{t_1}, {}^t v_{f_1}, J_1.e () \Downarrow {}^t v_{t_1} \Downarrow^J {}^t v_{f_1} \wedge (T - t, {}^s v, {}^t v_{f_1}) \in [\tau']_{\mathcal{V}}^{H'} \wedge p - K_1^{matP} - (p' + K_2^{matP}) \leq J_1$$

(F-MP1)

In order to prove (F-MP0) we choose ${}^t v_t$ as ${}^t v_{t_1}$, ${}^t v_f$ as ${}^t v_{f_1}$, J as $J_1 + K_1^{matP} + K_2^{matP}$ and t_1 as $t - 1$ and it suffices to prove that

- $E_t () \Downarrow {}^t v_t \Downarrow^J {}^t v_f$:
We get the desired from E-app, E-bind, E-release, E-store, E-tick, E-return and (F-MP1)
- $(T - t, {}^s v, {}^t v_f) \in [\tau']_{\mathcal{V}}^{H'}$:
From (F-MP1)
- $p - p' \leq J$:
We get this directly from (F-MP1)

15. Augment:

$$\frac{\Sigma; \Gamma \vdash_{q'}^q e : \tau \rightsquigarrow e_a}{\Sigma; \Gamma, x : \tau' \vdash_{q'}^q e : \tau \rightsquigarrow e_a} \text{ Augment}$$

$$\text{Given: } (T, V \cup \{x \mapsto {}^s v_x\}, \delta_t \cup \{x \mapsto {}^t v_x\}) \in [\Gamma, x : \tau']_{\mathcal{V}}^H$$

$$\text{To prove: } (T, e, e_a () \delta_t \cup \{x \mapsto {}^t v_x\} \delta_{t_f}) \in [\tau]_{\mathcal{E}}^{V \cup \{x \mapsto {}^s v_x\}, H}$$

This means from Definition 84 it suffices to prove that

$$\forall H', {}^s v, p, p', t < T . V \cup \{x \mapsto {}^s v_x\}, H \vdash_{p'}^p e_s \Downarrow_t {}^s v, H' \implies \exists {}^t v_t, {}^t v_f, J.e_a () \delta_t \cup \{x \mapsto {}^t v_x\} \delta_{t_f} \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in [\tau]_{\mathcal{V}}^{H'} \wedge p - p' \leq J$$

This means given some $H', {}^s v, p, p', t < T$ s.t $V \cup \{x \mapsto {}^s v_x\}, H \vdash_{p'}^p e_s \Downarrow_t {}^s v, H'$ it suffices to prove that

$$\exists {}^t v_t, {}^t v_f, J.e_a () \delta_t \cup \{x \mapsto {}^t v_x\} \delta_{t_f} \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t, {}^s v, {}^t v_f) \in [\tau]_{\mathcal{V}}^{H'} \wedge p - p' \leq J$$

(F-Ag0)

$$\text{Since we are given that } (T, V \cup \{x \mapsto {}^s v_x\}, \delta_t \cup \{x \mapsto {}^t v_x\}) \in [\Gamma, x : \tau']_{\mathcal{V}}^H$$

therefore from Definition 85 we know that

$$(T, V, \delta_t) \in [\Gamma]_{\mathcal{V}}^H$$

$$\underline{\text{IH:}} (T, e, e_a () \delta_t \delta_{t_f}) \in [\tau]_{\mathcal{E}}^{V, H}$$

This means from Definition 84 we have

$$\forall H'_i, {}^s v_i, p_i, p'_i, t_1 < T . V, H \vdash_{p'_i}^{p_i} e \Downarrow_{t_1} {}^s v, H' \implies \exists {}^t v_t, {}^t v_f, J.e_a () \delta_t \delta_{t_f} \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t_1, {}^s v_i, {}^t v_f) \in [\tau]_{\mathcal{V}}^{H'} \wedge p_i - p'_i \leq J \quad (\text{F-Ag1})$$

Since we are given $V \cup \{x \mapsto {}^s v_x\}, H \vdash_{p'}^p e_s \Downarrow_t {}^s v, H'$ and since $x \notin \text{free}(e)$ therefore we also have

$V, H \vdash_{p'}^p e_s \Downarrow_t {}^s v, H'$

Instantiating (F-Ag1) with the given $H', {}^s v, p, p', t$ we get

$$\exists {}^t v_t, {}^t v_f, J.e_a () \delta_t \delta_{t_f} \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T -t, {}^s v_i, {}^t v_f) \in [\tau]_{\mathcal{V}}^H \wedge p_i - p'_i \leq J$$

Also since $x \notin \text{free}(e)$ therefore we get

$$\exists {}^t v_t, {}^t v_f, J.e_a () \delta_t \cup \{x \mapsto {}^t v_x\} \delta_{t_f} \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T -t, {}^s v_i, {}^t v_f) \in [\tau]_{\mathcal{V}}^H \wedge p_i - p'_i \leq J$$

□

Lemma 93 (Value subtyping lemma). $\forall \tau, \tau', H, {}^s v, {}^t v, T.$

$$\tau <: \tau' \wedge (T, {}^s v, {}^t v) \in [\tau]_{\mathcal{V}}^H \implies (T, {}^s v, {}^t v) \in [\tau']_{\mathcal{V}}^H$$

Proof. Proof by induction on the subtyping relation of Univariate RAML

1. Unit:

$$\overline{\text{unit} <: \text{unit}}$$

Given: $(T, {}^s v, {}^t v) \in [\text{unit}]_{\mathcal{V}}^H$

To prove: $(T, {}^s v, {}^t v) \in [\text{unit}]_{\mathcal{V}}^H$

Trivial

2. Base:

$$\overline{\mathbf{b} <: \mathbf{b}}$$

Given: $(T, {}^s v, {}^t v) \in [\mathbf{b}]_{\mathcal{V}}^H$

To prove: $(T, {}^s v, {}^t v) \in [\mathbf{b}]_{\mathcal{V}}^H$

Trivial

3. Pair:

$$\frac{\tau_1 <: \tau'_1 \quad \tau_2 <: \tau'_2}{(\tau_1, \tau_2) <: (\tau'_1, \tau'_2)}$$

Given: $(T, {}^s v, {}^t v) \in [(\tau_1, \tau_2)]_{\mathcal{V}}^H$

To prove: $(T, {}^s v, {}^t v) \in [(\tau'_1, \tau'_2)]_{\mathcal{V}}^H$

From Definition 84 we know that ${}^s v = \ell$ s.t

$$H(\ell) = ({}^s v_1, {}^s v_2) \wedge (T, {}^s v_1, {}^t v_1) \in [\tau_1]_{\mathcal{V}} \wedge (T, {}^s v_2, {}^t v_2) \in [\tau_2]_{\mathcal{V}} \quad (\text{S-P0})$$

IH1 $(T, {}^s v_1, {}^t v_1) \in [\tau'_1]_{\mathcal{V}}^H$

IH2 $(T, {}^s v_2, {}^t v_2) \in [\tau'_2]_{\mathcal{V}}^H$

Again from Definition 84 it suffices to prove that

$$H(\ell) = ({}^s v_1, {}^s v_2) \wedge (T, {}^s v_1, {}^t v_1) \in [\tau'_1]_{\mathcal{V}} \wedge (T, {}^s v_2, {}^t v_2) \in [\tau'_2]_{\mathcal{V}}$$

We get this directly from (S-P0), IH1 and IH2

4. List:

$$\frac{\tau_1 <: \tau_2 \quad \vec{p} \geq \vec{q}}{L^{\vec{p}}\tau_1 <: L^{\vec{q}}\tau_2}$$

Given: $(T, {}^s v, {}^t v) \in [L^{\vec{p}}\tau_1]_{\mathcal{V}}^H$

To prove: $(T, {}^s v, {}^t v) \in [L^{\vec{q}}\tau_2]_{\mathcal{V}}^H$

From Definition 84 we know that ${}^s v = l_s$ and ${}^t v = \langle\langle(), l_t\rangle\rangle$ s.t $(T, l_s, l_t) \in [L \tau_1]_{\mathcal{V}}$

Similarly from Definition 84 it suffices to show that

$(T, l_s, l_t) \in [L \tau_2]_{\mathcal{V}}$

We induct on $(T, l_s, l_t) \in [L \tau_1]_{\mathcal{V}}$

- Base case:

In this case $l_s = \text{NULL}$ and $l_t = \text{nil}$:

It suffices to prove that $(T, \text{NULL}, \text{nil}) \in [L \tau_2]_{\mathcal{V}}$

This holds trivially from Definition 84

- Inductive case

In this case we have $l_s = \ell$ and $l_t = {}^t v_h :: l_{tt}$:

It suffices to prove that $(T, \ell, {}^t v_h :: l_{tt}) \in [L \tau_2]_{\mathcal{V}}$

Again from Definition 84 it suffices to show that

$\exists {}^s v_{h1}, \ell_{s1}. H(\ell) = ({}^s v_{h1}, \ell_{s1}) \wedge (T, {}^s v_{h1}, {}^t v_h) \in [\tau_2]_{\mathcal{V}} \wedge (T, \ell_{s1}, l_{tt}) \in [L \tau_2]_{\mathcal{V}}$

Since we are given that $(T, \ell, {}^t v_h :: l_{tt}) \in [L \tau_1]_{\mathcal{V}}$ therefore from Definition 84 we have

$\exists {}^s v_h, \ell_s. H(\ell) = ({}^s v_h, \ell_s) \wedge (T, {}^s v_h, {}^t v_h) \in [\tau_1]_{\mathcal{V}} \wedge (T, \ell_s, l_{tt}) \in [L \tau_1]_{\mathcal{V}} \quad (\text{S-L1})$

We choose ${}^s v_{h1}$ as ${}^s v_h$ and ℓ_{s1} as ℓ_s

- $H(\ell) = ({}^s v_h, \ell_s)$:
Directly from (S-L1)
- $(T, {}^s v_h, {}^t v_h) \in [\tau_1]_{\mathcal{V}}$:
From IH of outer induction
- $(T, \ell_s, l_{tt}) \in [L \tau_2]_{\mathcal{V}}$:
From IH of inner induction

□

Lemma 94 (Expression subtyping lemma). $\forall \tau, \tau', V, H, e_s, e_t.$

$$\tau <: \tau' \wedge (T, e_s, e_t) \in [\tau]_{\mathcal{E}}^{V, H} \implies (T, e_s, e_t) \in [\tau']_{\mathcal{E}}^{V, H}$$

Proof. From Definition 84 we are given that

$$\forall H', {}^s v, p, p', t < T. V, H \vdash_{p'}^p e_s \Downarrow_t {}^s v, H' \implies \exists {}^t v_t, {}^t v_f, J. e_t \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T -t, {}^s v, {}^t v_f) \in [\tau]_{\mathcal{V}}^H \wedge p - p' \leq J \quad (\text{SE0})$$

Also from Definition 84 it suffices to prove that

$$\forall H', {}^s v, p, p', t_1 < T. V, H \vdash_{p'}^p e_s \Downarrow_{t_1} {}^s v, H' \implies \exists {}^t v_t, {}^t v_f, J. e_t \Downarrow {}^t v_t \Downarrow^J {}^t v_f \wedge (T -t_1, {}^s v, {}^t v_f) \in [\tau']_{\mathcal{V}}^H \wedge p - p' \leq J$$

This means given some $H', {}^s v, p, p', t_1 < T$ s.t $V, H \vdash_{p'}^p e_s \Downarrow_{t_1} {}^s v, H'$ it suffices to prove that

$$\exists^t v_t, {}^t v_f, J.e_t \Downarrow^J {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t_1, {}^s v, {}^t v_f) \in [\tau']_{\mathcal{V}}^{H'} \wedge p - p' \leq J$$

We instantiate (SE0) with $H', {}^s v, p, p', t_1$ and we get

$$\exists^t v_t, {}^t v_f, J.e_t \Downarrow^J {}^t v_t \Downarrow^J {}^t v_f \wedge (T - t_1, {}^s v, {}^t v_f) \in [\tau]_{\mathcal{V}}^{H'} \wedge p - p' \leq J \quad (\text{SE1})$$

We get the desired from (SE1) and Lemma 93

□

2.5.3 Re-deriving Univariate RAML's soundness

Definition 95 (Translation of Univariate RAML stack). $\overline{(V : \Gamma)}_H \triangleq \forall x \in \text{dom}(\Gamma). \overline{(V(x))}_{H, \Gamma(x)}$

Definition 96 (Translation of Univariate RAML values).

$$\overline{({}^s v)}_{H, \tau} \triangleq \begin{cases} {}^s v & \tau = \text{unit} \\ !{}^s v & \tau = \mathbf{b} \\ \langle\langle () \rangle, \overline{({}^s v)}_{H, L \tau'} \rangle\rangle & \tau = L^- \tau' \\ \text{nil} & \tau = L \tau' \wedge {}^s v = \text{NULL} \\ \overline{(H(\ell) \downarrow_1)_{H, \tau'} :: (H(\ell) \downarrow_2)_{H, L \tau'}} & \tau = L \tau' \wedge {}^s v = \ell \\ \langle\langle (H(\ell) \downarrow_1)_{H, \tau_1}, (H(\ell) \downarrow_2)_{H, \tau_2} \rangle\rangle & \tau = (\tau_1, \tau_2) \wedge {}^s v = \ell \end{cases}$$

Lemma 97 (Irrelevance of T for translated value). $\forall {}^s v, \tau, H.$

$$H \models {}^s v \in \llbracket \tau \rrbracket \text{ in RAML} \implies \forall T. (\Phi_H({}^s v : \tau), T, \overline{({}^s v)}_{H, \tau}) \in \llbracket \llbracket \tau \rrbracket \rrbracket \text{ in } \lambda\text{-Amor}$$

Proof. By induction on τ

1. $\tau = \text{unit}$:

$$\text{To prove: } \forall T. (\Phi_H({}^s v : \tau), T, \overline{({}^s v)}_{H, \tau}) \in \llbracket \llbracket \text{unit} \rrbracket \rrbracket$$

This means given some T it suffices to prove that

$$(\Phi_H({}^s v : \text{unit}), T, \overline{({}^s v)}_{H, \text{unit}}) \in \llbracket \mathbf{1} \rrbracket$$

We know that $\Phi_H({}^s v : \text{unit}) = 0$ therefore it suffices to prove that

$$(0, T, {}^s v) \in \llbracket \mathbf{1} \rrbracket$$

Since we know that ${}^s v \in \llbracket \text{unit} \rrbracket$ therefore we know that ${}^s v = ()$

Therefore we get the desired directly from Definition 66

2. $\tau = \mathbf{b}$:

$$\text{To prove: } \forall T. (\Phi_H({}^s v : \tau), T, \overline{({}^s v)}_{H, \tau}) \in \llbracket \llbracket \mathbf{b} \rrbracket \rrbracket$$

This means given some T it suffices to prove that

$$(\Phi_H({}^s v : \mathbf{b}), T, \overline{({}^s v)}_{H, \tau}) \in \llbracket \llbracket \mathbf{b} \rrbracket \rrbracket$$

We know that $\Phi_H({}^s v : \mathbf{b}) = 0$ therefore it suffices to prove that

$$(0, T, !{}^s v) \in \llbracket \llbracket \mathbf{b} \rrbracket \rrbracket$$

From Definition 66 it suffices to prove that

$$(0, T, {}^s v) \in \llbracket \llbracket \mathbf{b} \rrbracket \rrbracket$$

Since we know that ${}^s v \in \llbracket \mathbf{b} \rrbracket$

Therefore we get the desired directly from Definition 66

3. $\tau = L^{\vec{q}}\tau'$:

By induction on ${}^s v$

- ${}^s v = NULL = []$:

To prove: $\forall T. (\Phi_H({}^s v : \tau), T, \overline{({}^s v)_{H, L^{\vec{q}}\tau'}}) \in \llbracket (L^{\vec{q}}\tau') \rrbracket$

This means given some T it suffices to prove that

$(\Phi_H([] : L^{\vec{q}}\tau'), T, \langle\langle(), nil\rangle\rangle) \in \llbracket \exists s. ([\phi(\vec{q}, s)] \mathbf{1} \otimes L^s(\tau')) \rrbracket$

We know that $\Phi_H([] : L^{\vec{q}}\tau') = 0$ therefore it suffices to prove that

$(0, T, \langle\langle(), nil\rangle\rangle) \in \llbracket \exists s. ([\phi(\vec{q}, s)] \mathbf{1} \otimes L^s(\tau')) \rrbracket$

From Definition 66 it suffices to prove that

$\exists s'. (0, T, \langle\langle(), nil\rangle\rangle) \in \llbracket ([\phi(\vec{q}, s)] \mathbf{1} \otimes L^s(\tau'))[s'/s] \rrbracket$

We choose s' as 0 and it suffices to prove that

$(0, T, \langle\langle(), nil\rangle\rangle) \in \llbracket ([\phi(\vec{q}, 0)] \mathbf{1} \otimes L[0](\tau')) \rrbracket$

From Definition 66 it further suffices to prove that

$\exists p_1, p_2. p_1 + p_2 \leq 0 \wedge (p_1, T, ()) \in \llbracket ([\phi(\vec{q}, 0)] \mathbf{1}) \wedge (p_1, T, nil) \in \llbracket L[0](\tau') \rrbracket \rrbracket$

We choose p_1 and p_2 as 0 and we get the desired directly from Definition 66

- ${}^s v = \ell = [{}^s v_1, \dots, {}^s v_n]$:

To prove: $\forall T. (\Phi_H([{}^s v_1 \dots {}^s v_n] : L^{\vec{q}}\tau'), T, \overline{({}^s v)_{H, \tau}}) \in \llbracket \exists s. ([\phi(\vec{q}, s)] \mathbf{1} \otimes L^s(\tau')) \rrbracket$

This means given some T it suffices to prove that

$(\Phi_H([{}^s v_1 \dots {}^s v_n] : L^{\vec{q}}\tau'), T, \overline{({}^s v)_{H, \tau}}) \in \llbracket \exists s. ([\phi(\vec{q}, s)] \mathbf{1} \otimes L^s(\tau')) \rrbracket$

We know that $\Phi_H([{}^s v_1 \dots {}^s v_n] : L^{\vec{q}}\tau') = (\Phi(n, \vec{q}) + \sum_{1 \leq i \leq n} \Phi_H({}^s v_i : \tau'))$ therefore it suffices to prove that

$((\Phi(n, \vec{q}) + \sum_{1 \leq i \leq n} \Phi_H({}^s v_i : \tau')), T, \overline{({}^s v)_{H, \tau}}) \in \llbracket \exists s. ([\phi(\vec{q}, s)] \mathbf{1} \otimes L^s(\tau')) \rrbracket$

From Definition 66 it suffices to prove that

$\exists s'. ((\Phi(n, \vec{q}) + \sum_{1 \leq i \leq n} \Phi_H({}^s v_i : \tau')), T, \overline{({}^s v)_{H, \tau}}) \in \llbracket ([\phi(\vec{q}, s)] \mathbf{1} \otimes L^s(\tau'))[s'/s] \rrbracket$

We choose s' as n and it suffices to prove that

$((\Phi(n, \vec{q}) + \sum_{1 \leq i \leq n} \Phi_H({}^s v_i : \tau')), T, \overline{({}^s v)_{H, \tau}}) \in \llbracket ([\phi(\vec{q}, n)] \mathbf{1} \otimes L^n(\tau')) \rrbracket$

From Definition 96 we know that $\overline{({}^s v)_{H, \tau}} = \langle\langle(), (\overline{H(\ell) \downarrow_1})_{H, \tau'} :: (\overline{H(\ell) \downarrow_2})_{H, L^{\vec{q}}\tau'} \rangle\rangle$

From Definition 66 it further suffices to prove that

$\exists p_1, p_2. p_1 + p_2 \leq (\Phi(n, \vec{q}) + \sum_{1 \leq i \leq n} \Phi_H({}^s v_i : \tau')) \wedge (p_1, T, ()) \in \llbracket ([\phi(\vec{q}, n)] \mathbf{1}) \wedge (p_2, T, (\overline{H(\ell) \downarrow_1})_{H, \tau'} :: (\overline{H(\ell) \downarrow_2})_{H, L^{\vec{q}}\tau'}) \in \llbracket L^n(\tau') \rrbracket \rrbracket \quad (\text{L0})$

III

$(\Phi_H([{}^s v_2 \dots {}^s v_n] : L^{\triangleleft \vec{q}}\tau'), T, \overline{(\overline{H(\ell) \downarrow_2})_{H, L^{\triangleleft \vec{q}}\tau'}}) \in \llbracket \exists s. ([\phi(\triangleleft \vec{q}, s)] \mathbf{1} \otimes L^s(\tau')) \rrbracket$

We know that $\Phi_H([{}^s v_2 \dots {}^s v_n] : L^{\triangleleft \vec{q}}\tau') = (\Phi(n-1, \triangleleft \vec{q}) + \sum_{2 \leq i \leq n} \Phi_H({}^s v_i : \tau'))$ this means we have

$((\Phi(n-1, \triangleleft \vec{q}) + \sum_{2 \leq i \leq n} \Phi_H({}^s v_i : \tau')), T, \overline{(\overline{H(\ell) \downarrow_2})_{H, L^{\triangleleft \vec{q}}\tau'}}) \in \llbracket \exists s. ([\phi(\triangleleft \vec{q}, s)] \mathbf{1} \otimes L^s(\tau')) \rrbracket$

From Definition 66 this means we have

$\exists s'. ((\Phi(n-1, \triangleleft \vec{q}) + \sum_{2 \leq i \leq n} \Phi_H({}^s v_i : \tau')), T, \overline{(\overline{H(\ell) \downarrow_2})_{H, L^{\triangleleft \vec{q}}\tau'}}) \in \llbracket ([\phi(\triangleleft \vec{q}, s)] \mathbf{1} \otimes L^s(\tau'))[s'/s] \rrbracket$

We know that s' as $n-1$ and we have

$$((\Phi(n-1, \triangleleft \vec{q}) + \sum_{2 \leq i \leq n} \Phi_H({}^s v_i : \tau')), T, \overline{(H(\ell) \downarrow_2)_{H, L \triangleleft \vec{q} \tau'}}) \in \llbracket [\phi(\triangleleft \vec{q}, n-1)] \mathbf{1} \otimes L^{n-1}(\tau') \rrbracket$$

From Definition 96 we know that $\overline{(H(\ell) \downarrow_2)_{H, L \triangleleft \vec{q} \tau'}} = \langle\langle (\cdot), l_t \rangle\rangle$

This means from Definition 66 we have

$$\begin{aligned} \exists p'_1, p'_2. p'_1 + p'_2 \leq (\Phi(n-1, \triangleleft \vec{q}) + \sum_{2 \leq i \leq n} \Phi_H({}^s v_i : \tau')) \wedge (p'_1, T, (\cdot)) \in \llbracket [\phi(\triangleleft \vec{q}, n)] \mathbf{1} \rrbracket \wedge \\ (p'_2, T, l_t) \in \llbracket L^{n-1}(\tau') \rrbracket \quad (\text{L1}) \end{aligned}$$

In order to prove (L0) we choose p_1 as $p'_1 + q_1$ and p_2 as $p'_2 + \Phi_H({}^s v_1 : \tau')$

$$- p_1 + p_2 \leq (\Phi(n, \vec{q}) + \sum_{1 \leq i \leq n} \Phi_H({}^s v_i : \tau')):$$

It suffices to prove that

$$p'_1 + q_1 + p'_2 + \Phi_H({}^s v_1 : \tau') \leq (\Phi(n, \vec{q}) + \sum_{1 \leq i \leq n} \Phi_H({}^s v_i : \tau'))$$

Since from (L1) we know that $p'_1 \leq \Phi(n-1, \triangleleft \vec{q})$ therefore we also know that

$$p'_1 + q_1 \leq \Phi(n, \vec{q}) \quad (\text{L2})$$

Similarly since from (L1) we know that $p'_2 \leq \sum_{2 \leq i \leq n} \Phi_H({}^s v_i : \tau')$

Therefore we also have

$$p'_2 + \Phi_H({}^s v_1 : \tau') \leq \sum_{1 \leq i \leq n} \Phi_H({}^s v_i : \tau') \quad (\text{L3})$$

Combining (L2) and (L3) we get the desired

$$- (p_1, T, (\cdot)) \in \llbracket [\phi(\vec{q}, n)] \mathbf{1} \rrbracket:$$

It suffices to prove that $(p'_1 + q_1, T, (\cdot)) \in \llbracket [\phi(\vec{q}, n)] \mathbf{1} \rrbracket$

Since from (L1) we are given that

$$(p'_1, T, (\cdot)) \in \llbracket [\phi(\triangleleft \vec{q}, n)] \mathbf{1} \rrbracket$$

Therefore we also have

$$(p'_1 + q_1, T, (\cdot)) \in \llbracket [\phi(\vec{q}, n)] \mathbf{1} \rrbracket$$

$$- (p_2, T, \overline{(H(\ell) \downarrow_1)_{H, \tau'}} :: \overline{(H(\ell) \downarrow_2)_{H, L \tau'}}) \in \llbracket L^n(\tau') \rrbracket:$$

It suffices to prove that

$$(p'_2 + \Phi_H({}^s v_1 : \tau'), T, \overline{(H(\ell) \downarrow_1)_{H, \tau'}} :: \overline{(H(\ell) \downarrow_2)_{H, L \tau'}}) \in \llbracket L^n(\tau') \rrbracket$$

From Definition 66 it suffices to show that

$$\begin{aligned} \exists p''_1, p''_2. p''_1 + p''_2 \leq \Phi_H({}^s v_1 : \tau') + p'_2 \wedge (p''_1, T, \overline{(H(\ell) \downarrow_1)_{H, \tau'}}) \in \llbracket \tau' \rrbracket \wedge (p''_2, T, \\ \overline{(H(\ell) \downarrow_2)_{H, L \tau'}}) \in \llbracket L^{n-1} \tau' \rrbracket \end{aligned}$$

We choose p''_1 as $\Phi_H({}^s v_1 : \tau')$ and p''_2 as p'_2 and it suffices to prove that

$$* (p''_1, T, \overline{(H(\ell) \downarrow_1)_{H, \tau'}}) \in \llbracket \tau' \rrbracket:$$

This means we need to prove that

$$(\Phi_H({}^s v_1 : \tau'), T, \overline{(H(\ell) \downarrow_1)_{H, \tau'}}) \in \llbracket \tau' \rrbracket$$

We get this from IH of outer induction

$$* (p''_2, T, \overline{(H(\ell) \downarrow_2)_{H, L \tau'}}) \in \llbracket L^{n-1} \tau' \rrbracket:$$

This means we need to prove that

$$(p'_2, T, \overline{(H(\ell) \downarrow_2)_{H, L \tau'}}) \in \llbracket L^{n-1} \tau' \rrbracket$$

Since we know that $\overline{(H(\ell) \downarrow_2)_{H, L \tau'}} = l_t$ therefore we get the desired from (L1)

4. $\tau = (\tau_1, \tau_2)$:

$$\text{To prove: } \forall T. (\Phi_H(({}^s v_1, {}^s v_2) : (\tau_1, \tau_2)), T, \overline{({}^s v_1, {}^s v_2)_{H, (\tau_1, \tau_2)}}) \in \llbracket ((\tau_1, \tau_2)) \rrbracket$$

This means given some T it suffices to prove that

$$(\Phi_H(({}^s v_1, {}^s v_2) : (\tau_1, \tau_2)), T, \overline{({}^s v_1, {}^s v_2)_{H, (\tau_1, \tau_2)}}) \in \llbracket (\tau_1) \otimes (\tau_2) \rrbracket$$

We know that $\Phi_H((^s v_1, ^s v_2) : (\tau_1, \tau_2)) = \Phi_H(^s v_1 : \tau_1) + \Phi_H(^s v_2 : \tau_2)$ therefore it suffices to prove that

$$(\Phi_H(^s v_1 : \tau_1) + \Phi_H(^s v_2 : \tau_2), T, \overline{((H(\ell) \downarrow_1)_{H, \tau_1}, (H(\ell) \downarrow_2)_{H, \tau_2})}) \in \llbracket (\tau_1) \otimes (\tau_2) \rrbracket$$

From Definition 66 it suffices to prove that

$$\exists p_1, p_2. p_1 + p_2 \leq (\Phi_H(^s v_1 : \tau_1) + \Phi_H(^s v_2 : \tau_2)) \wedge (p_1, T, \overline{(H(\ell) \downarrow_1)_{H, \tau_1}}) \in \llbracket (\tau_1) \rrbracket \wedge (p_2, T, \overline{(H(\ell) \downarrow_2)_{H, \tau_2}}) \in \llbracket (\tau_2) \rrbracket$$

Choosing p_1 as $\Phi_H(^s v_1 : \tau_1)$ and p_2 as $\Phi_H(^s v_2 : \tau_2)$ and it suffices to prove that

$$(\Phi_H(^s v_1 : \tau_1), T, \overline{(H(\ell) \downarrow_1)_{H, \tau_1}}) \in \llbracket (\tau_1) \rrbracket \wedge (\Phi_H(^s v_2 : \tau_2), T, \overline{(H(\ell) \downarrow_2)_{H, \tau_2}}) \in \llbracket (\tau_2) \rrbracket$$

We get this directly from IH1 and IH2

□

Lemma 98 (Irrelevance of T for translated Γ). $\forall^s v, \tau, H.$

$$H \models V : \Gamma \text{ in RAML} \implies \forall T. (\Phi_{V, H}(\Gamma), T, \overline{(V : \Gamma)_H}) \in \llbracket (\Gamma) \rrbracket \text{ in } \lambda\text{-Amor}$$

Proof. To prove: $\forall T. (\Phi_{V, H}(\Gamma), T, \overline{(V : \Gamma)_H}) \in \llbracket (\Gamma) \rrbracket$

This means given soem T it suffices to prove that

$$(\Phi_{V, H}(\Gamma), T, \overline{(V : \Gamma)_H}) \in \llbracket (\Gamma) \rrbracket$$

From Definition 67 it suffices to prove that

$$\exists f : \mathcal{Vars} \rightarrow \mathcal{Pots}. (\forall x \in \text{dom}(\llbracket \Gamma \rrbracket). (f(x), T, \overline{(V : \Gamma)_H}(x)) \in \llbracket (\Gamma)(x) \rrbracket_{\mathcal{E}} \wedge (\sum_{x \in \text{dom}(\llbracket \Gamma \rrbracket)} f(x) \leq \Phi_{V, H}(\Gamma)))$$

We choose $f(x)$ as $\Phi_H(V(x) : \Gamma(x))$ for every $x \in \text{dom}(\Gamma)$ and it suffices to prove that

$$\bullet (\forall x \in \text{dom}(\llbracket \Gamma \rrbracket). (\Phi_H(V(x) : \Gamma(x)), T, \overline{(V : \Gamma)_H}(x)) \in \llbracket (\Gamma)(x) \rrbracket_{\mathcal{E}}):$$

This means given some $x \in \text{dom}(\llbracket \Gamma \rrbracket)$ it suffices to prove that

$$(\Phi_H(V(x) : \Gamma(x)), T, \overline{(V : \Gamma)_H}(x)) \in \llbracket (\Gamma)(x) \rrbracket_{\mathcal{E}}$$

From Definition 95 it suffices to prove that

$$(\Phi_H(V(x) : \Gamma(x)), T, \overline{(V(x))_{H, \Gamma(x)}}) \in \llbracket (\Gamma)(x) \rrbracket_{\mathcal{E}}$$

From Lemma 97 we know that

$$(\Phi_H(V(x) : \Gamma(x)), T, \overline{(V(x))_{H, \Gamma(x)}}) \in \llbracket (\Gamma)(x) \rrbracket$$

And finally from Definition 66 we have

$$(\Phi_H(V(x) : \Gamma(x)), T, \overline{(V(x))_{H, \Gamma(x)}}) \in \llbracket (\Gamma)(x) \rrbracket_{\mathcal{E}}$$

$$\bullet (\sum_{x \in \text{dom}(\llbracket \Gamma \rrbracket)} f(x) \leq \Phi_{V, H}(\Gamma)):$$

Since we know that $\Phi_{V, H}(\Gamma) = \sum_{x \in \text{dom}(\Gamma)} \Phi_H(V(x) : \Gamma(x))$ therefore we are done

□

Lemma 99 (RAML's stack and its translation are in the cross-lang relation). $\forall H, V, \Gamma.$

$$H \models V : \Gamma \implies \forall T. (T, V, \overline{(V : \Gamma)_H}) \in \llbracket \Gamma \rrbracket_V^H$$

Proof. Given some T , it suffices to prove that $(T, V, \overline{(V : \Gamma)_H}) \in [\Gamma]_{\mathcal{V}}^H$

From Definition 85 it suffices to prove that
 $\forall x : \tau \in \text{dom}(\Gamma). (T, V(x), \overline{(V : \Gamma)_H(x)}) \in [\tau]_{\mathcal{V}}^H$

This means given some $x : \tau \in \text{dom}(\Gamma)$ and we need to prove that
 $(T, V(x), \overline{(V : \Gamma)_H(x)}) \in [\tau]_{\mathcal{V}}^H$

Since we are given that $H \models V : \Gamma$, it means we have $\forall x \in \text{dom}(\Gamma). H \models V(x) \in \llbracket \Gamma(x) \rrbracket$

Therefore we get the desired from Lemma 100

□

Lemma 100 (RAML's value and its translation are in the cross-lang relation). $\forall H, {}^s v, \tau.$
 $H \models {}^s v \in \llbracket \tau \rrbracket \implies \forall T. (T, {}^s v, \overline{({}^s v)_{H,\tau}}) \in [\tau]_{\mathcal{V}}^H$

Proof. By induction on τ

1. $\tau = \text{unit}$:

To prove: $\forall T. (T, {}^s v, \overline{({}^s v)_{H,\tau}}) \in [\text{unit}]_{\mathcal{V}}^H$

This means given some T , from Definition 96 it suffices to prove that

$(T, {}^s v, {}^s v) \in [\text{unit}]_{\mathcal{V}}^H$

We get this directly from Definition 84

2. $\tau = \mathbf{b}$:

To prove: $\forall T. (T, {}^s v, \overline{({}^s v)_{H,\tau}}) \in [\mathbf{b}]_{\mathcal{V}}^H$

This means given some T , from Definition 96 it suffices to prove that

$(T, {}^s v, !{}^s v) \in [\mathbf{b}]_{\mathcal{V}}^H$

We get this directly from Definition 84

3. $\tau = L^{\vec{q}}\tau'$:

By induction on ${}^s v$

- ${}^s v = \text{NULL}$:

To prove: $\forall T. (T, \text{NULL}, \overline{({}^s v)_{H,\tau}}) \in [\mathbf{b}]_{\mathcal{V}}^H$

Given some T , from Definition 96 it suffices to prove that

$(T, \text{NULL}, \langle\langle (), \text{nil} \rangle\rangle) \in [L^{\vec{q}}\tau']_{\mathcal{V}}^H$

We get this directly from Definition 84

- ${}^s v = \ell = [{}^s v_1 \dots {}^s v_n]$:

To prove: $\forall T. (T, \ell, \overline{({}^s v)_{H,\tau}}) \in [\mathbf{b}]_{\mathcal{V}}^H$

Given some T , from Definition 96 it suffices to prove that

$(T, \ell, \langle\langle (), \overline{(H(\ell) \downarrow_1)_{H,\tau'}} :: \overline{(H(\ell) \downarrow_2)_{H,L\tau'}} \rangle\rangle) \in [L^{\vec{q}}\tau']_{\mathcal{V}}^H$

From Definition 84 it further suffices to prove that

$(T, H(\ell) \downarrow_1, \overline{(H(\ell) \downarrow_1)_{H,\tau'}}) \in [\tau']_{\mathcal{V}} \wedge (T, H(\ell) \downarrow_2, \overline{(H(\ell) \downarrow_2)_{H,L\tau'}}) \in [L \tau']_{\mathcal{V}}$

We get $(T, H(\ell) \downarrow_1, \overline{(H(\ell) \downarrow_1)_{H,\tau'}}) \in [\tau']_{\mathcal{V}}$ from IH of outer induction

and $(T, H(\ell) \downarrow_2, \overline{(H(\ell) \downarrow_2)_{H,L\tau'}}) \in [L \tau']_{\mathcal{V}}$ from IH of inner induction

4. $\tau = (\tau_1, \tau_2)$:

To prove: $\forall T . (T, \ell, \overline{(\ell)_{H,(\tau_1, \tau_2)}}} \in [(\tau_1, \tau_2)]_{\mathcal{V}}^H$

Given some T , from Definition 96 it suffices to prove that

$(T, \ell, \langle\langle \overline{(H(\ell) \downarrow_1)_{H, \tau_1}}, \overline{(H(\ell) \downarrow_2)_{H, \tau_2}} \rangle\rangle) \in [(\tau_1, \tau_2)]_{\mathcal{V}}^H$

From Definition 84 it suffices to prove that

$(T, H(\ell) \downarrow_1, \overline{(H(\ell) \downarrow_1)_{H, \tau_1}}) \in [\tau_1]_{\mathcal{V}} \wedge (T, H(\ell) \downarrow_2, \overline{(H(\ell) \downarrow_2)_{H, \tau_2}}) \in [\tau_2]_{\mathcal{V}}$

We get this directly from IH

□

Lemma 101. $\forall {}^s v, {}^t v, \tau, H, T.$

$$(T, {}^s v, {}^t v) \in [\tau]_{\mathcal{V}}^H \implies {}^t v = \overline{({}^s v)_{H, \tau}}$$

Proof. Proof by induction on the $[\cdot]_{\mathcal{V}}$ relation

1. $[unit]_{\mathcal{V}}^H$:

Given: $(T, {}^s v, {}^s v) \in [unit]_{\mathcal{V}}^H$

To prove: ${}^s v = \overline{({}^s v)_{H, unit}}$

Directly from Definition 96

2. $[b]_{\mathcal{V}}^H$:

Given: $(T, {}^s v, !{}^s v) \in [b]_{\mathcal{V}}^H$

To prove: $!{}^s v = \overline{({}^s v)_{H, \tau}}$

Directly from Definition 96

3. $[(\tau_1, \tau_2)]_{\mathcal{V}}^H$:

Given: $(T, \ell, \langle\langle {}^t v_1, {}^t v_2 \rangle\rangle) \in [(\tau_1, \tau_2)]_{\mathcal{V}}^H$

This means from Definition 84 we have

$$H(\ell) = ({}^s v_1, {}^s v_2) \wedge (T, {}^s v_1, {}^t v_1) \in [\tau_1]_{\mathcal{V}} \wedge (T, {}^s v_2, {}^t v_2) \in [\tau_2]_{\mathcal{V}} \quad (\text{R0})$$

To prove: $\langle\langle {}^t v_1, {}^t v_2 \rangle\rangle = \overline{(\ell)_{H,(\tau_1, \tau_2)}}$

From Definition 96 we know that

$$\overline{(\ell)_{H,(\tau_1, \tau_2)}} = \langle\langle \overline{(H(\ell) \downarrow_1)_{H, \tau_1}}, \overline{(H(\ell) \downarrow_2)_{H, \tau_2}} \rangle\rangle$$

From (R0) we know that $H(\ell) \downarrow_1 = {}^s v_1$ and $H(\ell) \downarrow_2 = {}^s v_2$ therefore we have

$$\overline{(\ell)_{H,(\tau_1, \tau_2)}} = \langle\langle \overline{(H(\ell) \downarrow_1)_{H, \tau_1}}, \overline{(H(\ell) \downarrow_2)_{H, \tau_2}} \rangle\rangle = \langle\langle \overline{{}^s v_1}, \overline{{}^s v_2} \rangle\rangle \quad (\text{R1})$$

Since from (R0) we know that $(T, {}^s v_1, {}^t v_1) \in [\tau_1]_{\mathcal{V}}$ therefore we have

$${}^t v_1 = \overline{{}^s v_1} \quad (\text{IH1})$$

Similarly we also have

$${}^t v_2 = \overline{{}^s v_2} \quad (\text{IH2})$$

We get the desired from IH1, IH2 and (R1)

4. $[L^{\bar{q}}\tau']_{\mathcal{V}}^H$:

Given: $(T, \ell_s, \langle\langle(), l_t\rangle\rangle) \in [L^{\bar{q}}\tau']_{\mathcal{V}}^H$ where $(T, \ell_s, l_t) \in [L\tau']_{\mathcal{V}}^H$

To prove: $\langle\langle(), l_t\rangle\rangle = \overline{(\ell_s)_{H,\tau}}$

From Definition 96 we know that

$$\overline{(\ell_s)_{H,L^{-\tau'}}} = \langle\langle(), \overline{(\ell_s)_{H,L\tau'}}\rangle\rangle$$

Therefore it suffices to prove that $l_t = \overline{(\ell_s)_{H,L\tau'}}$

We induct on $(T, \ell_s, l_t) \in [L\tau']_{\mathcal{V}}^H$

(a) $\ell_s = NULL$:

In this case we know that $l_t = nil$

From Definition 96 we get the desired

(b) $\ell_s = \ell \neq NULL$:

In this case we know that $l_t = {}^t v_h :: l'_t$ s.t

$$H(\ell) = ({}^s v', \ell'_s) \wedge (T, {}^s v', {}^t v_h) \in [\tau']_{\mathcal{V}} \wedge (T, \ell'_s, l'_t) \in [L\tau']_{\mathcal{V}}$$

We get the desired from Definition 96, IH of outer induction and IH of inner induction

□

Definition 102 (Top level RAML program translation). Given a top-level RAML program

$P \triangleq F, e_{main}$ where $F \triangleq f_1(x) = e_{f_1}, \dots, f_n(x) = e_{f_n}$ s.t

$$\Sigma, x : \tau_{f_1} \vdash_{q'_1}^{q_1} e_{f_1} : \tau'_{f_1}$$

...

$$\Sigma, x : \tau_{f_n} \vdash_{q'_n}^{q_n} e_{f_n} : \tau'_{f_n}$$

$$\Sigma, \Gamma \vdash_{q'}^q e_{main} : \tau$$

where $\Sigma = f_1 : \tau_{f_1} \xrightarrow{q_1/q'_1} \tau'_{f_1}, \dots, f_n : \tau_{f_n} \xrightarrow{q_n/q'_n} \tau'_{f_n}$

Translation of P denoted by \overline{P} is defined as \overline{F}, e_t where

$\overline{F} = \text{fix}f_1.\lambda u.\lambda x.e_{t1}, \dots, \text{fix}f_n.\lambda u.\lambda x.e_{tn}$ s.t

$$\Sigma, x : \tau_{f_1} \vdash_{q'_1}^{q_1} e_{f_1} : \tau'_{f_1} \rightsquigarrow e_{t1}$$

...

$$\Sigma, x : \tau_{f_n} \vdash_{q'_n}^{q_n} e_{f_n} : \tau'_{f_n} \rightsquigarrow e_{tn}$$

and

$$\Sigma, \Gamma \vdash_{q'}^q e_{main} : \tau \rightsquigarrow e_t$$

Theorem 103 (RAML univariate soundness). $\forall H, H', V, \Gamma, \Sigma, e, \tau, {}^s v, p, p', q, q', t.$

$P = F, e$ and \overline{P} be a RAML top-level program and its translation respectively (as defined in Definition 102)

$$H \models V : \Gamma \wedge \Sigma, \Gamma \vdash_{q'}^q e : \tau \wedge V, H \vdash_{p'}^p e \Downarrow_t {}^s v, H'$$

\implies

$$p - p' \leq (\Phi_{H,V}(\Gamma) + q) - (q' + \Phi_H({}^s v : \tau))$$

Proof. From Definition 102 we are given that

$F \triangleq f_1(x) = e_{f_1}, \dots, f_n(x) = e_{f_n}$ s.t

$$\Sigma, x : \tau_{f_1} \vdash_{q'_1}^{q_1} e_{f_1} : \tau'_{f_1} \rightsquigarrow e_{t1}$$

...

$$\Sigma, x : \tau_{f_n} \vdash_{q'_n}^{q_n} e_{f_n} : \tau'_{f_n} \rightsquigarrow e_{tn}$$

Let $\forall i \in [1 \dots n]. \delta_{sf}(f_i) = (f_i(x) = e_{f_i})$ and $\forall i \in [1 \dots n]. \delta_{tf}(f_i) = (\text{fix } f_i. \lambda u. \lambda x. e_{t_i})$

Claim: $\forall T. (T, \delta_{sf}, \delta_{tf}) \in [\Sigma]_{\mathcal{V}'}^H$

Proof.

This means given some T , it suffices to prove that

$(T, \delta_{sf}, \delta_{tf}) \in [\Sigma]_{\mathcal{V}'}^H$

We induct on T

Base case: Trivial

Inductive case:

IH: $\forall T'' < T. (T'', \delta_{sf}, \delta_{tf}) \in [\Sigma]_{\mathcal{V}'}^H$

From Definition 86 it suffices to prove that

$\forall f_i \in \text{dom}(\Sigma). (T, f_i(x) = e_{f_i} \delta_{sf}, \text{fix } f_i. \lambda u. \lambda x. e_{t_i} \delta_{tf}) \in [\tau_{f_i}^{q_i/q'_i} \tau'_{f_i}]_{\mathcal{V}'}^H$

Given some $f_i \in \text{dom}(\Sigma)$ it suffices to prove that

$(T, f_i(x) = e_{f_i} \delta_{sf}, \text{fix } f_i. \lambda u. \lambda x. e_{t_i} \delta_{tf}) \in [\tau_{f_i}^{q_i/q'_i} \tau'_{f_i}]_{\mathcal{V}'}^H$

From Definition 84 it suffices to prove that

$\forall {}^s v', {}^t v', T' < T. (T', {}^s v', {}^t v') \in [\tau_{f_i}]_{\mathcal{V}}^H \implies (T', e_{f_i} \delta_{sf}, e_{t_i} \delta_{tf} [(\cdot)/u][{}^t v'/x][\text{fix } f_i. \lambda u. \lambda x. e_{t_i} \delta_{tf}/f_i]) \in [\tau'_{f_i}]_{\mathcal{E}}^{\{\{x \mapsto {}^s v'\}, H\}}$

This means given some ${}^s v', {}^t v', T' < T$ s.t $(T', {}^s v', {}^t v') \in [\tau_{f_i}]_{\mathcal{V}}^H$ it suffices to prove that

$(T', e_{f_i} \delta_{sf}, e_{t_i} \delta_{tf} [(\cdot)/u][{}^t v'/x][\text{fix } f_i. \lambda u. \lambda x. e_{t_i} \delta_{tf}/f_i]) \in [\tau'_{f_i}]_{\mathcal{E}}^{\{\{x \mapsto {}^s v'\}, H\}}$

Since $\delta_{tf} = \delta_{tf} \cup \{f_i \mapsto \text{fix } f_i. \lambda u. \lambda x. e_{t_i} \delta_{tf}\}$, therefore it suffices to prove that

$(T', e_{f_i} \delta_{sf}, e_{t_i} \delta_{tf} [(\cdot)/u][{}^t v'/x]) \in [\tau'_{f_i}]_{\mathcal{E}}^{\{\{x \mapsto {}^s v'\}, H\}}$ (C0)

Also since are given $(T', {}^s v', {}^t v') \in [\tau_{f_i}]_{\mathcal{V}}^H$ therefore we have

$(T', \{x \mapsto {}^s v'\}, \{x \mapsto {}^t v'\}) \in [x : \tau_{f_i}]_{\mathcal{V}}^H$

Also from IH we have $(T', \delta_{sf}, \delta_{tf}) \in [\Sigma]_{\mathcal{V}'}^{V, H}$

We can apply Theorem 92 to get

$(T', e_{f_i} \delta_{sf}, e_{t_i} (\cdot) \{x \mapsto {}^t v'\} \delta_{tf}) \in [\tau'_{f_i}]_{\mathcal{E}}^{\{\{x \mapsto {}^s v'\}, H\}}$

And this prove (C0)

□

From Theorem 82 we know that $\exists e_t$ s.t

$\Sigma, \Gamma \vdash_{q'}^q e : \tau \rightsquigarrow e_t$ and $.; ; (\Sigma); (\Gamma) \vdash e_t : [q] \mathbf{1} \multimap \mathbb{M} 0 [q'] (\tau)$

From Lemma 99 we know that $\forall T. (T, V, \overline{(V : \Gamma)_H}) \in [\Gamma]_{\mathcal{V}}^H$

Also from the Claim proved above we know that $\forall T. (T, \delta_{sf}, \delta_{tf}) \in [\Sigma]_{\mathcal{V}'}^H$

Therefore from Theorem 92 we know that $\forall T. (T, e \delta_{sf}, e_t (\cdot) \overline{(V : \Gamma)_H} \delta_{tf}) \in [\tau]_{\mathcal{E}}^{V, H}$

This means from Definition 84 we have

$\forall T. \exists H'_1, {}^s v_1, p_1, p'_1, t' < T. V, H \vdash_{p'_1}^{p_1} e \delta_{sf} \Downarrow_{t'} {}^s v_1, H'_1 \implies \exists {}^t v_t, {}^t v_f, J. e_t (\cdot) \overline{(V : \Gamma)_H} \delta_{tf} \Downarrow_{t'} {}^t v_t \Downarrow_{t'} {}^t v_f \wedge (T - t', {}^s v, {}^t v_f) \in [\tau]_{\mathcal{V}}^{H'_1} \wedge p_1 - p'_1 \leq J$ (RD-0.0)

We are given that $V, H \vdash_{p'}^p e \Downarrow_t {}^s v, H'$

Therefore instantiating (RD-0.0) with $t + 1, H', {}^s v, p, p', t$ we get

$\exists {}^t v_t, {}^t v_f, J. e_t (\cdot) \overline{(V : \Gamma)_H} \delta_{tf} \Downarrow_{-t} {}^t v_t \Downarrow_{-t} {}^t v_f \wedge (1, {}^s v, {}^t v_f) \in [\tau]_{\mathcal{V}}^{H'} \wedge p - p' \leq J$ (RD-0)

From reduction rules we know that $\exists t_1, t_2$ s.t $e_t (\cdot) \overline{(V : \Gamma)_H} \delta_{tf} \Downarrow_{t_1} {}^t v_t \Downarrow_{t_2} {}^t v_f$

Since from Lemma 98 we know that $\forall T. (\Phi_{V,H}(\Gamma), T, \overline{(V : \Gamma)_H}) \in \llbracket (\Gamma) \rrbracket$

Therefore we also have $(\Phi_{V,H}(\Gamma), t_1 + t_2 + 1, \overline{(V : \Gamma)_H}) \in \llbracket (\Gamma) \rrbracket$

Therefore from Theorem 80 we get

$$\exists p_v. (p_v, 1, {}^t v_f) \in \llbracket (\tau) \rrbracket \wedge J \leq (q + \Phi_{V,H}(\Gamma)) - (q' + p_v) \quad (\text{RD-1})$$

Since we have $(1, {}^s v, {}^t v_f) \in [\tau]_{\mathbb{V}}^{H'}$ therefore from Lemma 101 we know that ${}^t v_f = \overline{({}^s v)_{H',\tau}}$

From Lemma 97 we know that $\forall T. (\Phi_H({}^s v : \tau), T, \overline{({}^s v)_{H',\tau}}) \in \llbracket (\tau) \rrbracket$

Therefore we have $(\Phi_H({}^s v : \tau), 1, \overline{({}^s v)_{H',\tau}}) \in \llbracket (\tau) \rrbracket \quad (\text{RD-2})$

From (RD-1), (RD-2) and Lemma 69 we know that $p_v \geq \Phi_H({}^s v : \tau)$

Since from (RD-1) we know that $J \leq (q + \Phi_{V,H}(\Gamma)) - (q' + p_v)$ therefore we also have $J \leq (q + \Phi_{V,H}(\Gamma)) - (q' + \Phi_H({}^s v : \tau)) \quad (\text{RD-3})$

Finally from (RD-0) and (RD-3) we get the desired. □

3 Examples

3.1 Strict functional queue

enqueue : $\forall m, n. [3] \mathbf{1} \multimap \tau \multimap L^n([2] \tau) \multimap L^m \tau \multimap \mathbb{M} 0 (L^{n+1}([2] \tau) \otimes L^m \tau)$

enqueue $\triangleq \Lambda. \lambda p a l_1 l_2. \text{release } - = p \text{ in bind } x = \text{store } a \text{ in bind } - = \uparrow^1 \text{ in ret} \langle\langle x :: l_1, l_2 \rangle\rangle$

Typing derivation for *enqueue enqueue*

$$T_0 = \forall m, n. [3] \mathbf{1} \multimap \tau \multimap L^n([2] \tau) \multimap L^m \tau \multimap \mathbb{M} 0 (L^{n+1}([2] \tau) \otimes L^m \tau)$$

$$T_1 = [3] \mathbf{1} \multimap \tau \multimap L^n([2] \tau) \multimap L^m \tau \multimap \mathbb{M} 0 (L^{n+1}([2] \tau) \otimes L^m \tau)$$

$$T_{1,0} = [3] \mathbf{1}$$

$$T_2 = \tau \multimap L^n([2] \tau) \multimap L^m \tau \multimap \mathbb{M} 0 (L^{n+1}([2] \tau) \otimes L^m \tau)$$

$$T_3 = L^n([2] \tau) \multimap L^m \tau \multimap \mathbb{M} 0 (L^{n+1}([2] \tau) \otimes L^m \tau)$$

$$T_{3,1} = L^n([2] \tau)$$

$$T_{3,2} = L^m \tau$$

$$T_4 = \mathbb{M} 0 (L^{n+1}([2] \tau) \otimes L^m \tau)$$

$$T_5 = \mathbb{M} 1 (L^{n+1}([2] \tau) \otimes L^m \tau)$$

$$T_6 = \mathbb{M} 3 (L^{n+1}([2] \tau) \otimes L^m \tau)$$

$$\text{enqueue} = \Lambda. \lambda p a l_1 l_2. \text{release } - = p \text{ in bind } x = \text{store } a \text{ in bind } - = \uparrow^1 \text{ in ret} \langle\langle x :: l_1, l_2 \rangle\rangle$$

$$E_1 = \lambda p a l_1 l_2. \text{release } - = p \text{ in bind } x = \text{store } a \text{ in bind } - = \uparrow^1 \text{ in ret} \langle\langle x :: l_1, l_2 \rangle\rangle$$

$$E_2 = \text{release } - = p \text{ in bind } x = \text{store } a \text{ in bind } - = \uparrow^1 \text{ in ret} \langle\langle x :: l_1, l_2 \rangle\rangle$$

$$E_3 = \text{bind } x = \text{store } a \text{ in bind } - = \uparrow^1 \text{ in ret} \langle\langle x :: l_1, l_2 \rangle\rangle$$

$$E_4 = \text{bind } - = \uparrow^1 \text{ in ret} \langle\langle x :: l_1, l_2 \rangle\rangle$$

$$E_5 = \text{ret} \langle\langle x :: l_1, l_2 \rangle\rangle$$

D2:

$$\frac{}{.; m, n; .; .; x : [2] \tau, l_1 : L^n([2] \tau), l_2 : L^m \tau \vdash E_5 : T_4}$$

D1:

$$\frac{\frac{}{.; m, n; .; .; \vdash \uparrow^1 : \mathbb{M} 1 \mathbf{1}} \quad D2}{.; m, n; .; .; x : [2] \tau, l_1 : L^n([2] \tau), l_2 : L^m \tau \vdash E_4 : T_5}}$$

D0:

$$\frac{\frac{.; m, n; .; .; a : \tau \vdash \text{store } a : \mathbb{M}2([2]\tau)}{.; m, n; .; .; a : \tau, l_1 : L^n([2]\tau), l_2 : L^m\tau \vdash E_3 : T_6}}{D1}$$

Main derivation:

$$\frac{\frac{\frac{.; m, n; .; .; p : T_{1.0} \vdash p : T_{1.0}}{.; m, n; .; .; p : T_{1.0}, a : \tau, l_1 : L^n([2]\tau), l_2 : L^m\tau \vdash E_2 : T_4}}{.; m, n; .; .; . \vdash E_1 : T_1}}{.; .; .; . \vdash \text{enqueue} : T_0} \quad D0$$

$$Dq : \forall m, n. (m + n > 0) \Rightarrow [1] \mathbf{1} \multimap L^m([2]\tau) \multimap L^n\tau \multimap \mathbb{M}0(\exists m', n'. ((m' + n' + 1) = (m + n)) \& (L^{m'}[2]\tau \otimes L^{n'}\tau))$$

$$\begin{aligned} Dq &\triangleq \Lambda.\Lambda.\Lambda.\lambda p l_1 l_2. \text{match } l_2 \text{ with } | \text{nil} \mapsto E_1 \mid h_2 :: l'_2 \mapsto E_2 \\ E_1 &= \text{bind } l_r = M \square\square l_1 \text{ nil in match } l_r \text{ with } | \text{nil} \mapsto - \mid h_r :: l'_r \mapsto E_{1.1} \\ E_{1.1} &= \text{release } - = p \text{ in bind } - = \uparrow^1 \text{ in ret } \Lambda.\langle\langle \text{nil}, l'_r \rangle\rangle \\ E_2 &= \text{release } - = p \text{ in bind } - = \uparrow^1 \text{ in ret } \Lambda.\langle\langle l_1, l'_2 \rangle\rangle \end{aligned}$$

Typing derivation for dequeue Dq

$$\begin{aligned} T_0 &= \forall m, n. (m + n > 0) \Rightarrow [1] \mathbf{1} \multimap L^m([2]\tau) \multimap L^n\tau \multimap \mathbb{M}0(\exists m', n'. (m' + n' + 1) = (m + n) \& (L^{m'}[2]\tau \otimes L^{n'}\tau)) \\ T_1 &= (m + n > 0) \Rightarrow [1] \mathbf{1} \multimap L^m([2]\tau) \multimap L^n\tau \multimap \mathbb{M}0(\exists m', n'. (m' + n' + 1) = (m + n) \& (L^{m'}[2]\tau \otimes L^{n'}\tau)) \\ T_2 &= [1] \mathbf{1} \multimap L^m([2]\tau) \multimap L^n\tau \multimap \mathbb{M}0(\exists m', n'. (m' + n' + 1) = (m + n) \& (L^{m'}[2]\tau \otimes L^{n'}\tau)) \\ T_{2.1} &= L^m([2]\tau) \\ T_3 &= L^n\tau \multimap \mathbb{M}0(\exists m', n'. (m' + n' + 1) = (m + n) \& (L^{m'}[2]\tau \otimes L^{n'}\tau)) \\ T_{3.1} &= L^n\tau \\ T_4 &= \mathbb{M}0(\exists m', n'. (m' + n' + 1) = (m + n) \& (L^{m'}[2]\tau \otimes L^{n'}\tau)) \\ T_{4.1} &= \mathbb{M}1(\exists m', n'. (m' + n' + 1) = (m + n) \& (L^{m'}[2]\tau \otimes L^{n'}\tau)) \\ T_5 &= (\exists m', n'. (m' + n' + 1) = (m + n) \& (L^{m'}[2]\tau \otimes L^{n'}\tau)) \\ T_{5.1} &= (\exists m', n'. (m' + n' + 1) = (m + n) \& (L^{m'}[2]\tau \otimes L^{n'}\tau))[m/m'][i/n'] \\ T_{5.2} &= (L^m[2]\tau \otimes L^n\tau) \\ T_6 &= (m' + n' + 1) = (m + n) \& (L^{m'}[2]\tau \otimes L^{n'}\tau)[0/m'][i/n'] \\ T_7 &= (L^0[2]\tau \otimes L^{n-1}\tau) \end{aligned}$$

$$\begin{aligned} E_0 &= \Lambda.\Lambda.\Lambda.\lambda l_1 l_2. \text{match } l_2 \text{ with } | \text{nil} \mapsto E_1 \mid h_2 :: l'_2 \mapsto E_2 \\ E_{0.1} &= \lambda p l_1 l_2. \text{match } l_2 \text{ with } | \text{nil} \mapsto E_1 \mid h_2 :: l'_2 \mapsto E_2 \\ E_{0.2} &= \text{match } l_2 \text{ with } | \text{nil} \mapsto E_1 \mid h_2 :: l'_2 \mapsto E_2 \\ E_1 &= \text{bind } l_r = M \square\square l_1 \text{ nil in match } l_r \text{ with } | \text{nil} \mapsto - \mid h_r :: l'_r \mapsto E_{1.1} \\ E_{1.1} &= \text{release } - = p \text{ in bind } x = \uparrow^1 \text{ in } \Lambda. \text{ret}\langle\langle \text{nil}, l'_r \rangle\rangle \\ E_2 &= \text{release } - = p \text{ in bind } x = \uparrow^1 \text{ in } \Lambda. \text{ret}\langle\langle l_1, l'_2 \rangle\rangle \end{aligned}$$

D1.3:

$$\frac{.; m, n; (n > 0), (m + n) > 0; .; h_2 : \tau, l'_2 : L^{m-1}\tau, l_1 : T_{2.1} \vdash \langle\langle l_1, l'_2 \rangle\rangle : T_{5.2}}$$

D1.2:

$$\frac{\overline{.; m, n; (n > 0), (m + n) > 0 \models (m + i + 1) = (m + n)}}{D1.3} \\ \frac{\overline{.; m, n; (n > 0), (m + n) > 0; .; h_2 : \tau, l'_2 : L^{m-1}\tau, l_1 : T_{2.1} \vdash \Lambda.\langle\langle l_1, l'_2 \rangle\rangle : T_{5.1}}}{.; m, n; (n > 0), (m + n) > 0; .; h_2 : \tau, l'_2 : L^{m-1}\tau, l_1 : T_{2.1} \vdash \Lambda.\langle\langle l_1, l'_2 \rangle\rangle : T_5} \\ \frac{\overline{.; m, n; (n > 0), (m + n) > 0; .; h_2 : \tau, l'_2 : L^{m-1}\tau, l_1 : T_{2.1} \vdash \text{ret } \Lambda.\langle\langle l_1, l'_2 \rangle\rangle : T_4}}{.; m, n; (n > 0), (m + n) > 0; .; h_2 : \tau, l'_2 : L^{m-1}\tau, l_1 : T_{2.1} \vdash \text{ret } \Lambda.\langle\langle l_1, l'_2 \rangle\rangle : T_4}$$

D1.1:

$$\frac{\overline{.; m, n; (n > 0), (m + n) > 0; .; h_2 : \tau, l'_2 : L^{n-1}\tau \vdash \uparrow^1 : \mathbb{M} \mathbf{1} \mathbf{1}}}{D1.2} \\ \overline{.; m, n; (n > 0), (m + n) > 0; .; h_2 : \tau, l'_2 : L^{n-1}\tau, l_1 : T_{2.1} \vdash \text{bind } - = \uparrow^1 \text{ in ret } \Lambda.\langle\langle l_1, l'_2 \rangle\rangle : T_{4.1}}$$

D1:

$$\frac{\overline{.; m, n; (n > 0), (m + n) > 0; .; p : [\mathbf{1}] \mathbf{1} \vdash p : [\mathbf{1}] \mathbf{1}}}{D1.1} \\ \frac{\overline{.; m, n; (n > 0), (m + n) > 0; .; h_2 : \tau, l'_2 : L^{n-1}\tau, l_1 : T_{2.1} \vdash \text{release } - = p \text{ in bind } - = \uparrow^1 \text{ in ret } \Lambda.\langle\langle l_1, l'_2 \rangle\rangle : T_4}}{.; m, n; (n > 0), (m + n) > 0; .; h_2 : \tau, l'_2 : L^{n-1}\tau, l_1 : T_{2.1} \vdash E_2 : T_4}$$

D0.05:

$$\overline{.; m, n; (n = 0), (m > 0), (m + n) > 0; .; h_r : \tau, l'_r : L^{m-1}\tau \vdash \langle\langle \text{nil}, l'_r \rangle\rangle : T_7}$$

D0.04:

$$\frac{\overline{.; m, n; (n = 0), (m > 0), (m + n) > 0; .; h_r : \tau, l'_r : L^{m-1}\tau \vdash \Lambda.\langle\langle \text{nil}, l'_r \rangle\rangle : T_6}}{D0.05} \\ \frac{\overline{.; m, n; (n = 0), (m > 0), (m + n) > 0; .; h_r : \tau, l'_r : L^{m-1}\tau \vdash \Lambda.\langle\langle \text{nil}, l'_r \rangle\rangle : T_5}}{.; m, n; (n = 0), (m > 0), (m + n) > 0; .; h_r : \tau, l'_r : L^{m-1}\tau \vdash \text{ret } \Lambda.\langle\langle \text{nil}, l'_r \rangle\rangle : T_4}$$

D0.03:

$$\frac{\overline{.; m, n; (n = 0), (m > 0), (m + n) > 0; .; h_r : \tau, l'_r : L^{m-1}\tau \vdash \uparrow^1 : \mathbb{M} \mathbf{1} \mathbf{1}}}{D0.04} \\ \overline{.; m, n; (n = 0), (m > 0), (m + n) > 0; .; h_r : \tau, l'_r : L^{m-1}\tau \vdash \text{bind } - = \uparrow^1 \text{ in ret } \Lambda.\langle\langle \text{nil}, l'_r \rangle\rangle : T_{4.1}}$$

D0.02:

$$\frac{\overline{.; m, n; (n = 0), (m > 0), (m + n) > 0; .; p : [\mathbf{1}] \mathbf{1} \vdash p : [\mathbf{1}] \mathbf{1}}}{D0.03} \\ \overline{.; m, n; (n = 0), (m > 0), (m + n) > 0; .; h_r : \tau, l'_r : L^{m-1}\tau, p : [\mathbf{1}] \mathbf{1} \vdash E_{1.1} : T_4}$$

D0.01:

$$\overline{.; m, n; (n = 0), (m + n) > 0; .; . \vdash \text{fix } x.x : T_4}$$

D0.0:

$$\frac{\overline{.; m, n; (n = 0), (m + n) > 0; .; l_r : L^m\tau \vdash l_r : L^m\tau}}{D0.01 \quad D0.02} \\ \overline{.; m, n; (n = 0), (m + n) > 0; .; l_r : L^m\tau, p : [\mathbf{1}] \mathbf{1} \vdash \text{match } l_r \text{ with } |\text{nil} \mapsto - | h_r :: l'_r \mapsto E_{1.1} : T_4}$$

D0:

$$\frac{.; m, n; (n = 0), (m + n) > 0; .; l_1 : T_{2.1} \vdash M \square \square l_1 \text{ nil} : \mathbb{M}0(L^m \tau)}{.; m, n; (n = 0), (m + n) > 0; .; l_1 : T_{2.1}, p : [1] \mathbf{1} \vdash E_1 : T_4} \quad D0.0$$

Main derivation:

$$\frac{\frac{.; m, n; (m + n) > 0; .; l_2 : T_{3.1} \vdash l_2 : T_{3.1}}{.; m, n; (m + n) > 0; .; l_1 : T_{2.1}, l_2 : T_{3.1}, p : [1] \mathbf{1} \vdash E_{0.2} : T_0} \quad D0 \quad D1}{.; m, n; (m + n) > 0; .; \cdot \vdash E_{0.1} : T_0}}{.; ; ; .; \cdot \vdash E_0 : T_0}$$

$Move : \forall m, n. L^m([2] \tau) \multimap L^n \tau \multimap \mathbb{M}0(L^{m+n} \tau)$

$Move \triangleq \text{fix } M \Lambda. \Lambda. \lambda l_1 l_2. \text{match } l_1 \text{ with } | \text{nil} \mapsto E_1 | h_1 :: l'_1 \mapsto E_2$

$E_1 = \text{ret}(l_2)$

$E_2 = \text{release } h'_1 = h_1 \text{ in bind } - = \uparrow^2 \text{ in } M \square \square l'_1 (h_1 :: l_2)$

Typing derivation for $Move$

$T_0 = \forall m, n. L^m([2] \tau) \multimap L^n \tau \multimap \mathbb{M}0(L^{m+n} \tau)$

$T_1 = L^m([2] \tau) \multimap L^n \tau \multimap \mathbb{M}0(L^{m+n} \tau)$

$T_{1.1} = L^m([2] \tau)$

$T_2 = L^n \tau \multimap \mathbb{M}0(L^{m+n} \tau)$

$T_{2.1} = L^n \tau$

$T_3 = \mathbb{M}0(L^{m+n} \tau)$

$T_5 = \mathbb{M}2(L^{m+n} \tau)$

$E_0 = \text{fix } M \Lambda. \Lambda. \lambda l_1 l_2. \text{match } l_1 \text{ with } | \text{nil} \mapsto E_1 | h_1 :: l'_1 \mapsto E_2$

$E_{0.0} = \Lambda. \Lambda. \lambda l_1 l_2. \text{match } l_1 \text{ with } | \text{nil} \mapsto E_1 | h_1 :: l'_1 \mapsto E_2$

$E_{0.1} = \lambda l_1 l_2. \text{match } l_1 \text{ with } | \text{nil} \mapsto E_1 | h_1 :: l'_1 \mapsto E_2$

$E_{0.2} = \text{match } l_1 \text{ with } | \text{nil} \mapsto E_1 | h_1 :: l'_1 \mapsto E_2$

$E_1 = \text{ret}(l_2)$

$E_2 = \text{release } - = h \text{ in bind } - = \uparrow^2 \text{ in } M \square \square l'_1 (h_1 :: l_2)$

$E_{2.1} = \text{bind } - = \uparrow^2 \text{ in } M \square \square l'_1 (h_1 :: l_2)$

$E_{2.2} = M \square \square l'_1 (h_1 :: l_2)$

D3:

$$\frac{.; m, n; (m > 0); M : T_0; l'_1 : L^{m-1}[2] \tau, l_2 : T_{2.1}, h'_1 : \tau \vdash M \square \square l'_1 (h'_1 :: l_2) : T_3}$$

D2:

$$\frac{.; m, n; (m > 0); M : T_0; \cdot \vdash \uparrow^2 : \mathbb{M}2 \mathbf{1}}{.; m, n; (m > 0); M : T_0; l'_1 : L^{m-1}[2] \tau, l_2 : T_{2.1}, h'_1 : \tau \vdash E_{2.1} : T_5} \quad D3$$

D1:

$$\frac{.; m, n; (m > 0); M : T_0; h_1 : [2] \tau \vdash h_1 : [2] \tau}{.; m, n; (m > 0); M : T_0; h_1 : [2] \tau, l'_1 : L^{m-1}[2] \tau, l_2 : T_{2.1} \vdash E_2 : T_3} \quad D2$$

D0:

$$\frac{}{.; m, n; m = 0; M : T_0; l_2 : T_{2.1} \vdash E_1 : T_3}$$

Main derivation:

$$\frac{\frac{\frac{\frac{\frac{\frac{}{.; m, n; .; M : T_0; l_1 : T_{1.1} \vdash l_1 : T_{1.1}}{.; m, n; .; M : T_0; l_1 : T_{1.1}, l_2 : T_{2.1} \vdash E_{0.2} : T_1}}{.; m, n; .; M : T_0; . \vdash E_{0.2} : T_1}}{.; .; .; M : T_0; . \vdash E_{0.1} : T_{0.0}}{.; .; .; . \vdash E_0 : T_0}}{.; .; .; . \vdash Move : T_0}}{D0} \quad D1$$

3.2 Church numerals

$\text{Nat} = \lambda_t n. \forall \alpha : \mathbb{N} \rightarrow \text{Type}. \forall C : \mathbb{N} \rightarrow \mathbb{N}.$

$!(\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M}0(\alpha(j_n + 1)))) \multimap \mathbb{M}0((\alpha 0 \otimes [(\sum_{i < n} C i) + n] \mathbf{1}) \multimap \mathbb{M}0(\alpha n))$

$$e_1 \uparrow^1 e_2 \triangleq \text{bind} - = \uparrow^1 \text{ in } e_1 e_2$$

$$\frac{\frac{\Psi; \Theta; \Delta; \Omega_1; \Gamma_1 \vdash e_1 : \tau_1 \multimap \mathbb{M}(n) \tau_2}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash e_1 \uparrow^1 e_2 : \mathbb{M}(n+1) \tau_2} \quad \frac{\Psi; \Theta; \Delta; \Omega_2; \Gamma_2 \vdash e_2 : \tau_1}{\Psi; \Theta; \Delta; \Omega_1 \oplus \Omega_2; \Gamma_1 \oplus \Gamma_2 \vdash e_1 \uparrow^1 e_2 : \mathbb{M}(n+1) \tau_2}}$$

Type derivation for $\bar{0}$

$$\bar{0} = \Lambda. \Lambda. \lambda f. \text{ret } \lambda x. \text{let} \langle\langle y_1, y_2 \rangle\rangle = x \text{ in ret } y_1 : \text{Nat } 0$$

$$\begin{aligned} T_0 &= \\ \forall \alpha. \forall C. &!(\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M}0(\alpha(j_n + 1)))) \multimap \mathbb{M}0((\alpha 0 \otimes [0] \mathbf{1}) \multimap \mathbb{M}0(\alpha 0)) \\ T_{0.1} &= \forall C. !(\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M}0(\alpha(j_n + 1)))) \multimap \mathbb{M}0((\alpha 0 \otimes [0] \mathbf{1}) \multimap \mathbb{M}0(\alpha 0)) \\ T_{0.2} &= !(\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M}0(\alpha(j_n + 1)))) \multimap \mathbb{M}0((\alpha 0 \otimes [0] \mathbf{1}) \multimap \mathbb{M}0(\alpha 0)) \\ T_{0.3} &= !(\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M}0(\alpha(j_n + 1)))) \\ T_1 &= \mathbb{M}0((\alpha 0 \otimes [0] \mathbf{1}) \multimap \mathbb{M}0(\alpha 0)) \\ T_{1.1} &= ((\alpha 0 \otimes [0] \mathbf{1}) \multimap \mathbb{M}0(\alpha 0)) \\ T_2 &= (\alpha 0 \otimes [0] \mathbf{1}) \\ T_{2.1} &= \alpha 0 \\ T_{2.2} &= [0] \mathbf{1} \\ T_3 &= \mathbb{M}0(\alpha 0) \\ TI &= \alpha : \mathbb{N} \rightarrow \text{Type}; C : \mathbb{N} \rightarrow \text{Sort} \end{aligned}$$

D1:

$$\frac{}{TI; .; .; f : T_{0.3}, y_1 : T_{2.1}, y_2 : T_{2.2} \vdash \text{ret } y_1 : \mathbb{M}0 T_{2.1}}$$

D0:

$$\frac{}{TI; .; .; f : T_{0.3}, x : T_2 \vdash x : T_2}$$

Main derivation:

$$\begin{array}{c}
D0 \quad D1 \\
\hline
TI; .; .; f : T_{0.3}, x : T_2 \vdash \text{let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in ret } y_1 : T_3 \\
\hline
TI; .; .; f : T_{0.3} \vdash \lambda x. \text{let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in ret } y_1 : T_{1.1} \\
\hline
TI; .; .; f : T_{0.3} \vdash \text{ret } \lambda x. \text{let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in ret } y_1 : T_1 \\
\hline
TI; .; .; . \vdash \lambda f. \text{ret } \lambda x. \text{let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in ret } y_1 : T_{0.2} \\
\hline
\alpha : \mathbb{N} \rightarrow \text{Type}; .; .; . \vdash \Lambda. \lambda f. \text{ret } \lambda x. \text{let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in ret } y_1 : T_{0.1} \\
\hline
.; .; .; . \vdash \Lambda. \Lambda. (\lambda f. \text{ret } \lambda x. \text{let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in ret } y_1) : T_0
\end{array}$$

Type derivation for $\bar{1}$

$$\begin{array}{l}
\bar{1} = \Lambda. \Lambda. \lambda f. \text{ret } \lambda x. \text{let } ! f_u = f \text{ in let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in release } - = y_2 \text{ in } E_1 : \text{Nat } 1 \\
\text{where} \\
E_1 = \text{bind } a = \text{store}() \text{ in } f_u \square \uparrow^1 \langle\langle y_1, a \rangle\rangle \\
T_0 = \forall \alpha : \mathbb{N} \rightarrow \text{Type}. \forall C : \mathbb{N} \rightarrow \text{Sort}. \\
!(\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M}0(\alpha (j_n + 1)))) \multimap ((\alpha 0 \otimes [C 0 + 1] \mathbf{1}) \multimap \mathbb{M}0(\alpha 1)) \\
T_{0.1} = \forall C. !(\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M}0(\alpha (j_n + 1)))) \multimap ((\alpha 0 \otimes [C 0 + 1] \mathbf{1}) \multimap \\
\mathbb{M}0(\alpha 1)) \\
T_{0.2} = !(\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M}0(\alpha (j_n + 1)))) \multimap ((\alpha 0 \otimes [C 0 + 1] \mathbf{1}) \multimap \mathbb{M}0(\alpha 1)) \\
T_{0.3} = !(\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M}0(\alpha (j_n + 1)))) \\
T_{0.4} = (\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M}0(\alpha (j_n + 1)))) \\
T_{0.5} = (\alpha 0 \otimes [C 0] \mathbf{1}) \multimap \mathbb{M}0(\alpha (0 + 1)) \\
T_1 = \mathbb{M}0((\alpha 0 \otimes [C 0 + 1] \mathbf{1}) \multimap \mathbb{M}0(\alpha 1)) \\
T_{1.1} = ((\alpha 0 \otimes [C 0 + 1] \mathbf{1}) \multimap \mathbb{M}0(\alpha 1)) \\
T_2 = (\alpha 0 \otimes [C 0 + 1] \mathbf{1}) \\
T_{2.1} = \alpha 0 \\
T_{2.2} = [C 0 + 1] \mathbf{1} \\
T_3 = \mathbb{M}0(\alpha 1) \\
TI = \alpha : \mathbb{N} \rightarrow \text{Type}; C : \mathbb{N} \rightarrow \text{Sort} \\
D7:
\end{array}$$

$$\overline{TI; .; f_u : T_{0.4}; y_1 : T_{2.1}, a : [C 0] \mathbf{1} \vdash \langle\langle y_1, a \rangle\rangle : (T_{2.1} \otimes [C 0] \mathbf{1})}$$

D6:

$$\overline{TI; .; f_u : T_{0.4}; . \vdash f_u \square : T_{0.5}}$$

D5:

$$\begin{array}{c}
D6 \quad D7 \\
\hline
TI; .; f_u : T_{0.4}; y_1 : T_{2.2}, a : [C 0] \mathbf{1} \vdash f_u \square \uparrow^1 \langle\langle y_1, a \rangle\rangle : \mathbb{M}1 \alpha 1
\end{array}$$

D4:

$$\begin{array}{c}
D4 \\
\hline
\overline{TI; .; f_u : T_{0.4}; y_1 : T_{2.1}, y_2 : T_{2.2} \vdash \text{store}() : \mathbb{M}(C 0) [C 0] \mathbf{1}} \quad D5 \\
\hline
TI; .; f_u : T_{0.4}; y_1 : T_{2.1}, y_2 : T_{2.2} \vdash \text{bind } a = \text{store}() \text{ in } f_u \square \uparrow^1 \langle\langle y_1, a \rangle\rangle : \mathbb{M}(C 0 + 1) \alpha 1
\end{array}$$

D3:

$$\begin{array}{c}
D4 \\
\hline
\overline{TI; .; f_u : T_{0.4}; y_1 : T_{2.1} \vdash E_1 : \mathbb{M}(C 0 + 1) \alpha 1}
\end{array}$$

D2:

$$\frac{\frac{}{TI; \cdot; f_u : T_{0.4}; y_2 : T_{2.2} \vdash y_2 : T_{2.2}}{D3}}{TI; \cdot; f_u : T_{0.4}; y_1 : T_{2.1}, y_2 : T_{2.2} \vdash \text{release } - = y_2 \text{ in } E_1 : T_3}$$

D1:

$$\frac{\frac{}{TI; \cdot; f_u : T_{0.4}; x : T_2 \vdash x : T_2}{D2}}{TI; \cdot; f_u : T_{0.4}; x : T_2 \vdash \text{let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in release } - = y_2 \text{ in } E_1 : T_3}$$

D0:

$$\frac{}{TI; \cdot; \cdot; f : T_{0.3} \vdash f : T_{0.3}}$$

Main derivation:

$$\frac{\frac{\frac{\frac{\frac{}{D0} \quad \frac{}{D1}}{TI; \cdot; \cdot; f : T_{0.3}, x : T_2 \vdash \text{let! } f_u = f \text{ in let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in release } - = y_2 \text{ in } E_1 : T_3}}{TI; \cdot; \cdot; f : T_{0.3} \vdash \lambda x. \text{let! } f_u = f \text{ in let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in release } - = y_2 \text{ in } E_1 : T_{1.1}}}{TI; \cdot; \cdot; f : T_{0.3} \vdash \text{ret } \lambda x. \text{let! } f_u = f \text{ in let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in release } - = y_2 \text{ in } E_1 : T_1}}{TI; \cdot; \cdot; \cdot \vdash \lambda f. \text{ret } \lambda x. \text{let! } f_u = f \text{ in let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in release } - = y_2 \text{ in } E_1 : T_{0.2}}}{\cdot; \alpha : \mathbb{N} \rightarrow \text{Type}; \cdot; \cdot \vdash \Lambda. \lambda f. \text{ret } \lambda x. \text{let! } f_u = f \text{ in let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in release } - = y_2 \text{ in } E_1 : T_{0.1}}}{\cdot; \cdot; \cdot; \cdot \vdash \Lambda. \Lambda. \lambda f. \text{ret } \lambda x. \text{let! } f_u = f \text{ in let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in release } - = y_2 \text{ in } E_1 : T_0}$$

Type derivation for $\bar{2}$

$\bar{2} = \Lambda. \Lambda. \lambda f. \text{ret } \lambda x. \text{let! } f_u = f \text{ in let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in release } - = y_2 \text{ in bind } b = E_1 \text{ in } E_2 : \text{Nat } 2$
where

$E_1 = \text{bind } a = \text{store}() \text{ in } f_u \square \uparrow^1 \langle\langle y_1, a \rangle\rangle$
 $E_2 = \text{bind } c = \text{store}() \text{ in } f_u \square \uparrow^1 \langle\langle b, c \rangle\rangle$

$T_0 =$

$\forall \alpha : \mathbb{N} \rightarrow \text{Type}. \forall C. !(\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M} 0(\alpha (j_n + 1)))) \multimap ((\alpha 0 \otimes [C 0 + C 1 + 2] \mathbf{1}) \multimap \mathbb{M} 0(\alpha 2))$

$T_{0.1} = \forall C. !(\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M} 0(\alpha (j_n + 1)))) \multimap ((\alpha 0 \otimes [C 0 + C 1 + 2] \mathbf{1}) \multimap \mathbb{M} 0(\alpha 2))$

$T_{0.2} = !(\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M} 0(\alpha (j_n + 1)))) \multimap ((\alpha 0 \otimes [C 0 + C 1 + 2] \mathbf{1}) \multimap \mathbb{M} 0(\alpha 2))$

$T_{0.3} = !(\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M} 0(\alpha (j_n + 1))))$

$T_{0.4} = (\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M} 0(\alpha (j_n + 1))))$

$T_{0.5} = (\alpha 0 \otimes [C 0] \mathbf{1}) \multimap \mathbb{M} 0(\alpha 1)$

$T_{0.6} = (\alpha 1 \otimes [C 1] \mathbf{1}) \multimap \mathbb{M} 0(\alpha 2)$

$T_1 = \mathbb{M} 0((\alpha 0 \otimes [C 0 + C 1 + 2] \mathbf{1}) \multimap \mathbb{M} 0(\alpha 2))$

$T_{1.1} = ((\alpha 0 \otimes [C 0 + C 1 + 2] \mathbf{1}) \multimap \mathbb{M} 0(\alpha 2))$

$T_2 = (\alpha 0 \otimes [C 0 + C 1 + 2] \mathbf{1})$

$T_{2.1} = \alpha 0$

$T_{2.2} = [C 0 + C 1 + 2] \mathbf{1}$

$T_3 = \mathbb{M} 1(\alpha 2)$

$T_{3.1} = \mathbb{M}(C 0 + C 1 + 2)(\alpha 2)$

$TI = \alpha : \mathbb{N} \rightarrow \text{Type}; C : \mathbb{N} \rightarrow \text{Sort}$

D5.22

$$\overline{TI; \cdot; f_u : T_{0.4}; b : \alpha \ 1, c : [(C \ 1)] \ \mathbf{1} \vdash \langle\langle b, c \rangle\rangle : (\alpha \ 1 \otimes [(C \ 1)] \ \mathbf{1})}$$

D5.21

$$\overline{TI; \cdot; f_u : T_{0.4}; \cdot \vdash f_u \ \square : T_{0.6}}$$

D5.2

$$\frac{D5.21 \quad D5.22}{\overline{TI; \cdot; f_u : T_{0.4}; b : \alpha \ 1, c : [(C \ 1)] \ \mathbf{1} \vdash f_u \ \square \ \uparrow^1 \langle\langle b, c \rangle\rangle : T_3}}$$

D5.1

$$\overline{TI; \cdot; f_u : T_{0.4}; \cdot \vdash \text{store}() : \mathbb{M}(C \ 1) [(C \ 1)] \ \mathbf{1}}$$

D5:

$$\frac{D5.1 \quad D5.2}{\overline{TI; \cdot; f_u : T_{0.4}; b : \alpha \ 1 \vdash \text{bind } c = \text{store}() \text{ in } f_u \ \square \ \langle\langle b, c \rangle\rangle : \mathbb{M}(C \ 1 + 1) (\alpha \ 2)}}{\overline{TI; \cdot; f_u : T_{0.4}; b : \alpha \ 1 \vdash E_2 : \mathbb{M}(C \ 1 + 1) (\alpha \ 2)}}$$

D4.12:

$$\overline{TI; \cdot; f_u : T_{0.4}; y_1 : T_{2.1}, a : [(C \ 0)] \ \mathbf{1} \vdash \langle\langle y_1, a \rangle\rangle : (T_{2.1} \otimes [(C \ 0)] \ \mathbf{1})}$$

D4.11:

$$\overline{TI; \cdot; f_u : T_{0.4}; \cdot \vdash f_u \ \square : T_{0.5}}$$

D4.1:

$$\frac{D4.11 \quad D4.12}{\overline{TI; \cdot; f_u : T_{0.4}; y_1 : T_{2.1}, a : [(C \ 0)] \ \mathbf{1} \vdash f_u \ \square \ \uparrow^1 \langle\langle y_1, a \rangle\rangle : \mathbb{M} \ 1 (\alpha \ 1)}}$$

D4:

$$\frac{\overline{TI; \cdot; f_u : T_{0.4}; \cdot \vdash \text{store}() : \mathbb{M}(C \ 0) [(C \ 0)] \ \mathbf{1}} \quad D4.1}{\overline{TI; \cdot; f_u : T_{0.4}; y_1 : T_{2.1} \vdash \text{bind } a = \text{store}() \text{ in } f_u \ \square \ \uparrow^1 \langle\langle y_1, a \rangle\rangle : \mathbb{M}(C \ 0 + 1) (\alpha \ 1)}}{\overline{TI; \cdot; f_u : T_{0.4}; y_1 : T_{2.1} \vdash E_1 : \mathbb{M}(C \ 0 + 1) (\alpha \ 1)}}$$

D3.2:

$$\frac{D4 \quad D5}{\overline{TI; \cdot; f_u : T_{0.4}; y_1 : T_{2.1} \vdash \text{bind } b = E_1 \text{ in } E_2 : T_{3.1}}}$$

D3.1:

$$\overline{TI; \cdot; f_u : T_{0.4}; y_2 : T_{2.2} \vdash y_2 : T_{2.2}}$$

D3:

$$\frac{D3.1 \quad D3.2}{\overline{TI; \cdot; f_u : T_{0.4}; y_1 : T_{2.1}, y_2 : T_{2.2} \vdash \text{release } - = y_2 \text{ in } \text{bind } b = E_1 \text{ in } E_2 : T_3}}$$

D2:

$$\overline{TI; \cdot; f_u : T_{0.4}; x : T_2 \vdash x : T_2}$$

D1:

$$\frac{D2 \quad D3}{TI; .; f_u : T_{0.4}; x : T_2 \vdash \text{let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in release } - = y_2 \text{ in bind } b = E_1 \text{ in } E_2 : T_3}$$

D0:

$$\frac{}{TI; .; .; f : T_{0.3} \vdash f : T_{0.3}}$$

D0.0:

$$\frac{\frac{\frac{D0 \quad D1}{TI; .; .; f : T_{0.3}, x : T_2 \vdash \text{let! } f_u = f \text{ in let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in release } - = y_2 \text{ in bind } b = E_1 \text{ in } E_2 : T_3}}{TI; .; .; f : T_{0.3} \vdash \lambda x. \text{let! } f_u = f \text{ in let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in release } - = y_2 \text{ in bind } b = E_1 \text{ in } E_2 : T_{1.1}}}{TI; .; .; f : T_{0.3} \vdash \text{ret } \lambda x. \text{let! } f_u = f \text{ in let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in release } - = y_2 \text{ in bind } b = E_1 \text{ in } E_2 : T_1}}{TI; .; .; . \vdash \lambda f. \text{ret } \lambda x. \text{let! } f_u = f \text{ in let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in release } - = y_2 \text{ in bind } b = E_1 \text{ in } E_2 : T_{0.2}}$$

Main derivation:

$$\frac{\frac{D0.0}{.; \alpha : \mathbb{N} \rightarrow \text{Type}; .; . \vdash \Lambda C. \lambda f. \text{ret } \lambda x. \text{let! } f_u = f \text{ in let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in release } - = y_2 \text{ in bind } b = E_1 \text{ in } E_2 : T_{0.1}}{.; .; .; . \vdash \Lambda. \Lambda. \lambda f. \text{ret } \lambda x. \text{let! } f_u = f \text{ in let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in release } - = y_2 \text{ in bind } b = E_1 \text{ in } E_2 : T_0}}$$

Type derivation for $\text{succ} : \forall n. [2] \mathbf{1} \multimap \mathbb{M}0(\text{Nat } n \multimap \mathbb{M}0(\text{Nat } (n + 1)))$

$\text{succ} = \Lambda. \lambda p. \text{ret } \lambda \bar{N}. \text{ret } \Lambda. \Lambda. \lambda f. \text{ret } \lambda x. \text{let! } f_u = f \text{ in let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in release } - = y_2 \text{ in } E_0$

where

$E_0 = \text{release } - = p \text{ in bind } a = E_1 \text{ in } E_2$

$E_1 = \text{bind } b = \text{store}() \text{ in bind } b_1 = (\bar{N} \square \square \uparrow^1!f_u) \text{ in } b_1 \uparrow^1\langle\langle y_1, b \rangle\rangle$

$E_2 = \text{bind } c = \text{store}() \text{ in ret } f_u \square \uparrow^1\langle\langle a, c \rangle\rangle$

$T_p = [2] \mathbf{1}$

$T_0 = \forall n. T_p \multimap \mathbb{M}0(\text{Nat}[n] \multimap \mathbb{M}0(\text{Nat}[n + 1]))$

$T_{0.0} = T_p \multimap \mathbb{M}0(\text{Nat}[n] \multimap \mathbb{M}0(\text{Nat}[n + 1]))$

$T_{0.01} = \mathbb{M}0(\text{Nat}[n] \multimap \mathbb{M}0(\text{Nat}[n + 1]))$

$T_{0.1} = \text{Nat}[n] \multimap \mathbb{M}0(\text{Nat}[n + 1])$

$T_{0.2} = \mathbb{M}0(\text{Nat}[n + 1])$

$T_{0.11} = \text{Nat}[n]$

$T_{0.12} =$

$\forall \alpha : \mathbb{N} \rightarrow \text{Type}. \forall C!. (\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap$

$\mathbb{M}0(\alpha (j_n + 1)))) \multimap \mathbb{M}0((\alpha 0 \otimes [C 0 + \dots + C (n - 1) + n] \mathbf{1}) \multimap \mathbb{M}0(\alpha n))$

$T_{0.13} = \forall C!. (\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M}0(\alpha (j_n + 1)))) \multimap$

$\mathbb{M}0((\alpha 0 \otimes [C 0 + \dots + C (n - 1) + n] \mathbf{1}) \multimap \mathbb{M}0(\alpha n))$

$T_{0.14} = !(\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M}0(\alpha (j_n + 1)))) \multimap$

$\mathbb{M}0((\alpha 0 \otimes [C 0 + \dots + C (n - 1) + n] \mathbf{1}) \multimap \mathbb{M}0(\alpha n))$

$T_{0.15} = \mathbb{M}0((\alpha 0 \otimes [C 0 + \dots + C (n - 1) + n] \mathbf{1}) \multimap \mathbb{M}0(\alpha n))$

$$\begin{aligned}
T_{0.151} &= \mathbb{M}1((\alpha 0 \otimes [C 0 + \dots + C(n-1) + n] \mathbf{1}) \multimap \mathbb{M}0(\alpha n)) \\
T_{0.16} &= ((\alpha 0 \otimes [C 0 + \dots + C(n-1) + n] \mathbf{1}) \multimap \mathbb{M}0(\alpha n)) \\
T_{0.2} &= \text{Nat}[n+1] \\
T_1 &= \\
\forall \alpha : \mathbb{N} &\rightarrow \text{Type}. \forall C. !(\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M}0(\alpha(j_n+1)))) \multimap \\
\mathbb{M}0 &((\alpha 0 \otimes [(C 0 + \dots + C(n) + (n+1))] \mathbf{1}) \multimap \mathbb{M}0(\alpha(n+1))) \\
T_{1.1} &= \forall C. !(\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M}0(\alpha(j_n+1)))) \multimap \\
\mathbb{M}0 &((\alpha 0 \otimes [(C 0 + \dots + C(n) + (n+1))] \mathbf{1}) \multimap \mathbb{M}0(\alpha(n+1))) \\
T_{1.2} &= !(\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M}0(\alpha(j_n+1)))) \multimap \\
\mathbb{M}0 &((\alpha 0 \otimes [(C 0 + \dots + C(n) + (n+1))] \mathbf{1}) \multimap \mathbb{M}0(\alpha(n+1))) \\
T_{1.3} &= !(\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M}0(\alpha(j_n+1)))) \\
T_{1.31} &= (\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M}0(\alpha(j_n+1)))) \\
T_{1.40} &= \mathbb{M}0((\alpha 0 \otimes [(C 0 + \dots + C(n) + (n+1))] \mathbf{1}) \multimap \mathbb{M}0(\alpha(n+1))) \\
T_{1.4} &= ((\alpha 0 \otimes [(C 0 + \dots + C(n) + (n+1))] \mathbf{1}) \multimap \mathbb{M}0(\alpha(n+1))) \\
T_{1.41} &= (\alpha 0 \otimes [(C 0 + \dots + C(n) + (n+1))] \mathbf{1}) \\
T_{1.411} &= \alpha 0 \\
T_{1.412} &= [(C 0 + \dots + C(n) + (n+1))] \mathbf{1} \\
T_{1.42} &= \mathbb{M}0(\alpha(n+1)) \\
T_{1.43} &= \mathbb{M}(C 0 + \dots + C(n) + (n+1))(\alpha(n+1)) \\
T_{1.431} &= \mathbb{M}(C 0 + \dots + C(n) + (n+1) + 2)(\alpha(n+1)) \\
T_{1.44} &= \mathbb{M}(C 0 + \dots + C(n-1) + n + 2)(\alpha n) \\
T_{1.45} &= \mathbb{M}(C n + 1)(\alpha(n+1)) \\
TI &= \alpha; n, C
\end{aligned}$$

D3.1:

$$\overline{TI; ; f_u : T_{1.31}; a : \alpha n, c : [(C n)] \mathbf{1} \vdash f_u \square \uparrow^1 \langle\langle a, c \rangle\rangle : \mathbb{M}1 \alpha(n+1)}$$

D3:

$$\frac{\overline{TI; ; f_u : T_{1.31}; . \vdash \text{store}() : \mathbb{M}(C n) [(C n)] \mathbf{1}} \quad D3.1}{\overline{TI; ; f_u : T_{1.31}; a : \alpha n \vdash \text{bind } c = \text{store}() \text{ in } f_u \square \uparrow^1 \langle\langle a, c \rangle\rangle : T_{1.45}}$$

D2.3:

$$\frac{\overline{TI; ; f_u : T_{1.31}; y_1 : T_{1.411}, b : [n * C] \mathbf{1}, b_1 : T_{0.16} \vdash b_1 : T_{0.16}}}{\overline{TI; ; f_u : T_{1.31}; y_1 : T_{1.411}, b : [(C 0 + \dots + C(n-1) + (n))] \mathbf{1}, b_1 : T_{0.16} \vdash \langle\langle y_1, b \rangle\rangle : (T_{1.411} \otimes [(C 0 + \dots + C(n-1) + (n))] \mathbf{1})}}{\overline{TI; ; f_u : T_{1.31}; y_1 : T_{1.411}, b : [(C 0 + \dots + C(n-1) + (n))] \mathbf{1}, b_1 : T_{0.16} \vdash b_1 \uparrow^1 \langle\langle y_1, b \rangle\rangle : \mathbb{M}1 \alpha n}}$$

D2.2

$$\overline{TI; ; f_u : T_{1.31}; \overline{N} : T_{0.11} \vdash \overline{N} \square \square \uparrow^1 f_u : T_{0.151}}$$

D2.1:

$$\frac{\overline{D2.2} \quad \overline{D2.3}}{\overline{TI; ; f_u : T_{1.31}; \overline{N} : T_{0.11}, y_1 : T_{1.411}, b : [(C 0 + \dots + C(n-1) + (n))] \mathbf{1} \vdash \text{bind } b_1 = (\overline{N} \square \square \uparrow^1 f_u) \text{ in } b_1 \uparrow^1 \langle\langle y_1, b \rangle\rangle : \mathbb{M}2 \alpha n}}$$

D2:

$$\frac{TI; \cdot; f_u : T_{1.31}; \cdot \vdash \text{store}() : \mathbb{M}(C \ 0 + \dots + C \ (n-1) + (n)) [(C \ 0 + \dots + C \ (n-1) + (n))] \mathbf{1}}{D2.1}$$

$$TI; \cdot; f_u : T_{1.31}; \bar{N} : T_{0.11}, y_1 : T_{1.411} \vdash \text{bind } b = \text{store}() \text{ in } \text{bind } b_1 = (\bar{N} \ \square \ \square \ \uparrow^1!f_u) \text{ in } b_1 \ \uparrow^1\langle\langle y_1, b \rangle\rangle : T_{1.44}$$

D1.5:

$$\frac{\frac{D2}{TI; \cdot; f_u : T_{1.31}; \bar{N} : T_{0.11}, y_1 : T_{1.411} \vdash E_1 : T_{1.44}} \quad \frac{D3}{TI; \cdot; f_u : T_{1.31}; a : \alpha \ n \vdash E_2 : T_{1.45}}}{TI; \cdot; f_u : T_{1.31}; y_1 : T_{1.411} \vdash \text{bind } a = E_1 \text{ in } E_2 : T_{1.431}}$$

D1.4:

$$\frac{}{TI; \cdot; f_u : T_{1.31}; p : T_p \vdash p : T_p}$$

D1.3

$$\frac{\frac{D1.4 \quad D1.5}{TI; \cdot; f_u : T_{1.31}; \bar{N} : T_{0.11}, p : T_p, y_1 : T_{1.411} \vdash \text{release } - = p \text{ in } \text{bind } a = E_1 \text{ in } E_2 : T_{1.43}}}{TI; \cdot; f_u : T_{1.31}; \bar{N} : T_{0.11}, p : T_p, y_1 : T_{1.411} \vdash E_0 : T_{1.43}}$$

D1.2

$$\frac{\frac{D1.3}{TI; \cdot; f_u : T_{1.31}; y_2 : T_{1.412} \vdash y_2 : T_{1.412}}}{TI; \cdot; f_u : T_{1.31}; \bar{N} : T_{0.11}, p : T_p, y_1 : T_{1.411}, y_2 : T_{1.412} \vdash \text{release } - = y_2 \text{ in } E_0 : T_{1.42}}$$

D1.1

$$\frac{}{TI; \cdot; f_u : T_{1.31}; x : T_{1.41} \vdash x : T_{1.41}}$$

D1:

$$\frac{\frac{D1.1 \quad D1.2}{TI; \cdot; f_u : T_{1.31}; \bar{N} : T_{0.11}, p : T_p, x : T_{1.41} \vdash \text{let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in } \text{release } - = y_2 \text{ in } E_0 : T_{1.42}}}{}$$

D0:

$$\frac{}{TI; \cdot; \cdot; f : T_{1.3} \vdash f : T_{1.3}}$$

D0.0:

$$\frac{\frac{\frac{D0 \quad D1}{TI; \cdot; \cdot; \bar{N} : T_{0.11}, p : T_p, f : T_{1.31}, x : T_{1.41} \vdash \text{let}!f_u = f \text{ in } \text{let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in } \text{release } - = y_2 \text{ in } E_0 : T_{1.42}}}{TI; \cdot; \cdot; \bar{N} : T_{0.11}, p : T_p, f : T_{1.31} \vdash \lambda x. \text{let}!f_u = f \text{ in } \text{let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in } \text{release } - = y_2 \text{ in } E_0 : T_{1.4}}}{TI; \cdot; \cdot; \bar{N} : T_{0.11}, p : T_p, f : T_{1.31} \vdash \text{ret } \lambda x. \text{let}!f_u = f \text{ in } \text{let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in } \text{release } - = y_2 \text{ in } E_0 : T_{1.40}}}{TI; \cdot; \cdot; \bar{N} : T_{0.11}, p : T_p \vdash \lambda f. \text{ret } \lambda x. \text{let}!f_u = f \text{ in } \text{let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in } \text{release } - = y_2 \text{ in } E_0 : T_{1.2}}}{\cdot; n; \cdot; \cdot; \bar{N} : T_{0.11}, p : T_p \vdash \Lambda. \Lambda. \lambda f. \text{ret } \lambda x. \text{let}!f_u = f \text{ in } \text{let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in } \text{release } - = y_2 \text{ in } E_0 : T_1}}{\cdot; n; \cdot; \cdot; \bar{N} : T_{0.11}, p : T_p \vdash \text{ret } \Lambda. \Lambda. \lambda f. \text{ret } \lambda x. \text{let}!f_u = f \text{ in } \text{let}\langle\langle y_1, y_2 \rangle\rangle = x \text{ in } \text{release } - = y_2 \text{ in } E_0 : T_{0.2}}$$

Main derivation:

D0.0

$$\frac{.; n; .; .; p : T_p \vdash \lambda \bar{N}. \text{ret } \Lambda. \Lambda. \lambda f. \text{ret } \lambda x. \text{let } ! f_u = f \text{ in let } \langle\langle y_1, y_2 \rangle\rangle = x \text{ in release } - = y_2 \text{ in } E_0 : T_{0.1}}{.; n; .; .; p : T_p \vdash \text{ret } \lambda \bar{N}. \text{ret } \Lambda. \Lambda. \lambda f. \text{ret } \lambda x. \text{let } ! f_u = f \text{ in let } \langle\langle y_1, y_2 \rangle\rangle = x \text{ in release } - = y_2 \text{ in } E_0 : T_{0.01}}$$

$$\frac{.; n; .; .; . \vdash \lambda p. \text{ret } \lambda \bar{N}. \text{ret } \Lambda. \Lambda. \lambda f. \text{ret } \lambda x. \text{let } ! f_u = f \text{ in let } \langle\langle y_1, y_2 \rangle\rangle = x \text{ in release } - = y_2 \text{ in } E_0 : T_{0.0}}{.; .; .; . \vdash \Lambda. \lambda p. \text{ret } \lambda \bar{N}. \text{ret } \Lambda. \Lambda. \lambda f. \text{ret } \lambda x. \text{let } ! f_u = f \text{ in let } \langle\langle y_1, y_2 \rangle\rangle = x \text{ in release } - = y_2 \text{ in } E_0 : T_0}$$

Type derivation for add

$$\text{add} : \forall n_1, n_2. [(n_1 * 3 + n_1 + 2)] \mathbf{1} \multimap \mathbb{M}0(\text{Nat } n_1 \multimap \mathbb{M}0(\text{Nat } n_2 \multimap \mathbb{M}0(\text{Nat } (n_1 + n_2))))$$

$$\text{add} = \Lambda. \Lambda. \lambda p. \text{ret } \lambda \bar{N}_1. \text{ret } \lambda \bar{N}_2. E_0$$

where

$$E_0 = \text{release } - = p \text{ in bind } a = E_1 \text{ in } E_2$$

$$E_{0.1} = \text{release } - = y_2 \text{ in bind } b_1 = (\text{bind } b_2 = \text{store } () \text{ in succ } [] b_2) \text{ in } b_1 \uparrow^1 y_1$$

$$E_1 = \bar{N}_1 [] [] \uparrow^1 (\Lambda. \lambda t. \text{let } \langle\langle y_1, y_2 \rangle\rangle = t \text{ in } E_{0.1})$$

$$E_2 = \text{bind } b = \text{store}() \text{ in } a \uparrow^1 \langle\langle \bar{N}_2, b \rangle\rangle$$

$$T_p = [(n_1 * 3 + n_1 + 2)] \mathbf{1}$$

$$T_0 = \forall n_1, n_2. T_p \multimap \mathbb{M}0(\text{Nat } n_1 \multimap \mathbb{M}0(\text{Nat } n_2 \multimap \mathbb{M}0(\text{Nat } (n_1 + n_2))))$$

$$T_{0.1} = \forall n_2. T_p \multimap \mathbb{M}0(\text{Nat } n_1 \multimap \mathbb{M}0(\text{Nat } n_2 \multimap \mathbb{M}0(\text{Nat } (n_1 + n_2))))$$

$$T_{0.2} = T_p \multimap \mathbb{M}0(\text{Nat } n_1 \multimap \mathbb{M}0(\text{Nat } n_2 \multimap \mathbb{M}0(\text{Nat } (n_1 + n_2))))$$

$$T_{0.20} = \mathbb{M}0(\text{Nat } n_1 \multimap \mathbb{M}0(\text{Nat } n_2 \multimap \mathbb{M}0(\text{Nat}[n_1 + n_2])))$$

$$T_{0.21} = (\text{Nat } n_1 \multimap \mathbb{M}0(\text{Nat } n_2 \multimap \mathbb{M}0(\text{Nat}[n_1 + n_2])))$$

$$T_{0.3} = \mathbb{M}0(\text{Nat } n_2 \multimap \mathbb{M}0(\text{Nat } (n_1 + n_2)))$$

$$T_{0.31} = \text{Nat } n_2 \multimap \mathbb{M}0(\text{Nat } (n_1 + n_2))$$

$$T_{0.4} = \mathbb{M}1(\text{Nat } (n_1 + n_2))$$

$$T_{0.40} = \mathbb{M}0(\text{Nat } (n_1 + n_2))$$

$$T_{0.5} = \mathbb{M}(n_1 * 3 + n_1 + 1)(\text{Nat } (n_1 + n_2))$$

$$T_{0.6} = \mathbb{M}(n_1 * 3 + n_1 + 2)(\text{Nat } (n_1 + n_2))$$

$$T_1 =$$

$$\forall \alpha : \mathbb{N} \rightarrow \text{Type}. \forall C. !(\forall k. ((\alpha k \otimes [C k] \mathbf{1}) \multimap \mathbb{M}0(\alpha (k + 1)))) \multimap$$

$$\mathbb{M}0((\alpha 0 \otimes [(C 0 + \dots + C (n_1 - 1) + n_1)] \mathbf{1}) \multimap \mathbb{M}0(\alpha (n_1)))$$

$$a_f = \lambda k. \text{Nat } (n_2 + k)$$

$$T_{1.1} = \forall C. !(\forall k. ((a_f k \otimes [C k] \mathbf{1}) \multimap \mathbb{M}0(a_f (k + 1)))) \multimap$$

$$\mathbb{M}0((a_f 0 \otimes [(C 0 + \dots + C (n_1 - 1) + n_1)] \mathbf{1}) \multimap \mathbb{M}0(a_f n_1))$$

$$T_{1.2} = \forall C. !(\forall k. ((\text{Nat } (n_2 + k) \otimes [C k] \mathbf{1}) \multimap \mathbb{M}0(\text{Nat } (n_2 + (k + 1)))) \multimap$$

$$\mathbb{M}0((\text{Nat } (n_2 + 0) \otimes [(C 0 + \dots + C (n_1 - 1) + n_1)] \mathbf{1}) \multimap \mathbb{M}0(\text{Nat } (n_2 + n_1)))$$

$$T_{1.21} = !(\forall k. ((\text{Nat } (n_2 + k) \otimes [C k] \mathbf{1}) \multimap \mathbb{M}0(\text{Nat } (n_2 + (k + 1)))) \multimap$$

$$\mathbb{M}0((\text{Nat } (n_2 + 0) \otimes [(C 0 + \dots + C (n_1 - 1) + n_1)] \mathbf{1}) \multimap \mathbb{M}0(\text{Nat } (n_2 + n_1)))[(\lambda_s - .3)/C]$$

$$T_{1.22} = !(\forall k. ((\text{Nat } (n_2 + k) \otimes [3] \mathbf{1}) \multimap \mathbb{M}0(\text{Nat}[n_2 + (k + 1)])))$$

$$T_{1.23} = (\forall k. ((\text{Nat } (n_2 + k) \otimes [3] \mathbf{1}) \multimap \mathbb{M}0(\text{Nat}[n_2 + (k + 1)])))$$

$$T_{1.24} = ((\text{Nat } (n_2 + k) \otimes [3] \mathbf{1}) \multimap \mathbb{M}0(\text{Nat } (n_2 + (k + 1))))$$

$$T_{1.241} = (\text{Nat } (n_2 + k) \otimes [3] \mathbf{1})$$

$$T_{1.2411} = (\text{Nat } (n_2 + k))$$

$$T_{1.2412} = [3] \mathbf{1}$$

$$T_{1.242} = \mathbb{M}0(\text{Nat } (n_2 + (k + 1)))$$

$$T_{1.3} = \mathbb{M}0((\text{Nat } (n_2 + 0) \otimes [(n_1 * 3 + n_1)] \mathbf{1}) \multimap \mathbb{M}0(\text{Nat } (n_2 + n_1)))$$

$$T_{1.30} = \mathbb{M}1((\text{Nat } (n_2 + 0) \otimes [(n_1 * 3 + n_1)] \mathbf{1}) \multimap \mathbb{M}0(\text{Nat } (n_2 + n_1)))$$

$$\begin{aligned}
T_{1.31} &= ((\text{Nat } (n_2 + 0) \otimes [(n_1 * 3 + n_1)] \mathbf{1}) \multimap \mathbb{M}0 (\text{Nat } (n_2 + n_1))) \\
T_2 &= \text{Nat } n_2 \\
T_3 &= (\text{Nat } (n_2 + k) \multimap \mathbb{M}0 (\text{Nat } (n_2 + k + 1)))
\end{aligned}$$

D3:

$$\overline{.; n_1, n_2; .; .; \overline{N_1} : T_1 \vdash \overline{N_1} : T_1}$$

D2.10:

$$\begin{array}{c}
D3 \\
\hline
.; n_1, n_2; .; .; \cdot \vdash (\lambda_t k. \text{Nat}[n_2 + k]) : \mathbb{N} \rightarrow \text{Type} \\
\hline
.; n_1, n_2; .; .; \overline{N_1} : T_1 \vdash \overline{N_1} \square : T_{1.1} \\
\hline
.; n_1, n_2; .; .; \overline{N_1} : T_1 \vdash \overline{N_1} \square : T_{1.2}
\end{array}$$

D2:

$$\begin{array}{c}
D2.10 \\
\hline
.; n_1, n_2; .; .; \cdot \vdash (\lambda_s . 3) : \mathbb{N} \rightarrow \mathbb{N} \\
\hline
.; n_1, n_2; .; .; \overline{N_1} : T_1 \vdash \overline{N_1} \square \square : T_{1.21}
\end{array}$$

D1.32:

$$\overline{.; n_1, n_2, k; .; .; b_2 : [2] \mathbf{1} \vdash \text{succ} \square b_2 : \mathbb{M}0 T_3}$$

D1.31:

$$\begin{array}{c}
\overline{.; n_1, n_2, k; .; .; \cdot \vdash \text{store}() : \mathbb{M}2 [2] \mathbf{1}} \quad D1.32 \\
\hline
.; n_1, n_2, k; .; .; \cdot \vdash (\text{bind } b_2 = \text{store}() \text{ in } \text{succ} \square b_2) : \mathbb{M}2 T_3
\end{array}$$

D1.3:

$$\begin{array}{c}
D1.31 \\
\hline
.; n_1, n_2, k; .; .; y_1 : T_{1.2411}, b_1 : T_3 \vdash b_1 \uparrow^1 y_1 : \mathbb{M}1 \text{Nat}[n_2 + k + 1] \\
\hline
.; n_1, n_2, k; .; .; y_1 : T_{1.2411} \vdash \text{bind } b_1 = (\text{bind } b_2 = \text{store}() \text{ in } \text{succ} \square b_2) \text{ in } b_1 \uparrow^1 y_1 : \mathbb{M}(3) \text{Nat}[n_2 + k + 1]
\end{array}$$

D1.2:

$$\begin{array}{c}
\overline{.; n_1, n_2, k; .; .; y_2 : T_{1.2412} \vdash y_2 : T_{1.2412}} \quad D1.3 \\
\hline
.; n_1, n_2, k; .; .; y_1 : T_{1.2411}, y_2 : T_{1.2412} \vdash \\
\text{release } - = y_2 \text{ in } \text{bind } b_1 = (\text{bind } b_2 = \text{store}() \text{ in } \text{succ} \square b_2) \text{ in } b_1 \uparrow^1 y_1 : \mathbb{M}0 \text{Nat}[n_2 + k + 1]
\end{array}$$

D1.1:

$$\begin{array}{c}
D1.2 \\
\hline
\overline{.; n_1, n_2, k; .; .; t : T_{1.241} \vdash t : T_{1.241}} \quad \overline{.; n_1, n_2, k; .; .; y_1 : T_{1.2411}, y_2 : T_{1.2412} \vdash E0.1 : T_{1.242}} \\
\hline
.; n_1, n_2, k; .; .; t : T_{1.241} \vdash \text{let}\langle\langle y_1, y_2 \rangle\rangle = t \text{ in } E0.1 : T_{1.242} \\
\hline
.; n_1, n_2, k; .; .; \cdot \vdash \lambda t. \text{let}\langle\langle y_1, y_2 \rangle\rangle = t \text{ in } E0.1 : T_{1.24}
\end{array}$$

D1:

$$\begin{array}{c}
D1.1 \\
\hline
D2 \\
\hline
.; n_1, n_2; .; .; \cdot \vdash (\Lambda. \lambda t. \text{let}\langle\langle y_1, y_2 \rangle\rangle = t \text{ in } E0.1) : T_{1.23} \\
.; n_1, n_2; .; .; \cdot \vdash !(\Lambda. \lambda t. \text{let}\langle\langle y_1, y_2 \rangle\rangle = t \text{ in } E0.1) : T_{1.22} \\
\hline
.; n_1, n_2; .; .; \overline{N_1} : T_1 \vdash \overline{N_1} \square \square \uparrow^1 !(\Lambda. \lambda t. \text{let}\langle\langle y_1, y_2 \rangle\rangle = t \text{ in } E0.1) : T_{1.30}
\end{array}$$

D0.1

$$\begin{array}{c}
D1 \\
\hline
.; n_1, n_2; .; .; \overline{N_1} : T_1, \overline{N_2} : T_2 \vdash E_1 : T_{1.30}
\end{array}$$

D2.1:

$$\frac{}{.; n_1, n_2; .; .; \overline{N_2} : T_2, a : T_{1.31}, b : [(n_1 * 3 + n_1)] \mathbf{1} \vdash a \uparrow^1 \langle\langle \overline{N_2}, b \rangle\rangle : T_{0.4}}$$

D2.0:

$$\frac{\frac{}{.; n_1, n_2; .; .; \cdot \vdash \text{store}() : \mathbb{M}(n_1 * 3 + n_1) [(n_1 * 3 + n_1)] \mathbf{1}}{.; n_1, n_2; .; .; \overline{N_1} : T_1, \overline{N_2} : T_2, a : T_{1.31} \vdash \text{bind } b = \text{store}() \text{ in } a \uparrow^1 \langle\langle \overline{N_2}, b \rangle\rangle : T_{0.5}} \quad D2.1}$$

D0.2:

$$\frac{D2.0}{.; n_1, n_2; .; .; \overline{N_1} : T_1, \overline{N_2} : T_2, a : T_{1.31} \vdash E_2 : T_{0.5}}$$

D0:

$$\frac{D0.1 \quad D0.2}{.; n_1, n_2; .; .; \overline{N_1} : T_1, \overline{N_2} : T_2 \vdash \text{bind } a = E_1 \text{ in } E_2 : T_{0.6}}$$

D0.0

$$\frac{\frac{}{.; n_1, n_2; .; .; p : T_p \vdash p : T_p} \quad D0}{.; n_1, n_2; .; .; p : T_p, \overline{N_1} : T_1, \overline{N_2} : T_2 \vdash \text{release } - = p \text{ in } \text{bind } a = E_1 \text{ in } E_2 : T_{0.40}}$$

Main derivation:

$$\frac{D0.0}{.; n_1, n_2; .; .; p : T_p, \overline{N_1} : T_1 \vdash \lambda \overline{N_2}. E_0 : T_{0.31}} \\ \frac{}{.; n_1, n_2; .; .; p : T_p, \overline{N_1} : T_1 \vdash \text{ret } \lambda \overline{N_2}. E_0 : T_{0.3}} \\ \frac{}{.; n_1, n_2; .; .; p : T_p \vdash \lambda \overline{N_1}. \text{ret } \lambda \overline{N_2}. E_0 : T_{0.21}} \\ \frac{}{.; n_1, n_2; .; .; p : T_p \vdash \text{ret } \lambda \overline{N_1}. \text{ret } \lambda \overline{N_2}. E_0 : T_{0.20}} \\ \frac{}{.; n_1, n_2; .; .; \cdot \vdash \lambda p. \text{ret } \lambda \overline{N_1}. \text{ret } \lambda \overline{N_2}. E_0 : T_{0.2}} \\ \frac{}{.; n_1; .; .; \cdot \vdash \Lambda. \lambda p. \text{ret } \lambda \overline{N_1}. \text{ret } \lambda \overline{N_2}. E_0 : T_{0.1}} \\ \frac{}{.; .; .; .; \cdot \vdash \Lambda. \Lambda. \lambda p. \text{ret } \lambda \overline{N_1}. \text{ret } \lambda \overline{N_2}. E_0 : T_0}$$

Type derivation for *mult*

$mult : \forall n_1, n_2.$

$[(n_1 * (n_2 * 3 + n_2 + 4) + n_1 + 2)] \mathbf{1} \multimap \mathbb{M}0(\text{Nat } n_1 \multimap \mathbb{M}0(\text{Nat } n_2 \multimap \mathbb{M}0(\text{Nat } (n_1 * n_2))))$

$mult = \Lambda. \Lambda. \lambda p. \text{ret } \lambda \overline{N_1}. \text{ret } (\lambda \overline{N_2}. E_0)$

where

$E_0 = \text{release } - = p \text{ in } \text{bind } a = E_1 \text{ in } E_2$

$E_{0.1} = \text{release } - = y_2 \text{ in } \text{bind } b_1 = (\text{bind } b_2 = \text{store}() \text{ in } \text{add } [] [] b_2 \uparrow^1 \overline{N_2}) \text{ in } b_1 \uparrow^1 y_1$

$E_1 = \overline{N_1} [] [] \uparrow^1 !(\Lambda. \lambda t. \text{let } \langle\langle y_1, y_2 \rangle\rangle = t \text{ in } E_{0.1})$

$E_2 = \text{bind } b = \text{store}() \text{ in } a \uparrow^1 \langle\langle 0, b \rangle\rangle$

$T_p = [(n_1 * (n_2 * 3 + n_2 + 4) + n_1 + 2)] \mathbf{1}$

$T_0 = \forall n_1, n_2. T_p \multimap \mathbb{M}0(\text{Nat } n_1 \multimap \mathbb{M}0(\text{Nat } n_2 \multimap \mathbb{M}0(\text{Nat } (n_1 * n_2))))$

$T_{0.1} = \forall n_2. T_p \multimap \mathbb{M}0(\text{Nat } n_1 \multimap \mathbb{M}0(\text{Nat } n_2 \multimap \mathbb{M}0(\text{Nat } (n_1 * n_2))))$

$T_{0.2} = T_p \multimap \mathbb{M}0(\text{Nat } n_1 \multimap \mathbb{M}0(\text{Nat } n_2 \multimap \mathbb{M}0(\text{Nat } (n_1 * n_2))))$

$T_{0.21} = \mathbb{M}0(\text{Nat } n_1 \multimap \mathbb{M}0(\text{Nat } n_2 \multimap \mathbb{M}0(\text{Nat } [n_1 * n_2])))$

$$\begin{aligned}
T_{0.22} &= (\text{Nat } n_1 \multimap \mathbb{M} 0 (\text{Nat } n_2 \multimap \mathbb{M} 0 (\text{Nat } (n_1 * n_2)))) \\
T_{0.3} &= \mathbb{M} 0 (\text{Nat } n_2 \multimap \mathbb{M} 0 (\text{Nat } (n_1 * n_2))) \\
T_{0.31} &= (\text{Nat } n_2 \multimap \mathbb{M} 0 (\text{Nat } (n_1 * n_2))) \\
T_{0.4} &= \mathbb{M} 1 (\text{Nat } (n_1 * n_2)) \\
T_{0.5} &= \mathbb{M}(n_1 * (n_2 * 3 + n_2 + 4) + n_1 + 1) (\text{Nat } (n_1 * n_2)) \\
T_{0.6} &= \mathbb{M}(n_1 * (n_2 * 3 + n_2 + 4) + n_1 + 2) (\text{Nat } (n_1 * n_2)) \\
T_1 &= \\
\forall \alpha : \mathbb{N} \rightarrow \text{Type}. \forall C. &!(\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M} 0 (\alpha (j_n + 1)))) \multimap \\
\mathbb{M} 0 ((\alpha 0 \otimes [(C 0 + \dots + C (n_1 - 1) + n_1)] \mathbf{1}) &\multimap \mathbb{M} 0 (\alpha n_1)) \\
a_f = \lambda k. \text{Nat}[n_2 * k] & \\
T_{1.1} = \forall C. &!(\forall j_n. ((a_f j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M} 0 (a_f (j_n + 1)))) \multimap \\
\mathbb{M} 0 ((a_f 0 \otimes [(C 0 + \dots + C (n_1 - 1) + n_1)] \mathbf{1}) &\multimap \mathbb{M} 0 (a_f n_1)) \\
T_{1.2} = \forall C. &!(\forall j_n. ((\text{Nat}[n_2 * j_n] \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M} 0 (\text{Nat}[n_2 * (j_n + 1)]))) \multimap \\
\mathbb{M} 0 ((\text{Nat}[n_2 * 0] \otimes [(C 0 + \dots + C (n_1 - 1) + n_1)] \mathbf{1}) &\multimap \mathbb{M} 0 (\text{Nat } (n_2 * n_1))) \\
T_{1.21} = &!(\forall j_n. ((\text{Nat}[n_2 * j_n] \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M} 0 (\text{Nat}[n_2 * (j_n + 1)]))) \multimap \\
\mathbb{M} 0 ((\text{Nat}[n_2 * 0] \otimes [(C 0 + \dots + C (n_1 - 1) + n_1)] \mathbf{1}) &\multimap \mathbb{M} 0 (\text{Nat } (n_2 * n_1))) [C / (\lambda. (n_2 * 3 + n_2 + 4))] \\
T_{1.22} = &!(\forall j_n. ((\text{Nat}[n_2 * j_n] \otimes [(n_2 * 3 + n_2 + 4)] \mathbf{1}) \multimap \mathbb{M} 0 (\text{Nat}[n_2 * (j_n + 1)]))) \\
T_{1.23} = &(\forall j_n. ((\text{Nat}[n_2 * j_n] \otimes [(n_2 * 3 + n_2 + 4)] \mathbf{1}) \multimap \mathbb{M} 0 (\text{Nat}[n_2 * (j_n + 1)]))) \\
T_{1.24} = &((\text{Nat}[n_2 * k] \otimes [(n_2 * 3 + n_2 + 4)] \mathbf{1}) \multimap \mathbb{M} 0 (\text{Nat}[n_2 * (k + 1)])) \\
T_{1.241} = &(\text{Nat}[n_2 * k] \otimes [(n_2 * 3 + n_2 + 4)] \mathbf{1}) \\
T_{1.2411} = &(\text{Nat}[n_2 * k] \\
T_{1.2412} = &[(n_2 * 3 + n_2 + 4)] \mathbf{1} \\
T_{1.242} = &\mathbb{M} 0 (\text{Nat}[n_2 * (k + 1)]) \\
T_{1.3} = \mathbb{M} 0 &((\text{Nat}[n_2 * 0] \otimes [(n_1 * (n_2 * 3 + n_2 + 4) + n_1)] \mathbf{1}) \multimap \mathbb{M} 0 (\text{Nat } (n_2 * n_1))) \\
T_{1.30} = \mathbb{M} 1 &((\text{Nat}[n_2 * 0] \otimes [(n_1 * (n_2 * 3 + n_2 + 4) + n_1)] \mathbf{1}) \multimap \mathbb{M} 0 (\text{Nat } (n_2 * n_1))) \\
T_{1.31} = &((\text{Nat}[n_2 * 0] \otimes [(n_1 * (n_2 * 3 + n_2 + 4) + n_1)] \mathbf{1}) \multimap \mathbb{M} 0 (\text{Nat } (n_2 * n_1))) \\
T_2 = \text{Nat } n_2 & \\
T_3 = (\text{Nat}[n_2 * k] &\multimap \mathbb{M} 0 (\text{Nat}[n_2 * (k + 1)]))
\end{aligned}$$

D3:

$$\frac{}{.; n_1, n_2; .; .; \overline{N_1} : T_1 \vdash \overline{N_1} : T_1}$$

D2.10:

$$\begin{array}{c}
D3 \\
\frac{.; n_1, n_2; .; .; . \vdash (\lambda_t k. \text{Nat}[n_2 * k]) : \mathbb{N} \rightarrow \text{Type}}{.; n_1, n_2; .; .; \overline{N_1} : T_1 \vdash \overline{N_1} \square : T_{1.1}} \\
\frac{.; n_1, n_2; .; .; \overline{N_1} : T_1 \vdash \overline{N_1} \square : T_{1.1}}{.; n_1, n_2; .; .; \overline{N_1} : T_1 \vdash \overline{N_1} \square : T_{1.2}}
\end{array}$$

D2:

$$\begin{array}{c}
D2.10 \\
\frac{.; n_1, n_2; .; .; . \vdash (\lambda_s - .(n_2 * 3 + n_2 + 4)) : S \rightarrow S}{.; n_1, n_2; .; .; \overline{N_1} : T_1 \vdash \overline{N_1} \square \square : T_{1.21}} \text{T-iapp}
\end{array}$$

D1.32

$$\frac{}{.; n_1, n_2, k; .; .; . \vdash \text{add} \square \square b_2 \uparrow^1 \overline{N_2} : \mathbb{M} 1 T_3}$$

D1.31

$$\frac{.; n_1, n_2, k; .; .; . \vdash \text{store} () : \mathbb{M}(n_2 * 3 + n_2 + 2) [(n_2 * 3 + n_2 + 2)] \mathbf{1}}{.; n_1, n_2, k; .; .; y_1 : T_{1.241}, b_1 : T_3 \vdash (\text{bind } b_2 = \text{store}() \text{ in } \text{add} \square \square b_2 \uparrow^1 \overline{N_2}) : \mathbb{M}(n_2 * 3 + n_2 + 3) T_3} \text{D1.32}$$

D1.3

$$\frac{D1.31 \quad \frac{\frac{}{.; n_1, n_2, k; .; .; y_1 : T_{1.241}, b_1 : T_3 \vdash b_1 \uparrow^1 y_1 : \mathbb{M}1 \text{Nat}[n_2 * (k + 1)]}}{.; n_1, n_2, k; .; .; y_1 \vdash \text{bind } b_1 = (\text{bind } b_2 = \text{store } () \text{ in } \text{add } [] [] b_2 \uparrow^1 \overline{N}_2) \text{ in } b_1 \uparrow^1 y_1 : \mathbb{M}(n_2 * 3 + n_2 + 4) \text{Nat}[n_2 * (k + 1)]}}{D1.2:}$$

D1.2:

$$\frac{\frac{}{.; n_1, n_2, k; .; .; y_2 : T_{1.2412} \vdash y_2 : T_{1.2412}}{D1.3} \quad \frac{}{.; n_1, n_2, k; .; .; y_1 : T_{1.2411}, y_2 : T_{1.2412} \vdash \text{release } - = y_2 \text{ in } \text{bind } b_1 = (\text{bind } b_2 = \text{store } () \text{ in } \text{add } [] [] b_2 \uparrow^1 \overline{N}_2) \text{ in } b_1 \uparrow^1 y_1 : \mathbb{M}0 \text{Nat}[n_2 * (k + 1)]}}{D1.1}$$

D1.1

$$\frac{\frac{\frac{}{.; n_1, n_2, k; .; .; t : T_{1.241} \vdash t : T_{1.241}}{D1.2} \quad \frac{}{.; n_1, n_2, k; .; .; y_1 : T_{1.2411}, y_2 : T_{1.2412} \vdash E0.1 : T_{1.242}}{D1.2}}{.; n_1, n_2, k; .; .; t : T_{1.241} \vdash \text{let}\langle\langle y_1, y_2 \rangle\rangle = t \text{ in } E0.1) : T_{1.242}}{.; n_1, n_2, k; .; .; . \vdash \lambda t. \text{let}\langle\langle y_1, y_2 \rangle\rangle = t \text{ in } E0.1) : T_{1.24}}$$

D1:

$$\frac{D2 \quad \frac{\frac{}{.; n_1, n_2; .; .; . \vdash (\Lambda. \lambda t. \text{let}\langle\langle y_1, y_2 \rangle\rangle = t \text{ in } E0.1) : T_{1.23}}{.; n_1, n_2; .; .; . \vdash !(\Lambda. \lambda t. \text{let}\langle\langle y_1, y_2 \rangle\rangle = t \text{ in } E0.1) : T_{1.22}}{.; n_1, n_2; .; .; \overline{N}_1 : T_1 \vdash \overline{N}_1 [] [] \uparrow^1 !(\Lambda. \lambda t. \text{let}\langle\langle y_1, y_2 \rangle\rangle = t \text{ in } E0.1) : T_{1.30}}{D1.1}$$

D0.1:

$$\frac{D1 \quad \frac{}{.; n_1, n_2; .; .; \overline{N}_1 : T_1, \overline{N}_2 : T_2 \vdash E_1 : T_{1.30}}{D2.1}$$

D2.1:

$$\frac{}{.; n_1, n_2; .; .; \overline{N}_2 : T_2, a : T_{1.31}, b : [(n_1 * (n_2 * 3 + n_2 + 4) + n_1)] \mathbf{1} \vdash a \uparrow^1 \langle\langle \overline{0}, b \rangle\rangle : T_{0.4}}$$

D2.0:

$$\frac{\frac{}{.; n_1, n_2; .; .; . \vdash \text{store } () : \mathbb{M}(n_1 * (n_2 * 3 + n_2 + 4) + n_1) [(n_1 * (n_2 * 3 + n_2 + 4) + n_1)] \mathbf{1}}{D2.1}}{.; n_1, n_2; .; .; \overline{N}_1 : T_1, \overline{N}_2 : T_2, a : T_{1.31} \vdash \text{bind } b = \text{store } () \text{ in } a \uparrow^1 \langle\langle \overline{0}, b \rangle\rangle : T_{0.5}}$$

D0.2:

$$\frac{D2.0 \quad \frac{}{.; n_1, n_2; .; .; \overline{N}_1 : T_1, \overline{N}_2 : T_2, a : T_{1.31} \vdash E_2 : T_{0.5}}{D0.1} \quad \frac{}{D0.2}}$$

D0:

$$\frac{D0.1 \quad D0.2 \quad \frac{}{.; n_1, n_2; .; .; \overline{N}_1 : T_1, \overline{N}_2 : T_2 \vdash \text{bind } a = E_1 \text{ in } E_2 : T_{0.6}}{D0.0}}$$

D0.0

$$\frac{\frac{}{.; n_1, n_2; .; .; p : T_p \vdash p : T_p}}{D0} \quad \frac{}{.; n_1, n_2; .; .; p : T_p, \overline{N}_1 : T_1, \overline{N}_2 : T_2 \vdash \text{release } - = p \text{ in } \text{bind } a = E_1 \text{ in } E_2 : T_{0.4}}$$

Main derivation:

$$\begin{array}{c}
D0.0 \\
\hline
.; n_1, n_2; .; .; p : T_p, \overline{N_1} : T_1 \vdash \lambda \overline{N_2}. E_0 : T_{0.31} \\
\hline
.; n_1, n_2; .; .; p : T_p, \overline{N_1} : T_1 \vdash \text{ret } \lambda \overline{N_2}. E_0 : T_{0.3} \\
\hline
.; n_1, n_2; .; .; p : T_p \vdash \lambda \overline{N_1}. \text{ret } \lambda \overline{N_2}. E_0 : T_{0.22} \\
\hline
.; n_1, n_2; .; .; p : T_p \vdash \text{ret } \lambda \overline{N_1}. \text{ret } \lambda \overline{N_2}. E_0 : T_{0.21} \\
\hline
.; n_1, n_2; .; .; . \vdash \lambda p. \text{ret } \lambda \overline{N_1}. \text{ret } \lambda \overline{N_2}. E_0 : T_{0.2} \\
\hline
.; n_1; .; .; . \vdash \Lambda. \lambda p. \text{ret } \lambda \overline{N_1}. \text{ret } \lambda \overline{N_2}. E_0 : T_{0.1} \\
\hline
.; .; .; .; . \vdash \Lambda. \Lambda. \lambda p. \text{ret } \lambda \overline{N_1}. \text{ret } \lambda \overline{N_2}. E_0 : T_0
\end{array}$$

Type derivation for *exp*

$$\text{exp} : \forall n_1, n_2. [\sum_{i \in \{0, n_2 - 1\}} (\lambda k. (n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 4) (i)) + n_2 + 2] \mathbf{1} \multimap \mathbb{M} 0 (\text{Nat } n_1 \multimap \mathbb{M} 0 (\text{Nat } n_2 \multimap \mathbb{M} 0 (\text{Nat } (n_1^{n_2}))))$$

$$\text{exp} = \Lambda. \Lambda. \lambda p. \text{ret } \lambda \overline{N_1}. \text{ret } \lambda \overline{N_2}. E_0$$

where

$$E_0 = \text{release } - = p \text{ in bind } a = E_1 \text{ in } E_2$$

$$E_{0.1} = \text{release } - = y_2 \text{ in bind } b_1 = (\text{bind } b_2 = \text{store } () \text{ in } \text{mult } [] [] b_2 \uparrow^1 \overline{N_1}) \text{ in } b_1 \uparrow^1 y_1$$

$$E_1 = \overline{N_2} [] [] \uparrow^1 !(\Lambda. \lambda t. \text{let} \langle \langle y_1, y_2 \rangle \rangle = t \text{ in } E_{0.1})$$

$$E_2 = \text{bind } b = \text{store } \mathbf{1} \text{ in } a \uparrow^1 \langle \langle \overline{1}, b \rangle \rangle$$

$$P = \sum_{i \in \{0, n_2 - 1\}} (\lambda k. (n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 4) i) + n_2 + 2$$

$$T_p = [P] \mathbf{1}$$

$$T_b = [P - 1] \mathbf{1}$$

$$T_0 = \forall n_1, n_2. T_p \multimap \mathbb{M} 0 (\text{Nat } n_1 \multimap \mathbb{M} 0 (\text{Nat } n_2 \multimap \mathbb{M} 0 (\text{Nat } (n_1^{n_2}))))$$

$$T_{0.1} = \forall n_2. T_p \multimap \mathbb{M} 0 (\text{Nat } n_1 \multimap \mathbb{M} 0 (\text{Nat } n_2 \multimap \mathbb{M} 0 (\text{Nat } (n_1^{n_2}))))$$

$$T_{0.2} = T_p \multimap \mathbb{M} 0 (\text{Nat } n_1 \multimap \mathbb{M} 0 (\text{Nat } n_2 \multimap \mathbb{M} 0 (\text{Nat } (n_1^{n_2}))))$$

$$T_{0.20} = \mathbb{M} 0 (\text{Nat } n_1 \multimap \mathbb{M} 0 (\text{Nat } n_2 \multimap \mathbb{M} 0 (\text{Nat } [n_1^{n_2}])))$$

$$T_{0.21} = \text{Nat } n_1 \multimap \mathbb{M} 0 (\text{Nat } n_2 \multimap \mathbb{M} 0 (\text{Nat } (n_1^{n_2})))$$

$$T_{0.3} = \mathbb{M} 0 (\text{Nat } n_2 \multimap \mathbb{M} 0 (\text{Nat } (n_1^{n_2})))$$

$$T_{0.31} = (\text{Nat } n_2 \multimap \mathbb{M} 0 (\text{Nat } (n_1^{n_2})))$$

$$T_{0.4} = \mathbb{M} 1 (\text{Nat } (n_1^{n_2}))$$

$$T_{0.5} = \mathbb{M} (P - 1) (\text{Nat } (n_1^{n_2}))$$

$$T_{0.6} = \mathbb{M} 0 (\text{Nat } (n_1^{n_2}))$$

$$T_1 =$$

$$\forall \alpha : \mathbb{N} \rightarrow \text{Type}. \forall C. !(\forall j_n. ((\alpha j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M} 0 (\alpha (j_n + 1)))) \multimap$$

$$\mathbb{M} 0 ((\alpha 0 \otimes [(C 0 + \dots + C (n_2 - 1) + n_2)] \mathbf{1}) \multimap \mathbb{M} 0 (\alpha n_2))$$

$$a_f = \lambda k. \text{Nat} [n_2^k]$$

$$T_{1.1} =$$

$$\forall C. !(\forall j_n. ((a_f j_n \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M} 0 (a_f (j_n + 1)))) \multimap$$

$$\mathbb{M} 0 ((a_f 0 \otimes [(C 0 + \dots + C (n_2 - 1) + n_2)] \mathbf{1}) \multimap \mathbb{M} 0 (a_f n_2))$$

$$T_{1.2} =$$

$$\forall C. !(\forall j_n. ((\text{Nat} [n_2^{j_n}] \otimes [C j_n] \mathbf{1}) \multimap \mathbb{M} 0 (\text{Nat} [n_2^{(j_n + 1)}]))) \multimap$$

$$\mathbb{M} 0 ((\text{Nat} [n_2^0] \otimes [(C 0 + \dots + C (n_2 - 1) + n_2)] \mathbf{1}) \multimap \mathbb{M} 0 (\text{Nat } (n_1^{n_2})))$$

$$T_{1.21} =$$

$$!(\forall j_n. ((\text{Nat} [n_2^{j_n}] \otimes [(n_1 * (n_1^{j_n} * 3 + n_1^{j_n} + 4) + n_1 + 4)] \mathbf{1}) \multimap \mathbb{M} 0 (\text{Nat} [n_2^{(j_n + 1)}]))) \multimap$$

$$\mathbb{M} 0 ((\text{Nat} [n_2^0] \otimes [P] \mathbf{1}) \multimap \mathbb{M} 0 (\text{Nat } (n_1^{n_2})))$$

$$\begin{aligned}
P &= \\
(\lambda k. (n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 4)) 0 + \dots + (\lambda k. (n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 4)) (n_2 - 1) + n_2 \\
T_{1.22} &= !(\forall k. ((\text{Nat}[n_2^k] \otimes [(n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 4)] \mathbf{1}) \multimap \mathbb{M} 0 (\text{Nat}[n_2^{(k+1)}]))) \\
T_{1.23} &= (\forall k. ((\text{Nat}[n_2^k] \otimes [(n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 4)] \mathbf{1}) \multimap \mathbb{M} 0 (\text{Nat}[n_2^{(k+1)}]))) \\
T_{1.24} &= ((\text{Nat}[n_2^k] \otimes [(n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 4)] \mathbf{1}) \multimap \mathbb{M} 0 (\text{Nat}[n_2^{(k+1)}])) \\
T_{1.241} &= (\text{Nat}[n_2^k] \otimes [(n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 4)] \mathbf{1}) \\
T_{1.2411} &= \text{Nat}[n_2^k] \\
T_{1.2412} &= [(n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 4)] \mathbf{1} \\
T_{1.242} &= \mathbb{M} 0 (\text{Nat}[n_2^{(k+1)}]) \\
T_{1.3} &= \mathbb{M} 0 ((\text{Nat}[n_2^0] \otimes [P] \mathbf{1}) \multimap \mathbb{M} 0 (\text{Nat}(n_1^{n_2})) \\
T_{1.30} &= \mathbb{M} 1 ((\text{Nat}[n_2^0] \otimes [P] \mathbf{1}) \multimap \mathbb{M} 0 (\text{Nat}(n_1^{n_2})) \\
T_{1.31} &= ((\text{Nat}[n_2^0] \otimes [P] \mathbf{1}) \multimap \mathbb{M} 0 (\text{Nat}(n_1^{n_2})) \\
T_2 &= \text{Nat } n_1 \\
T_3 &= (\text{Nat}[n_1^k] \multimap \mathbb{M} 0 (\text{Nat}[n_1^{(k+1)}]))
\end{aligned}$$

D3:

$$\frac{}{.; n_1, n_2; .; .; \overline{N_2} : T_1 \vdash \overline{N_2} : T_1}$$

D2.1:

$$\begin{array}{c}
D3 \\
\frac{}{.; n_1, n_2; .; .; . \vdash (\lambda_t k. \text{Nat}[n_2^k]) : \mathbb{N} \rightarrow \text{Type}} \\
\frac{}{.; n_1, n_2; .; .; \overline{N_2} : T_1 \vdash \overline{N_2} \square : T_{1.1}} \\
\frac{}{.; n_1, n_2; .; .; \overline{N_2} : T_1 \vdash \overline{N_2} \square : T_{1.2}}
\end{array}$$

D2:

$$\begin{array}{c}
D2.1 \\
\frac{}{.; n_1, n_2; .; .; . \vdash (\lambda_s k. (n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 2)) : \mathbb{N} \rightarrow \mathbb{N}} \\
\frac{}{.; n_1, n_2; .; .; \overline{N_2} : T_1 \vdash \overline{N_2} \square \square : T_{1.21}}
\end{array}$$

D1.32

$$\frac{}{.; n_1, n_2, k; .; .; b_2 : [(n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 2)] \mathbf{1} \vdash \text{mult} \square \square b_2 \uparrow^1 \overline{N_1} : \mathbb{M} 1 T_3}$$

D1.31

$$\frac{}{.; n_1, n_2, k; .; .; . \vdash \text{store} () : \mathbb{M}((n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 2)) [(n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 2)] \mathbf{1}}$$

D1.32

$$\frac{}{.; n_1, n_2, k; .; .; y_1 : T_{1.241}, b_1 : T_3 \vdash (\text{bind } b_2 = \text{store} () \text{ in } \text{mult} \square \square b_2 \uparrow^1 \overline{N_1}) : \mathbb{M}(n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 3) T_3}$$

D1.3

$$\begin{array}{c}
D1.31 \\
\frac{}{.; n_1, n_2, k; .; .; y_1 : T_{1.241}, b_1 : T_3 \vdash b_1 \uparrow^1 y_1 : \mathbb{M} 1 \text{Nat}[n_2^{(k+1)}]} \\
\frac{}{.; n_1, n_2, k; .; .; y_1 \vdash \text{bind } b_1 = (\text{bind } b_2 = \text{store} () \text{ in } \text{mult} \square \square b_2 \uparrow^1 \overline{N_1}) \text{ in } b_1 \uparrow^1 y_1 : \mathbb{M}(n_1 * (n_1^k * 3 + n_1^k + 4) + n_1 + 4) \text{Nat}[n_2^{(k+1)}]}
\end{array}$$

D1.2:

$$\frac{\frac{\frac{}{.; n_1, n_2, k; .; .; y_2 : T_{1.2412} \vdash y_2 : T_{1.2412}}{D1.3}}{.; n_1, n_2, k; .; .; y_1 : T_{1.2411}, y_2 : T_{1.2412} \vdash \text{release } - = y_2 \text{ in bind } b_1 = (\text{bind } b_2 = \text{store } () \text{ in } \text{mult } b_2 \text{ } n_1 (n_1^k) \uparrow^1 \overline{N}_1) \text{ in } b_1 \uparrow^1 y_1 : \mathbb{M}0 \text{Nat}[n_2^{(k+1)}]}{D1.1}}{D1.2}$$

D1:

$$\frac{\frac{\frac{\frac{}{.; n_1, n_2; .; .; . \vdash (\Lambda. \lambda t. \text{let} \langle\langle y_1, y_2 \rangle\rangle = t \text{ in } E0.1) : T_{1.23}}{D2}}{.; n_1, n_2; .; .; . \vdash !(\Lambda. \lambda t. \text{let} \langle\langle y_1, y_2 \rangle\rangle = t \text{ in } E0.1) : T_{1.22}}{.; n_1, n_2; .; .; \overline{N}_1 : T_1 \vdash \overline{N}_1 \square \square \uparrow^1 (\Lambda. \lambda t. \text{let} \langle\langle y_1, y_2 \rangle\rangle = t \text{ in } E0.1) : T_{1.30}}{D1.1}}{D1}$$

D0.1:

$$\frac{\frac{}{D1}}{.; n_1, n_2; .; .; \overline{N}_1 : T_1, \overline{N}_2 : T_2 \vdash E_1 : T_{1.30}}$$

D2.1:

$$\frac{}{.; n_1, n_2; .; .; \overline{N}_2 : T_2, a : T_{1.31}, b : T_b \vdash a \uparrow^1 \langle\langle \overline{1}, b \rangle\rangle : T_{0.4}}$$

D2.0:

$$\frac{\frac{\frac{}{.; n_1, n_2; .; .; . \vdash \text{store } () : \mathbb{M}(P-2) T_b}}{D2.1}}{.; n_1, n_2; .; .; \overline{N}_1 : T_1, \overline{N}_2 : T_2, a : T_{1.31} \vdash \text{bind } b = \text{store } () \text{ in } a \uparrow^1 \langle\langle \overline{1}, b \rangle\rangle : T_{0.5}}$$

D0.2:

$$\frac{\frac{}{D2.0}}{.; n_1, n_2; .; .; \overline{N}_1 : T_1, \overline{N}_2 : T_2, a : T_{1.31} \vdash E_2 : T_{0.5}}$$

D0:

$$\frac{\frac{}{D0.1} \quad \frac{}{D0.2}}{.; n_1, n_2; .; .; \overline{N}_1 : T_1, \overline{N}_2 : T_2 \vdash \text{bind } a = E_1 \text{ in } E_2 : T_{0.5}}$$

D0.0

$$\frac{\frac{\frac{}{.; n_1, n_2; .; .; p : T_p \vdash p : T_p}}{D0}}{.; n_1, n_2; .; .; p : T_p, \overline{N}_1 : T_1, \overline{N}_2 : T_2 \vdash \text{release } - = p \text{ in bind } a = E_1 \text{ in } E_2 : T_{0.6}}$$

Main derivation:

$$\frac{\frac{\frac{\frac{\frac{\frac{\frac{}{.; n_1, n_2; .; .; p : T_p, \overline{N}_1 : T_1 \vdash \lambda \overline{N}_2. E_0 : T_{0.31}}{.; n_1, n_2; .; .; p : T_p, \overline{N}_1 : T_1 \vdash \text{ret } \lambda \overline{N}_2. E_0 : T_{0.3}}{.; n_1, n_2; .; .; p : T_p \vdash \lambda \overline{N}_1. \text{ret } \lambda \overline{N}_2. E_0 : T_{0.21}}{.; n_1, n_2; .; .; p : T_p \vdash \text{ret } \lambda \overline{N}_1. \text{ret } \lambda \overline{N}_2. E_0 : T_{0.20}}{.; n_1, n_2; .; .; . \vdash \lambda p. \text{ret } \lambda \overline{N}_1. \text{ret } \lambda \overline{N}_2. E_0 : T_{0.2}}{.; n_1; .; .; . \vdash \Lambda. \lambda p. \text{ret } \lambda \overline{N}_1. \text{ret } \lambda \overline{N}_2. E_0 : T_{0.1}}{.; .; .; . \vdash \Lambda. \Lambda. \lambda p. \text{ret } \lambda \overline{N}_1. \text{ret } \lambda \overline{N}_2. E_0 : T_0}}{D0.0}}{Main derivation}$$

3.3 Fold

$$\frac{}{\Psi; \Theta; \Delta; \cdot; \cdot \vdash 0 : \text{Nat}(0)} \text{T-nat} \qquad \frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash n : \text{Nat}(n)}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash n - 1 : \text{Nat}(n - 1)} \text{T-sub}$$

$foldr : \forall \alpha, \beta, n, C : \mathbb{N} \rightarrow \mathbb{R}^+.$

$!(\forall i. [C\ i] \mathbf{1} \multimap \text{Nat}(i) \multimap \alpha \multimap \beta \multimap \mathbb{M} 0 \beta) \multimap !\text{Nat}(n) \multimap \beta \multimap L^n \alpha \multimap [\sum_{i < n} C\ i] \mathbf{1} \multimap \mathbb{M} 0 \beta$

$foldr \triangleq \text{fix } f'. \Lambda. \Lambda. \Lambda. \Lambda. \lambda f\ c\ s\ ls\ p.$

let $!f_u = f$ in

let $!c_u = c$ in

match ls with

| nil \mapsto ret s

| $h :: t$ \mapsto release $_ = p$ in

bind $p' = \text{store}()$ in

bind $p'' = \text{store}()$ in

bind $tr = f' \ \square \square \square \square \ !f_u \ !c_u \ (c_u - 1) \ s \ t \ p''$ in
 $(f_u \ \square \ p' \ (c_u - 1) \ h \ tr)$

Listing 1: fold function

$E_0 = \text{fix } f'. E_1$

$E_1 = \Lambda. \Lambda. \Lambda. \Lambda. E_2$

$E_2 = \lambda f\ c\ s\ ls\ p. E_3$

$E_3 = \text{let } !f_u = f \text{ in } E_{4,0}$

$E_{4,0} = \text{let } !n_u = n \text{ in } E_4$

$E_4 = \text{match } ls \text{ with } |nil \mapsto \text{ret } s |h :: t \mapsto E_5$

$E_5 = \text{release } _ = p \text{ in } E_6$

$E_6 = \text{bind } p' = \text{store}() \text{ in } E_7$

$E_7 = \text{bind } p'' = \text{store}() \text{ in } E_8$

$E_8 = \text{bind } tr = f' \ \square \square \square \square \ !f_u \ !c_u \ (c_u - 1) \ s \ t \ p'' \text{ in } E_9$

$E_9 = (f \ \square \ p' \ (c_u - 1) \ h \ tr)$

$T_0 = \forall \alpha, \beta, n, C : \mathbb{N} \rightarrow \mathbb{R}^+. T_1$

$T_1 = !T_2 \multimap !T_{2,1} \multimap \beta \multimap T_3 \multimap T_4 \multimap T_5$

$T_2 = \forall i. [C\ i] \mathbf{1} \multimap \text{Nat}(i) \multimap \alpha \multimap \beta \multimap \mathbb{M} 0 \beta$

$T_{2,0} = [C\ (n - 1)] \mathbf{1} \multimap \text{Nat}(n - 1) \multimap \alpha \multimap \beta \multimap \mathbb{M} 0 \beta$

$T_{2,01} = \text{Nat}(n - 1) \multimap \alpha \multimap \beta \multimap \mathbb{M} 0 \beta$

$T_{2,02} = \alpha \multimap \beta \multimap \mathbb{M} 0 \beta$

$T_{2,03} = \beta \multimap \mathbb{M} 0 \beta$

$T_{2,1} = \text{Nat}(n)$

$T_3 = L^n \alpha$

$T_4 = [\sum_{i < n} C\ i] \mathbf{1}$

$T_{4,0} = \mathbb{M}(\sum_{i < n} C\ i) [(\sum_{i < n} C\ i)] \beta$

$T_{4,1} = [(C(n - 1))] \mathbf{1}$

$T_{4,10} = \mathbb{M}(C(n - 1)) [(C(n - 1))] \mathbf{1}$

$T_{4,2} = [\sum_{i < n-1} C\ i] \mathbf{1}$

$T_{4,20} = \mathbb{M}(\sum_{i < n-1} C\ i) [(\sum_{i < n-1} C\ i)] \mathbf{1}$

$T_5 = \mathbb{M} 0 \beta$

$T_6 = \mathbb{M}(\sum_{i < n-1} C\ i) \beta$

$T_6 = \mathbb{M}(\sum_{i < n} C\ i) \beta$

D6.5:

$$\frac{}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; \cdot \vdash f_u : T_2}$$

D6.4:

$$\frac{D6.5 \quad \frac{}{\alpha, \beta, n, C; n; n > 0 \vdash n - 1 : \mathbb{N}}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; h : \alpha, p' : T_{4.1}, tr : \beta \vdash f_u \square : T_{2.0}}$$

D6.3:

$$\frac{D6.4 \quad \frac{}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; p' : T_{4.1} \vdash p' : T_{4.1}}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; h : \alpha, p' : T_{4.1}, tr : \beta \vdash f_u \square p' : T_{2.01}}$$

D6.2:

$$D6.3 \quad \frac{\frac{}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; \cdot \vdash c_u : T_{2.1}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; \cdot \vdash (c_u - 1) : \text{Nat}(n - 1)}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; p' : T_{4.1} \vdash f_u \square p' (c_u - 1) : T_{2.02}}$$

D6.1:

$$\frac{D6.2 \quad \frac{}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; p' : T_{4.1} \vdash h : \alpha}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; h : \alpha, p' : T_{4.1} \vdash f_u \square p' (c_u - 1) h : T_{2.03}}$$

D6:

$$\frac{D6.1 \quad \frac{}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; tr : \beta \vdash tr : \beta}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; h : \alpha, p' : T_{4.1}, tr : \beta \vdash (f_u \square p' (c_u - 1) h tr) : T_5}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; h : \alpha, p' : T_{4.1}, tr : \beta \vdash E_9 : T_5}$$

D5.5:

$$\frac{\frac{}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; \cdot \vdash f' \square \square : \alpha, \beta, n, C; n; n > 0 \vdash n - 1 : \mathbb{N}}{\forall n, C : \mathbb{N} \rightarrow \mathbb{R}^+ .!T_2 \multimap !T_{2.1} \multimap \beta \multimap L^{n-1}\alpha \multimap T_{4.2} \multimap T_5}}{\frac{}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; \cdot \vdash f' \square \square \square : \forall C : \mathbb{N} \rightarrow \mathbb{R}^+ .!T_2 \multimap !T_{2.1} \multimap \beta \multimap L^{n-1}\alpha \multimap T_{4.2} \multimap T_5}}$$

D5.4:

$$\frac{D5.5 \quad \frac{}{\alpha, \beta, n, C; n; n > 0 \vdash C : \mathbb{N} \rightarrow \mathbb{R}^+}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; \cdot \vdash f' \square \square \square \square :!T_2 \multimap !T_{2.1} \multimap \beta \multimap L^{n-1}\alpha \multimap T_{4.2} \multimap T_5}$$

D5.3:

$$\frac{D5.4 \quad \frac{}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; \cdot \vdash !f_u :!T_2}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; \cdot \vdash f' \square \square \square \square !f_u :!T_{2.1} \multimap \beta \multimap L^{n-1}\alpha \multimap T_{4.2} \multimap T_5}$$

D5.21:

$$\frac{D5.3 \quad \frac{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; \cdot \vdash !(c_u - 1) : !T_{2.1}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; \cdot \vdash f' \square \square \square \square !f_u !(c_u - 1) : \beta \multimap L^{n-1}\alpha \multimap T_{4.2} \multimap T_5}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; \cdot \vdash f' \square \square \square \square !f_u !(c_u - 1) : \beta \multimap L^{n-1}\alpha \multimap T_{4.2} \multimap T_5}}$$

D5.2:

$$\frac{D5.21 \quad \frac{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; s : \beta \vdash s : \beta}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; s : \beta \vdash f' \square \square \square \square !f_u !(c_u - 1) s : L^{n-1}\alpha \multimap T_{4.2} \multimap T_5}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; s : \beta \vdash f' \square \square \square \square !f_u !(c_u - 1) s : L^{n-1}\alpha \multimap T_{4.2} \multimap T_5}}$$

D5.1:

$$\frac{D5.2 \quad \frac{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; t : L^{n-1}\alpha \vdash t : L^{n-1}\alpha}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; s : \beta, t : L^{n-1}\alpha \vdash f' \square \square \square \square !f_u !(c_u - 1) s t : T_{4.2} \multimap T_5}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; s : \beta, t : L^{n-1}\alpha \vdash f' \square \square \square \square !f_u !(c_u - 1) s t : T_{4.2} \multimap T_5}}$$

D5:

$$\frac{D5.1 \quad \frac{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; p'' : T_{4.2} \vdash p'' : T_{4.2}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; s : \beta, t : L^{n-1}\alpha, p'' : T_{4.2} \vdash f' \square \square \square \square !f_u !(c_u - 1) s t p'' : T_5}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; s : \beta, t : L^{n-1}\alpha, p'' : T_{4.2} \vdash f' \square \square \square \square !f_u !(c_u - 1) s t p'' : T_5}} \quad D6$$

$$\frac{\frac{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; s : \beta, h : \alpha, t : L^{n-1}\alpha, p' : T_{4.1}, p'' : T_{4.2} \vdash \text{bind } tr = f' \square \square \square \square !f_u !(c_u - 1) s t p'' \text{ in } E_9 : T_5}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; s : \beta, h : \alpha, t : L^{n-1}\alpha, p' : T_{4.1}, p'' : T_{4.2} \vdash E_8 : T_5}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; s : \beta, h : \alpha, t : L^{n-1}\alpha, p' : T_{4.1}, p'' : T_{4.2} \vdash E_8 : T_5}}$$

D4.1:

$$\frac{\frac{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; \cdot \vdash \text{store}() : T_{4.20}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; s : \beta, h : \alpha, t : L^{n-1}\alpha, p' : T_{4.1} \vdash \text{bind } p'' = \text{store}() \text{ in } E_8 : T_6}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; s : \beta, h : \alpha, t : L^{n-1}\alpha, p' : T_{4.1} \vdash E_7 : T_6}} \quad E_5$$

D4:

$$\frac{\frac{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; \cdot \vdash \text{store}() : T_{4.10}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; s : \beta, h : \alpha, t : L^{n-1}\alpha \vdash \text{bind } p' = \text{store}() \text{ in } E_7 : T_7}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; s : \beta, h : \alpha, t : L^{n-1}\alpha \vdash E_6 : T_7}} \quad D4.1$$

D3:

$$\frac{\frac{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; p : T_4 \vdash p : T_4}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; s : \beta, h : \alpha, t : L^{n-1}\alpha, p : T_4 \vdash \text{release } _ = p \text{ in } E_6 : T_5}}{\alpha, \beta, n, C; n; n > 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; s : \beta, h : \alpha, t : L^{n-1}\alpha, p : T_4 \vdash E_5 : T_5}} \quad D4$$

D2:

$$\frac{\alpha, \beta, n, C; n; n = 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; s : \beta, p : T_4 \vdash \text{ret } s : T_5}{\alpha, \beta, n, C; n; n = 0; c_u : T_{2.1}, f_u : T_2, f' : T_0; s : \beta, p : T_4 \vdash \text{ret } s : T_5}}$$

D1:

$$\frac{\frac{\alpha, \beta, n, C; n; \cdot; c_u : T_{2.1}, f_u : T_2, f' : T_0; s : \beta, ls : T_3, p : T_4 \vdash ls : T_3}{\alpha, \beta, n, C; n; \cdot; c_u : T_{2.1}, f_u : T_2, f' : T_0; s : \beta, ls : T_3, p : T_4 \vdash \text{match } ls \text{ with } |nil \mapsto \text{ret } s| h :: t \mapsto E_5 : T_5}}{\alpha, \beta, n, C; n; \cdot; c_u : T_{2.1}, f_u : T_2, f' : T_0; s : \beta, ls : T_3, p : T_4 \vdash E_4 : T_5}} \quad D2 \quad D3$$

D0.0:

$$\frac{\frac{\alpha, \beta, n, C; n; \cdot; f' : T_0; f : !T_2, c : !T_{2.1}, s : \beta, ls : T_3, p : T_4 \vdash c : !T_{2.1}}{D1}}{\alpha, \beta, n, C; n; \cdot; f_u : T_2, f' : T_0; c : !T_{2.1}, s : \beta, ls : T_3, p : T_4 \vdash \text{let } !c_u = c \text{ in } E_4 : D0.0}$$

D0:

$$\frac{\frac{\alpha, \beta, n, C; n; \cdot; f' : T_0; f : !T_2, c : !T_{2.1}, s : \beta, ls : T_3, p : T_4 \vdash f : !T_2}{D0.0}}{\alpha, \beta, n, C; n; \cdot; f' : T_0; f : !T_2, c : !T_{2.1}, s : \beta, ls : T_3, p : T_4 \vdash \text{let } !f_u = f \text{ in } E_{4.0} : T_5}$$

Main derivation:

$$\frac{\frac{\frac{\frac{\frac{\alpha, \beta, n, C; n; \cdot; f' : T_0; f : !T_2, c : !T_{2.1}, s : \beta, ls : T_3, p : T_4 \vdash E_3 : T_5}{D0}}{\alpha, \beta, n, C; \cdot; \cdot; f' : T_0; \cdot \vdash E_2 : T_1}}{\cdot; \cdot; \cdot; f' : T_0; \cdot \vdash E_1 : T_0}}{\cdot; \cdot; \cdot; \cdot \vdash E_0 : T_0}}$$

3.4 Append

$append : \forall s_1, s_2. L^{s_1}[1] \tau \multimap L^{s_2} \tau \multimap \mathbb{M} 0 (L^{s_1+s_2} \tau)$

$append \triangleq \text{fix } f. \Lambda. \Lambda. \lambda l_1 l_2. E_0$

$E_0 = \text{match } l_1 \text{ with } | \text{nil} \mapsto E_{0.1} \mid h :: t \mapsto E_{0.2}$

$E_{0.1} = \text{ret } \text{nil} :: l_2$

$E_{0.2} = \text{release } h_e = h \text{ in bind } t_e = f \square \square t l_2 \text{ in } E_{0.3}$

$E_{0.3} = \text{bind } - = \uparrow^1 \text{ in ret } h_e :: t_e$

Typing derivation

$E_0 = \text{match } l_1 \text{ with } | \text{nil} \mapsto E_{0.1} \mid h :: t \mapsto E_{0.2}$

$E_{0.1} = \text{ret } \text{nil} :: l_2$

$E_{0.2} = \text{release } h_e = h \text{ in bind } t_e = f \square \square t l_2 \text{ in } E_{0.3}$

$E_{0.3} = \text{bind } - = \uparrow^1 \text{ in ret } h_e :: t_e$

$T_0 = \forall s_1, s_2. L^{s_1}[1] \tau \multimap L^{s_2} \tau \multimap \mathbb{M} 0 (L^{s_1+s_2} \tau)$

$T_1 = L^{s_1}[1] \tau \multimap L^{s_2} \tau \multimap \mathbb{M} 0 (L^{s_1+s_2} \tau)$

$T_{1.1} = L^{s_1}[1] \tau$

$T_{1.2} = L^{s_2} \tau$

$T_{1.3} = \mathbb{M} 0 (L^{s_1+s_2} \tau)$

$T_2 = L^{s_2} \tau \multimap \mathbb{M} s_1 (L^{s_1+s_2} \tau)$

D1.2:

$$\frac{\cdot; s_1, s_2; s_1 > 0; f : T_0; h_e : \tau, t_e : L^{s_1-1+s_2} \tau \vdash (h_e :: t_e) : L^{s_1+s_2} \tau}{\cdot; s_1, s_2; s_1 > 0; f : T_0; h_e : \tau, t_e : L^{s_1-1+s_2} \tau \vdash \text{ret}(h_e :: t_e) : \mathbb{M} 0 (L^{s_1+s_2} \tau)}$$

D1.1:

$$\frac{\frac{\cdot; s_1, s_2; s_1 > 0; \cdot \vdash \uparrow^1 : \mathbb{M} 1 \mathbf{1}}{D1.2}}{\cdot; s_1, s_2; s_1 > 0; f : T_0; h_e : \tau, t_e : L^{s_1-1+s_2} \tau \vdash \text{bind } - = \uparrow^1 \text{ in ret } h_e :: t_e : \mathbb{M} 1 (L^{s_1+s_2} \tau)}$$

D1.0:

$$\frac{\overline{.; s_1, s_2; s_1 > 0; f : T_0; t : L^{s_1-1}\tau, l_2 : L^{s_2}\tau \vdash f[] t l_2 : \mathbb{M}(0) (L^{s_1-1+s_2}\tau)} \quad D1.1}{.; s_1, s_2; s_1 > 0; f : T_0; h_e : \tau, t : L^{s_1-1}\tau, l_2 : L^{s_2}\tau \vdash \text{bind } t_e = f[] t l_2 \text{ in ret}(h_e :: t_e) : \mathbb{M}1 (L^{s_1+s_2}\tau)}$$

D1:

$$\frac{\overline{.; s_1, s_2; s_1 > 0; f : T_0; h : [1]\tau \vdash h : [1]\tau} \quad D1.0}{.; s_1, s_2; s_1 > 0; f : T_0; h : [1]\tau, t : L^{s_1-1}\tau, l_2 : T_{1.2} \vdash E_{0.2} : \mathbb{M}0 (L^{(s_1+s_2)}\tau)}$$

D0:

$$\frac{\overline{.; s_1, s_2; s_1 = 0; f : T_0; l_2 : T_{1.2} \vdash l_2 : L^{s_2}\tau}}{.; s_1, s_2; s_1 = 0; f : T_0; l_2 : T_{1.2} \vdash \text{ret } l_2 : \mathbb{M}0 (L^{s_1+s_2})\tau}$$

Main derivation:

$$\frac{\overline{.; s_1, s_2; .; f : T_0; l_1 : T_{1.1} \vdash l_1 : T_{1.1}} \quad D0 \quad D1}{.; s_1, s_2; .; f : T_0; l_1 : T_{1.1}, l_2 : T_{1.2} \vdash E_0 : \mathbb{M}0 (L^{s_1+s_2}\tau)} \quad \frac{.; s_1, s_2; .; f : T_0; . \vdash \lambda l_1 l_2. E_0 : T_1}{.; .; .; f : T_0; . \vdash \Lambda. \Lambda. \lambda l_1 l_2. E_0 : T_0} \quad \frac{.; .; .; .; . \vdash \text{fix } f. \Lambda. \Lambda. \lambda l_1 l_2. E_0 : T_0}{.}$$

3.5 Map

$$\text{map} : \forall n, c.!(\tau_1 \multimap \mathbb{M} c \tau_2) \multimap L^n([c] \tau_1) \multimap \mathbb{M}0 (L^n \tau_2)$$

$$\text{map} \triangleq$$

$$\text{fix } f. \Lambda. \Lambda. \lambda gl. \text{let } !g_u = g \text{ in } E_0$$

$$E_0 = \text{match } l \text{ with } | \text{nil} \mapsto E_{0.1} \mid h :: t \mapsto E_{0.2}$$

$$E_{0.1} = \text{ret } \text{nil}$$

$$E_{0.2} = \text{release } h_e = h \text{ in } E_{0.3}$$

$$E_{0.3} = \text{bind } h_n = g_u h_e \text{ in } E_{0.4}$$

$$E_{0.4} = \text{bind } t_n = f[] !g_u t \text{ in ret } h_n :: t_n$$

Typing derivation

$$E = \text{fix } f. \Lambda. \Lambda. \lambda gl. \text{let } !g_u = g \text{ in } E_0$$

$$E_0 = \text{match } l \text{ with } | \text{nil} \mapsto E_{0.1} \mid h :: t \mapsto E_{0.2}$$

$$E_{0.1} = \text{ret } \text{nil}$$

$$E_{0.2} = \text{release } h_e = h \text{ in } E_{0.3}$$

$$E_{0.3} = \text{bind } h_n = g_u h_e \text{ in } E_{0.4}$$

$$E_{0.4} = \text{bind } t_n = f[] !g_u t \text{ in ret } h_n :: t_n$$

$$E_1 = \Lambda. \Lambda. \lambda gl. \text{let } !g_u = g \text{ in } E_0$$

$$E_2 = \lambda gl. \text{let } !g_u = g \text{ in } E_0$$

$$E_3 = \text{let } !g_u = g \text{ in } E_0$$

$$T_0 = \forall n, c.!(\tau_1 \multimap \mathbb{M} c \tau_2) \multimap L^n([c] \tau_1) \multimap \mathbb{M}0 (L^n \tau_2)$$

$$T_1 = !(\tau_1 \multimap \mathbb{M} c \tau_2) \multimap L^n([c] \tau_1) \multimap \mathbb{M}0 (L^n \tau_2)$$

$$T_{1.1} = (\tau_1 \multimap \mathbb{M} c \tau_2)$$

$$T_{1.2} = L^n([c] \tau_1)$$

$$T_{1.3} = \mathbb{M}0 (L^n \tau_2)$$

D1.2:

$$\frac{}{.; n, c; n > 0; f : T_0, g_u : T_{1.1}; h_n : \tau_2, t_n : L^{n-1}\tau_2 \vdash \mathbf{ret} h_n :: t_n : \mathbb{M} 0 L^n \tau_2}$$

D1.1:

$$\frac{.; n, c; n > 0; f : T_0, g_u : T_{1.1}; h_n : \tau_2 \vdash f \square \square !g_u t : \mathbb{M} 0 L^{n-1} \tau_2 \quad D1.2}{.; n, c; n > 0; f : T_0, g_u : T_{1.1}; h_n : \tau_2, t : L^{n-1}([c] \tau_1) \vdash E_{0.4} : \mathbb{M} 0 L^n \tau_2}$$

D1.0:

$$\frac{.; n, c; n > 0; f : T_0, g_u : T_{1.1}; h_e : \tau_1 \vdash (g_u h_e) : \mathbb{M} c \tau_2 \quad D1.1}{.; n, c; n > 0; f : T_0, g_u : T_{1.1}; h_e : \tau_1, t : L^{n-1}([c] \tau_1) \vdash E_{0.3} : \mathbb{M} c L^n \tau_2}$$

D1:

$$\frac{.; n, c; n > 0; f : T_0, g_u : T_{1.1}; h : [c] \tau_1 \vdash h : [c] \tau_1 \quad D1.0}{.; n, c; n > 0; f : T_0, g_u : T_{1.1}; h : [c] \tau_1, t : L^{n-1}([c] \tau_1) \vdash E_{0.2} : \mathbb{M} 0 L^n \tau_2}$$

D0:

$$\frac{.; n, c; n = 0; f : T_0, g_u : T_{1.1}; \cdot \vdash \mathbf{nil} : L^0 \tau_2}{.; n, c; n = 0; f : T_0, g_u : T_{1.1}; \cdot \vdash \mathbf{ret} \mathbf{nil} : \mathbb{M} 0 L^n \tau_2}$$

$$\frac{}{.; n, c; n = 0; f : T_0, g_u : T_{1.1}; \cdot \vdash E_{0.1} : \mathbb{M} 0 L^n \tau_2}$$

Main derivation:

$$\frac{\frac{.; n, c; f : T_0; g : !T_{1.1} \vdash g : !T_{1.1} \quad \frac{.; n, c; ; f : T_0, g_u : T_{1.1}; l : T_{1.2} \vdash l : T_{1.2} \quad D0 \quad D1}{.; n, c; ; f : T_0, g_u : T_{1.1}; l : T_{1.2} \vdash E_0 : \mathbb{M} 0 L^n \tau_2}}{.; n, c; ; f : T_0; g : !T_{1.1}, l : T_{1.2} \vdash E_3 : \mathbb{M} 0 L^n \tau_2}}{.; n, c; ; f : T_0; \cdot \vdash E_2 : T_1}}{.; ; ; f : T_0; \cdot \vdash E_1 : T_0}}{.; ; ; ; \cdot \vdash E : T_0}$$

3.6 Okasaki's implicit queue

Typing rules for value constructors and case analysis

$$\begin{array}{c}
\frac{}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash C0 : Queue \tau} \text{T-C0} \qquad \frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : \tau}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash C1 e : Queue \tau} \text{T-C1} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : [1] \mathbf{1} \multimap \mathbb{M}0(\tau \otimes Queue(\tau \otimes \tau))}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash C2 e : Queue \tau} \text{T-C2} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : [0] \mathbf{1} \multimap \mathbb{M}0((\tau \otimes Queue(\tau \otimes \tau)) \otimes \tau)}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash C3 e : Queue \tau} \text{T-C3} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : [2] \mathbf{1} \multimap \mathbb{M}0((\tau \otimes \tau) \otimes Queue(\tau \otimes \tau))}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash C4 e : Queue \tau} \text{T-C4} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash e : [1] \mathbf{1} \multimap \mathbb{M}0(((\tau \otimes \tau) \otimes Queue(\tau \otimes \tau)) \otimes \tau)}{\Psi; \Theta; \Delta; \Omega; \Gamma \vdash C5 e : Queue \tau} \text{T-C5} \\
\\
\frac{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \vdash e : (Queue \tau) \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2 \vdash e_0 : \tau' \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : \tau \vdash e_1 : \tau' \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : [1] \mathbf{1} \multimap \mathbb{M}0(\tau \otimes Queue(\tau \otimes \tau)) \vdash e_2 : \tau' \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : [0] \mathbf{1} \multimap \mathbb{M}0((\tau \otimes Queue(\tau \otimes \tau)) \otimes \tau) \vdash e_3 : \tau' \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : [2] \mathbf{1} \multimap \mathbb{M}0((\tau \otimes \tau) \otimes Queue(\tau \otimes \tau)) \vdash e_4 : \tau' \quad \Psi; \Theta; \Delta; \Omega; \Gamma_2, x : [1] \mathbf{1} \multimap \mathbb{M}0(((\tau \otimes \tau) \otimes Queue(\tau \otimes \tau)) \otimes \tau) \vdash e_5 : \tau'}{\Psi; \Theta; \Delta; \Omega; \Gamma_1 \oplus \Gamma_2 \vdash \text{case } e \text{ of } |C0 \mapsto e_0 |C1 x \mapsto e_1 |C2 x \mapsto e_2 |C3 x \mapsto e_3 |C4 x \mapsto e_4 |C5 x \mapsto e_5 : \tau'} \text{T-caseIQ} \\
\\
\text{snoc} : [2] \mathbf{1} \multimap \forall \alpha. Queue \alpha \multimap \alpha \multimap \mathbb{M}0 Queue \alpha \\
\text{fix snoc. } \lambda p. \Lambda. \lambda q. a. \\
- = \text{release } p \text{ in } - = \uparrow^1; \text{ret} \\
\text{case } q \text{ of} \\
|C0 \mapsto \text{ret } C1 \ a \\
|C1 \ x \mapsto \text{ret } C4 \ (\lambda p''. \text{ret} \langle \langle \langle x, a \rangle \rangle, C0 \rangle \rangle) \\
|C2 \ x \mapsto \\
\text{bind } p' = \text{store}() \text{ in} \\
\text{bind } x' = x \ p' \text{ in} \\
\text{let} \langle \langle f, m \rangle \rangle = x' \text{ in} \\
\text{ret}(C3 \ (\lambda p''. \langle \langle \langle f, m \rangle \rangle, a \rangle \rangle)) \\
|C3 \ x \mapsto \\
\text{bind } p' = \text{store}() \text{ in} \\
\text{bind } x' = x \ p' \text{ in let} \langle \langle fm, r \rangle \rangle = x' \text{ in} \\
\text{let} \langle \langle f, m \rangle \rangle = fm \text{ in bind } p_o = \text{store}() \text{ in} \\
\text{ret } C2 \ (\lambda p''. \\
- = \text{release } p_o \text{ in } - = \text{release } p'' \text{ in bind } p''' = \text{store}() \text{ in} \\
\text{bind } m' = \text{snoc } p''' \ m \ (r, a) \text{ in ret} \langle \langle f, m' \rangle \rangle)
\end{array}$$


```

| C4 x ↦
  bind p' = store() in
    ret C5 (λp''.
      – = release p' in – = release p'' in
      bind p''' = store() in let⟨⟨f, m⟩⟩ = x p''' in
      ret⟨⟨⟨f, m⟩⟩, a⟩)

```

```

| C5 x ↦
  bind p' = store() in
    bind x' = x p' in
      let⟨⟨fm, r⟩⟩ = x' in let⟨⟨f, m⟩⟩ = fm in
        ret(C4 (λp''.
          bind m' = snoc p'' m in ret⟨⟨f, m'⟩⟩)

```

Listing 2: snoc function

```

E0.0 = – = release p in E0.1
E0.1 = – = ↑1; E0.2
E0.2 = case q of | C0 ↦ E0 | C1 x ↦ E1 | C2 x ↦ E2 | C3 x ↦ E3 | C4 x ↦ E4 | C5 x ↦ E5
E0 = ret(C1 a)
E1 = ret C4 (λp''. ret⟨⟨⟨x, a⟩⟩, C0⟩)
E2 = bind p' = store() in E2.1
E2.1 = bind x' = x p' in E2.2
E2.2 = let⟨⟨f, m⟩⟩ = x' in E2.3
E2.3 = ret(C3 (λp''.⟨⟨⟨f, m⟩⟩, a⟩))
E3 = bind p' = store() in E3.1
E3.1 = bind x' = x p' in E3.2
E3.2 = let⟨⟨fm, r⟩⟩ = x' in E3.3
E3.3 = let⟨⟨f, m⟩⟩ = fm in E3.31
E3.31 = bind po = store() in E3.4
E3.4 = ret C2 (λp''. E3.41)
E3.41 = – = release po in – = release p'' in bind p''' = store() in E3.42
E3.42 = bind m' = snoc p''' m (r, a) in ret⟨⟨f, m'⟩⟩
E4 = bind p' = store() in E4.1
E4.1 = ret C5 (λp''. E4.11)
E4.11 = – = release p' in – = release p'' in E4.12
E4.12 = bind p''' = store() in let⟨⟨f, m⟩⟩ = x p''' in E4.13
E4.13 = ret⟨⟨⟨f, m⟩⟩, a⟩
E5 = bind p' = store() in E5.1
E5.1 = bind x' = x p' in E5.2
E5.2 = let⟨⟨fm, r⟩⟩ = x' in E5.3
E5.3 = let⟨⟨f, m⟩⟩ = fm in E5.4
E5.4 = ret(C4 (λp''. bind m' = snoc p'' m in ret⟨⟨f, m'⟩⟩))

```

```

T0.0 = [2] 1 ↦ ∃α. Queue α ↦ α ↦ M0 Queue α
T0 = M0 Queue α

```

$T_1 = \mathbb{M}1 \text{ Queue } \alpha$
 $T_2 = \mathbb{M}2 \text{ Queue } \alpha$
 $T_3 = \mathbb{M}0 (\alpha \otimes \text{Queue } (\alpha \otimes \alpha))$
 $T_{3.1} = (\alpha \otimes \text{Queue } (\alpha \otimes \alpha))$
 $T_{3.2} = \text{Queue } (\alpha \otimes \alpha)$
 $T_4 = \mathbb{M}0 (\alpha \otimes \text{Queue } (\alpha \otimes \alpha) \otimes \alpha)$
 $T_{4.1} = (\alpha \otimes \text{Queue } (\alpha \otimes \alpha) \otimes \alpha)$
 $T_{4.2} = \alpha \otimes \text{Queue } (\alpha \otimes \alpha)$
 $T_{4.3} = \text{Queue } (\alpha \otimes \alpha)$
 $T_5 = [2] \mathbf{1} \multimap \mathbb{M}0 (\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha)$
 $T_{5.1} = \mathbb{M}0 (\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha)$
 $T_{5.2} = (\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha)$
 $T_{5.3} = (\alpha \otimes \alpha)$
 $T_{5.4} = \text{Queue } (\alpha \otimes \alpha)$
 $T_6 = [1] \mathbf{1} \multimap \mathbb{M}0 ((\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha) \otimes \alpha)$
 $T_{6.1} = \mathbb{M}0 ((\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha) \otimes \alpha)$
 $T_{6.2} = ((\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha) \otimes \alpha)$
 $T_{6.3} = (\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha)$
 $T_{6.4} = (\alpha \otimes \alpha)$
 $T_{6.5} = \text{Queue } (\alpha \otimes \alpha)$
 $T_7 = \mathbb{M}0 (\alpha \otimes \text{Queue } (\alpha \otimes \alpha))$
 $T_{7.1} = \mathbb{M}1 (\alpha \otimes \text{Queue } (\alpha \otimes \alpha))$
 $T_{7.2} = \mathbb{M}2 (\alpha \otimes \text{Queue } (\alpha \otimes \alpha))$
 $T_8 = \mathbb{M}0 (((\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha)) \otimes \alpha)$
 $T_{8.1} = ((\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha)) \otimes \alpha$
 $T_9 = \mathbb{M}0 ((\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha))$
 $T_{9.1} = ((\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha))$

D5.5:

$$\frac{\alpha; .; .; S : T_{0.0}; a : \alpha, f : T_{6.4}, m : T_{6.5}, p'' : [2] \mathbf{1}, m' : \text{Queue } (\alpha \otimes \alpha) \vdash \langle\langle f, m' \rangle\rangle : T_{9.1}}{\alpha; .; .; S : T_{0.0}; a : \alpha, f : T_{6.4}, m : T_{6.5}, p'' : [2] \mathbf{1}, m' : \text{Queue } (\alpha \otimes \alpha) \vdash \text{ret}\langle\langle f, m' \rangle\rangle : T_9}$$

D5.4:

$$\frac{\alpha; .; .; S : T_{0.0}; r : \alpha, a : \alpha, m : T_{6.5}, p'' : [2] \mathbf{1} \vdash S p'' \square m \langle\langle r, a \rangle\rangle : \mathbb{M}0 (\text{Queue } (\alpha \otimes \alpha))}{D5.5}$$

$$\frac{\alpha; .; .; S : T_{0.0}; r : \alpha, a : \alpha, f : T_{6.4}, m : T_{6.5}, p'' : [2] \mathbf{1} \vdash \text{bind } m' = S p'' \square m \langle\langle r, a \rangle\rangle \text{ in } \text{ret}\langle\langle f, m' \rangle\rangle : T_9}{\alpha; .; .; S : T_{0.0}; r : \alpha, a : \alpha, f : T_{6.4}, m : T_{6.5} \vdash (\lambda p''. \text{bind } m' = S p'' \square m \langle\langle r, a \rangle\rangle \text{ in } \text{ret}\langle\langle f, m' \rangle\rangle) : [2] \mathbf{1} \multimap T_9}$$

$$\frac{\alpha; .; .; S : T_{0.0}; r : \alpha, a : \alpha, f : T_{6.4}, m : T_{6.5} \vdash (C4 (\lambda p''. \text{bind } m' = S p'' \square m \langle\langle r, a \rangle\rangle \text{ in } \text{ret}\langle\langle f, m' \rangle\rangle)) : \text{Queue } \alpha}{\alpha; .; .; S : T_{0.0}; r : \alpha, a : \alpha, f : T_{6.4}, m : T_{6.5} \vdash \text{ret}(C4 (\lambda p''. \text{bind } m' = S p'' \square m \langle\langle r, a \rangle\rangle \text{ in } \text{ret}\langle\langle f, m' \rangle\rangle)) : T_0}$$

$$\alpha; .; .; S : T_{0.0}; r : \alpha, a : \alpha, f : T_{6.4}, m : T_{6.5} \vdash E_{5.4} : T_0$$

D5.3:

$$\frac{\alpha; .; .; S : T_{0.0}; fm : T_{6.3} \vdash fm : T_{6.3}}{D5.4}$$

$$\frac{\alpha; .; .; S : T_{0.0}; a : \alpha, fm : T_{6.3}, r : \alpha \vdash \text{let}\langle\langle f, m \rangle\rangle = fm \text{ in } E_{5.4} : T_0}{\alpha; .; .; S : T_{0.0}; a : \alpha, fm : T_{6.3}, r : \alpha \vdash E_{5.3} : T_0}$$

D5.2:

$$\frac{\frac{\alpha; .; .; S : T_{0.0}; x' : T_{6.2} \vdash x' : T_{6.2}}{D5.3}}{\frac{\alpha; .; .; S : T_{0.0}; a : \alpha, x' : T_{6.2} \vdash \text{let}\langle\langle f, m, r \rangle\rangle = x' \text{ in } E_{5.3} : T_0}{\alpha; .; .; S : T_{0.0}; a : \alpha, x' : T_{6.2} \vdash E_{5.2} : T_0}}$$

D5.1:

$$\frac{\frac{\alpha; .; .; S : T_{0.0}; x : T_6, p' : [1] \mathbf{1} \vdash x p' : T_{6.1}}{D5.2}}{\frac{\alpha; .; .; S : T_{0.0}; a : \alpha, x : T_6, p' : [1] \mathbf{1} \vdash \text{bind } x' = x p' \text{ in } E_{5.2} : T_0}{\alpha; .; .; S : T_{0.0}; a : \alpha, x : T_6, p' : [1] \mathbf{1} \vdash E_{5.1} : T_0}}$$

D5:

$$\frac{\alpha; .; .; S : T_{0.0}; \cdot \vdash \text{store}() : \mathbb{M}1([1] \mathbf{1})}{D5.1}}{\alpha; .; .; S : T_{0.0}; a : \alpha, x : T_6 \vdash E_5 : T_1}$$

D4.5:

$$\frac{\frac{\alpha; .; .; S : T_{0.0}; a : \alpha, x : T_5, f : T_{5.3}, m : T_{5.4} \vdash \langle\langle\langle f, m \rangle\rangle, a \rangle\rangle : T_{8.1}}{\alpha; .; .; S : T_{0.0}; a : \alpha, x : T_5, f : T_{5.3}, m : T_{5.4} \vdash \text{ret}\langle\langle\langle f, m \rangle\rangle, a \rangle\rangle : T_8}}{\alpha; .; .; S : T_{0.0}; a : \alpha, x : T_5, f : T_{5.3}, m : T_{5.4} \vdash E_{4.13} : T_8}$$

D4.4:

$$\frac{\frac{\alpha; .; .; S : T_{0.0}; x : T_5, p''' : [2] \mathbf{1} \vdash x p''' : T_{5.1}}{D4.5}}{\alpha; .; .; S : T_{0.0}; a : \alpha, x : T_5, p''' : [2] \mathbf{1} \vdash \text{let}\langle\langle f, m \rangle\rangle = x p''' \text{ in } E_{4.13} : T_8}$$

D4.3:

$$\frac{\frac{\alpha; .; .; S : T_{0.0}; a : \alpha, x : T_5 \vdash \text{store}() : \mathbb{M}2([2] \mathbf{1})}{D4.4}}{\frac{\alpha; .; .; S : T_{0.0}; a : \alpha, x : T_5 \vdash \text{bind } p''' = \text{store}() \text{ in } \text{let}\langle\langle f, m \rangle\rangle = x p''' \text{ in } E_{4.13} : T_{8.2}}{\alpha; .; .; S : T_{0.0}; a : \alpha, x : T_5 \vdash E_{4.12} : T_{8.2}}}}$$

D4.2:

$$\frac{\frac{\alpha; .; .; S : T_{0.0}; p'' : [1] \mathbf{1} \vdash p'' : [1] \mathbf{1}}{D4.3}}{\alpha; .; .; S : T_{0.0}; a : \alpha, x : T_5, p'' : [1] \mathbf{1} \vdash - = \text{release } p'' \text{ in } E_{4.12} : T_{8.1}}$$

D4.11:

$$\frac{\frac{\alpha; .; .; S : T_{0.0}; p' : [1] \mathbf{1} \vdash p' : [1] \mathbf{1}}{D4.2}}{\alpha; .; .; S : T_{0.0}; a : \alpha, x : T_5, p' : [1] \mathbf{1}, p'' : [1] \mathbf{1} \vdash - = \text{release } p' \text{ in } - = \text{release } p'' \text{ in } E_{4.12} : T_8}$$

D4.1:

$$\frac{\frac{\frac{\alpha; .; .; S : T_{0.0}; a : \alpha, x : [2] \mathbf{1} \multimap \mathbb{M}0(\alpha \otimes \alpha) \otimes \text{Queue}(\alpha \otimes \alpha), p' : [1] \mathbf{1}, p'' : [1] \mathbf{1} \vdash E_{4.11} : T_8}{D4.11}}{\alpha; .; .; S : T_{0.0}; a : \alpha, x : [2] \mathbf{1} \multimap \mathbb{M}0(\alpha \otimes \alpha) \otimes \text{Queue}(\alpha \otimes \alpha), p' : [1] \mathbf{1} \vdash (\lambda p''. E_{4.11}) : [1] \mathbf{1} \multimap T_8}}{\frac{\alpha; .; .; S : T_{0.0}; a : \alpha, x : [2] \mathbf{1} \multimap \mathbb{M}0(\alpha \otimes \alpha) \otimes \text{Queue}(\alpha \otimes \alpha), p' : [1] \mathbf{1} \vdash C5(\lambda p''. E_{4.11}) : \text{Queue } \alpha}{\alpha; .; .; S : T_{0.0}; a : \alpha, x : [2] \mathbf{1} \multimap \mathbb{M}0(\alpha \otimes \alpha) \otimes \text{Queue}(\alpha \otimes \alpha), p' : [1] \mathbf{1} \vdash \text{ret } C5(\lambda p''. E_{4.11}) : T_0}}{\alpha; .; .; S : T_{0.0}; a : \alpha, x : [2] \mathbf{1} \multimap \mathbb{M}0(\alpha \otimes \alpha) \otimes \text{Queue}(\alpha \otimes \alpha), p' : [1] \mathbf{1} \vdash E_{4.1} : T_0}}$$

D4:

$$\frac{\frac{}{\alpha; .; .; S : T_{0.0}; . \vdash \text{store}() : \mathbb{M} 1 ([1] \mathbf{1})} \quad D4.1}{\alpha; .; .; S : T_{0.0}; a : \alpha, x : [2] \mathbf{1} \multimap \mathbb{M} 0 (\alpha \otimes \alpha) \otimes \text{Queue} (\alpha \otimes \alpha) \vdash E_4 : T_1}$$

D3.43:

$$\frac{}{\alpha; .; .; S : T_{0.0}; f : \alpha, m' : \text{Queue} (\alpha \otimes \alpha) \vdash \text{ret}\langle\langle f, m' \rangle\rangle : T_7}$$

D3.42:

$$\frac{\frac{}{\alpha; .; .; S : T_{0.0}; m : T_{4.3}, r : \alpha, a : \alpha, p''' : [2] \mathbf{1} \vdash S p''' \square m (r, a) : \mathbb{M} 0 (\text{Queue} (\alpha \otimes \alpha))} \quad D3.43}{\frac{\frac{}{\alpha; .; .; S : T_{0.0}; f : \alpha, m : T_{4.3}, r : \alpha, a : \alpha, p''' : [2] \mathbf{1} \vdash \text{bind } m' = S p''' \square m (r, a) \text{ in } \text{ret}\langle\langle f, m' \rangle\rangle : T_7}}{\alpha; .; .; S : T_{0.0}; f : \alpha, m : T_{4.3}, r : \alpha, a : \alpha, p''' : [2] \mathbf{1} \vdash E_{3.42} : T_7}}$$

D3.41:

$$\frac{\frac{}{\alpha; .; .; S : T_{0.0}; . \vdash \text{store}() : \mathbb{M} 2 ([2] \mathbf{1})} \quad D3.42}{\alpha; .; .; S : T_{0.0}; f : \alpha, m : T_{4.3}, r : \alpha, a : \alpha \vdash \text{bind } p''' = \text{store}() \text{ in } E_{3.42} : T_{7.2}}$$

D3.401:

$$\frac{\frac{}{\alpha; .; .; S : T_{0.0}; p'' : [1] \mathbf{1} \vdash p'' : [1] \mathbf{1}} \quad D3.41}{\alpha; .; .; S : T_{0.0}; f : \alpha, m : T_{4.3}, r : \alpha, a : \alpha, p'' : [1] \mathbf{1} \vdash - = \text{release } p'' \text{ in } \text{bind } p''' = \text{store}() \text{ in } E_{3.42} : T_{7.1}}$$

D3.40:

$$\frac{\frac{\frac{}{\alpha; .; .; S : T_{0.0}; p_o : [1] \mathbf{1} \vdash p_o : [1] \mathbf{1}} \quad D3.401}{\frac{\alpha; .; .; S : T_{0.0}; f : \alpha, m : T_{4.3}, r : \alpha, a : \alpha, p_o : [1] \mathbf{1}, p'' : [1] \mathbf{1} \vdash - = \text{release } p_o \text{ in } - = \text{release } p'' \text{ in } \text{bind } p''' = \text{store}() \text{ in } E_{3.42} : T_7}}{\frac{\alpha; .; .; S : T_{0.0}; f : \alpha, m : T_{4.3}, r : \alpha, a : \alpha, p_o : [1] \mathbf{1} \vdash \lambda p'' . - = \text{release } p_o \text{ in } - = \text{release } p'' \text{ in } \text{bind } p''' = \text{store}() \text{ in } E_{3.42} : [1] \mathbf{1} \multimap T_7}}{\alpha; .; .; S : T_{0.0}; f : \alpha, m : T_{4.3}, r : \alpha, a : \alpha, p_o : [1] \mathbf{1} \vdash (\lambda p'' . E_{3.41}) : [1] \mathbf{1} \multimap T_7}}$$

D3.4:

$$\frac{\frac{\frac{}{\alpha; .; .; S : T_{0.0}; a : \alpha, f : \alpha, m : T_{4.3}, r : \alpha, a : \alpha, p_o : [1] \mathbf{1} \vdash C2 (\lambda p'' . E_{3.41}) : \text{Queue } \alpha} \quad D3.40}{\frac{\alpha; .; .; S : T_{0.0}; a : \alpha, f : \alpha, m : T_{4.3}, r : \alpha, a : \alpha, p_o : [1] \mathbf{1} \vdash \text{ret } C2 (\lambda p'' . E_{3.41}) : T_1}}{\alpha; .; .; S : T_{0.0}; a : \alpha, f : \alpha, m : T_{4.3}, r : \alpha, a : \alpha, p_o : [1] \mathbf{1} \vdash E_{3.4} : T_1}}$$

D3.31:

$$\frac{\frac{}{\alpha; .; .; S : T_{0.0}; a : \alpha, f : \alpha, m : T_{4.3}, r : \alpha \vdash \text{store}() : \mathbb{M} 1 [1] \mathbf{1}} \quad D3.4}{\frac{\alpha; .; .; S : T_{0.0}; a : \alpha, f : \alpha, m : T_{4.3}, r : \alpha \vdash \text{bind } p_o = \text{store}() \text{ in } E_{3.4} : T_1}{\alpha; .; .; S : T_{0.0}; a : \alpha, f : \alpha, m : T_{4.3}, r : \alpha \vdash E_{3.31} : T_1}}$$

D3.3:

$$\frac{\frac{\alpha; .; .; S : T_{0.0}; fm : T_{4.2} \vdash fm : T_{4.2}}{D3.4}}{\alpha; .; .; S : T_{0.0}; a : \alpha, fm : T_{4.2}, r : \alpha \vdash \text{let}\langle\langle f, m \rangle\rangle = fm \text{ in } E_{3.31} : T_1} \frac{}{\alpha; .; .; S : T_{0.0}; a : \alpha, fm : T_{4.2}, r : \alpha \vdash E_{3.3} : T_1}$$

D3.2:

$$\frac{\frac{\alpha; .; .; S : T_{0.0}; x' : T_{4.1} \vdash x' : T_{4.1}}{D3.3}}{\alpha; .; .; S : T_{0.0}; \cdot \vdash \text{let}\langle\langle fm, r \rangle\rangle = x' \text{ in } E_{3.3} : T_1} \frac{}{\alpha; .; .; S : T_{0.0}; a : \alpha, x' : T_{4.1} \vdash E_{3.2} : T_1}$$

D3.1:

$$\frac{\frac{\alpha; .; .; S : T_{0.0}; x : [0] \mathbf{1} \multimap \mathbb{M} 0 (\alpha \otimes Queue (\alpha \otimes \alpha) \otimes \alpha), p' : [0] \mathbf{1} \vdash x p' : T_4}{D3.2}}{\alpha; .; .; S : T_{0.0}; a : \alpha, x : [0] \mathbf{1} \multimap \mathbb{M} 0 (\alpha \otimes Queue (\alpha \otimes \alpha) \otimes \alpha) \vdash E_{3.1} : T_1}$$

D3:

$$\frac{\frac{\alpha; .; .; S : T_{0.0}; \vdash \text{store}() : \mathbb{M} 0 ([0] \mathbf{1})}{D3.1}}{\alpha; .; .; S : T_{0.0}; a : \alpha, x : [0] \mathbf{1} \multimap \mathbb{M} 0 (\alpha \otimes Queue (\alpha \otimes \alpha) \otimes \alpha) \vdash E_3 : T_1}$$

D2.3:

$$\frac{\frac{\alpha; .; .; S : T_{0.0}; q : Queue \alpha, a : \alpha, f : \alpha, m : T_{3.2} \vdash (C\mathcal{I} (\lambda p'' . \langle\langle\langle f, m \rangle\rangle, a \rangle\rangle)) : Queue \alpha}{\alpha; .; .; S : T_{0.0}; q : Queue \alpha, a : \alpha, f : \alpha, m : T_{3.2} \vdash \text{ret}(C\mathcal{I} (\lambda p'' . \langle\langle\langle f, m \rangle\rangle, a \rangle\rangle)) : T_0}}{\alpha; .; .; S : T_{0.0}; q : Queue \alpha, a : \alpha, f : \alpha, m : T_{3.2} \vdash E_{2.3} : T_0}$$

D2.2:

$$\frac{\frac{\alpha; .; .; S : T_{0.0}; x' : T_{3.1} \vdash x' : T_{3.1}}{D2.3}}{\alpha; .; .; S : T_{0.0}; a : \alpha, x' : T_{3.1} \vdash \text{let}\langle\langle f, m \rangle\rangle = x' \text{ in } E_{2.3} : T_0} \frac{}{\alpha; .; .; S : T_{0.0}; a : \alpha, x' : T_{3.1} \vdash E_{2.2} : T_0}$$

D2.1:

$$\frac{\frac{\alpha; .; .; S : T_{0.0}; x : ([1] \mathbf{1} \multimap \mathbb{M} 0 (\alpha \otimes Queue (\alpha \otimes \alpha))), p' : [1] \mathbf{1} \vdash x p' : T_3}{D2.2}}{\alpha; .; .; S : T_{0.0}; a : \alpha, x : ([1] \mathbf{1} \multimap \mathbb{M} 0 (\alpha \otimes Queue (\alpha \otimes \alpha))), p' : [1] \mathbf{1} \vdash E_{2.1} : T_0}$$

D2:

$$\frac{\frac{\alpha; .; .; S : T_{0.0}; \cdot \vdash \text{store}() : \mathbb{M} 1 ([1] \mathbf{1})}{D2.1}}{\alpha; .; .; S : T_{0.0}; a : \alpha, x : ([1] \mathbf{1} \multimap \mathbb{M} 0 (\alpha \otimes Queue (\alpha \otimes \alpha))) \vdash E_2 : T_1}$$

D1:

$$\frac{\frac{\frac{\alpha; .; .; S : T_{0.0}; a : \alpha, x : \alpha \vdash C_4 (\lambda p'' . \text{ret}\langle\langle\langle x, a \rangle\rangle, C0 \rangle\rangle) : Queue \alpha}{\alpha; .; .; S : T_{0.0}; a : \alpha, x : \alpha \vdash \text{ret } C_4 (\lambda p'' . \text{ret}\langle\langle\langle x, a \rangle\rangle, C0 \rangle\rangle) : T_0}}{\alpha; .; .; S : T_{0.0}; a : \alpha, x : \alpha \vdash \text{ret } C_4 (\lambda p'' . \text{ret}\langle\langle\langle x, a \rangle\rangle, C0 \rangle\rangle) : T_1} \frac{}{\alpha; .; .; S : T_{0.0}; a : \alpha, x : \alpha \vdash E_1 : T_1}$$

D0:

$$\frac{\frac{\alpha; .; .; S : T_{0.0}; a : \alpha \vdash C1\ a : Queue\ \alpha}{\alpha; .; .; S : T_{0.0}; a : \alpha \vdash \text{ret}(C1\ a) : \mathbb{M}\ 1\ Queue\ \alpha}}{\alpha; .; .; S : T_{0.0}; a : \alpha \vdash E_0 : T_1}$$

D0.2:

$$\frac{\alpha; .; .; S : T_{0.0}; q : Queue\ \alpha \vdash q : Queue\ \alpha \quad D0 \quad D1 \quad D2 \quad D3 \quad D4 \quad D5}{\alpha; .; .; S : T_{0.0}; q : Queue\ \alpha, a : \alpha \vdash E_{0.2} : T_1}$$

D0.1:

$$\frac{\frac{\alpha; .; .; S : T_{0.0}; \cdot \vdash \uparrow^1 : \mathbb{M}\ 1\ \mathbf{1}}{\alpha; .; .; S : T_{0.0}; q : Queue\ \alpha, a : \alpha \vdash E_{0.1} : T_2} \quad D0.2}{\alpha; .; .; S : T_{0.0}; p : [2]\ \mathbf{1} \vdash p : [2]\ \mathbf{1}} \quad D0.1$$

Main derivation:

$$\frac{\frac{\alpha; .; .; S : T_{0.0}; p : [2]\ \mathbf{1} \vdash p : [2]\ \mathbf{1}}{\alpha; .; .; S : T_{0.0}; p : [2]\ \mathbf{1}, q : Queue\ \alpha, a : \alpha \vdash E_{0.0} : T_0} \quad D0.1}{.; .; .; .; \cdot \vdash \text{fix}f.\lambda p.\Lambda.\lambda q.\lambda a.E_{0.0} : T_{0.0}}$$

$head : [3]\ \mathbf{1} \multimap \forall \alpha. Queue\ \alpha \multimap \mathbb{M}\ 0\ \alpha$

$head \triangleq \lambda p.\Lambda.\lambda q.$

$\text{bind}\ ht = headTail\ p\ []\ q\ \text{in}\ \text{ret}\ \text{fst}(ht)$

Listing 3: head function

$E_0 = \text{bind}\ ht = headTail\ p\ []\ q\ \text{in}\ E_1$

$E_1 = \text{ret}(\text{fst}(ht))$

$T_0 = [3]\ \mathbf{1} \multimap \forall \alpha. Queue\ \alpha \multimap \mathbb{M}\ 0\ \alpha$

D0:

$$\frac{\frac{\alpha; .; .; .; q : Queue\ \alpha, ht : (\alpha \otimes Queue\ \alpha) \vdash \text{fst}(ht) : \alpha}{\alpha; .; .; .; q : Queue\ \alpha, ht : (\alpha \otimes Queue\ \alpha) \vdash \text{ret}(\text{fst}(ht)) : \mathbb{M}\ 0\ \alpha}}{\alpha; .; .; .; q : Queue\ \alpha, ht : (\alpha \otimes Queue\ \alpha) \vdash E_1 : \mathbb{M}\ 0\ \alpha}$$

Main derivation:

$$\frac{\frac{\alpha; .; .; .; q : Queue\ \alpha \vdash headTail\ p\ []\ q : \mathbb{M}\ 0\ (\alpha \otimes Queue\ \alpha)}{\alpha; .; .; .; q : Queue\ \alpha \vdash \text{bind}\ ht = headTail\ p\ []\ q\ \text{in}\ E_1 : \mathbb{M}\ 0\ \alpha} \quad D0}{\alpha; .; .; .; p : [3]\ \mathbf{1}, q : Queue\ \alpha \vdash E_0 : \mathbb{M}\ 0\ \alpha}}{.; .; .; .; \cdot \vdash \lambda p.\Lambda.\lambda q.E_0 : T_0}$$

$tail : [3] \mathbf{1} \multimap \forall \alpha. Queue \ \alpha \multimap \mathbb{M}0(Queue \ \alpha)$
 $tail \triangleq \lambda p. \Lambda. \lambda q.$
 $bind \ ht = headTail \ p \ [] \ q \text{ in } ret \ snd(ht)$

Listing 4: tail function

$E_0 = bind \ ht = headTail \ p \ [] \ q \text{ in } E_1$
 $E_1 = ret(snd(ht))$

$T_0 = [3] \mathbf{1} \multimap \forall \alpha. Queue \ \alpha \multimap \mathbb{M}0(Queue \ \alpha)$

D0:

$$\frac{\frac{\alpha; .; .; .; q : Queue \ \alpha, ht : (\alpha \otimes Queue \ \alpha) \vdash snd(ht) : Queue \ \alpha}{\alpha; .; .; .; q : Queue \ \alpha, ht : (\alpha \otimes Queue \ \alpha) \vdash ret(snd(ht)) : \mathbb{M}0(Queue \ \alpha)}}{\alpha; .; .; .; q : Queue \ \alpha, ht : (\alpha \otimes Queue \ \alpha) \vdash E_1 : \mathbb{M}0(Queue \ \alpha)}$$

Main derivation:

$$\frac{\frac{\frac{\alpha; .; .; .; q : Queue \ \alpha \vdash headTail \ p \ [] \ q : \mathbb{M}0(\alpha \otimes Queue \ \alpha)}{\alpha; .; .; .; q : Queue \ \alpha \vdash bind \ ht = headTail \ p \ [] \ q \text{ in } E_1 : \mathbb{M}0(Queue \ \alpha)}}{\alpha; .; .; .; p : [3] \mathbf{1}, q : Queue \ \alpha \vdash E_0 : \mathbb{M}0(Queue \ \alpha)}}{.; .; .; .; \vdash \lambda p. \Lambda. \lambda q. E_0 : T_0} \quad D0$$

$headTail : [3] \mathbf{1} \multimap \forall \alpha. Queue \ \alpha \multimap \mathbb{M}0(\alpha \otimes Queue \ \alpha)$
 $headTail \triangleq fix \ HT. \lambda p. \Lambda. \lambda q.$

$- = release \ p \text{ in } - = \uparrow^1; ret$

case q of

| $C0 \mapsto fix \ x. x$

| $C1 \ x \mapsto ret \langle \langle x, C0 \rangle \rangle$

| $C2 \ x \mapsto$

$bind \ p' = store() \text{ in } bind \ p_o = store() \text{ in}$

$bind \ x' = x \ p' \text{ in } let \langle \langle f, m \rangle \rangle = x' \text{ in}$

$ret \langle \langle f, (C4 \ (\lambda p''. - = release \ p_o \text{ in } - = release \ p'' \text{ in } bind \ p_r = store() \text{ in } HT \ p_r \ [] \ m)) \rangle \rangle$

| $C3 \ x \mapsto$

$bind \ p' = store() \text{ in } bind \ p_o = store() \text{ in}$

$bind \ x' = x \ p' \text{ in } let \langle \langle f, m, r \rangle \rangle = x' \text{ in } let \langle \langle f, m \rangle \rangle = f \ m \text{ in}$

$ret \langle \langle f, (C5 \ (\lambda p''. - = release \ p_o \text{ in } - = release \ p'' \text{ in}$

$bind \ p''' = store() \text{ in } bind \ ht = HT \ p''' \ [] \ m \text{ in } ret \langle \langle ht, r \rangle \rangle \rangle \rangle$

| $C4 \ x \mapsto$

$bind \ p' = store() \text{ in } bind \ x' = x \ p' \text{ in } let \langle \langle f, m \rangle \rangle = x' \text{ in } let \langle \langle f_1, f_2 \rangle \rangle = f \ \text{in}$

$ret \langle \langle f_1, C2 \ (\lambda p''. ret \langle \langle f_2, m \rangle \rangle) \rangle \rangle$

$|C5 \ x \mapsto$
 $\text{bind } p' = \text{store}() \text{ in bind } x' = x \ p' \text{ in let}\langle\langle fm, r \rangle\rangle = x' \text{ in let}\langle\langle f, m \rangle\rangle = fm \text{ in let}\langle\langle f_1, f_2 \rangle\rangle = f \text{ in}$
 $\text{ret}\langle\langle f_1, (C3 \ (\lambda p''. \text{ret}\langle\langle\langle f_2, m \rangle\rangle, r) \rangle\rangle)\rangle$

Listing 5: head and tail function

$E_{0.0} = \text{fix}HT.\lambda p.\Lambda.\lambda q.E_{0.1}$
 $E_{0.1} = - = \text{release } p \text{ in } - = \uparrow^1; E_{0.2}$
 $E_{0.2} = \text{case } q \text{ of } |C0 \mapsto E_0 |C1 \ x \mapsto E_1 |C2 \ x \mapsto E_2 |C3 \ x \mapsto E_3 |C4 \ x \mapsto E_4 |C5 \ x \mapsto E_5$
 $E_0 = \text{fix}x.x$
 $E_1 = \text{ret}\langle\langle x, C0 \rangle\rangle$
 $E_2 = \text{bind } p' = \text{store}() \text{ in } E_{2.0}$
 $E_{2.0} = \text{bind } p_o = \text{store}() \text{ in } E_{2.1}$
 $E_{2.1} = \text{bind } x' = x \ p' \text{ in } E_{2.11}$
 $E_{2.11} = \text{let}\langle\langle f, m \rangle\rangle = x' \text{ in } E_{2.2}$
 $E_{2.2} = \text{ret}\langle\langle f, (C4 \ (\lambda p''. E_{2.3})) \rangle\rangle$
 $E_{2.3} = - = \text{release } p_o \text{ in } E_{2.4}$
 $E_{2.4} = - = \text{release } p'' \text{ in } E_{2.5}$
 $E_{2.5} = \text{bind } p_r = \text{store}() \text{ in } HT \ p_r \ \square \ m$
 $E_3 = \text{bind } p' = \text{store}() \text{ in } E_{3.0}$
 $E_{3.0} = \text{bind } p_o = \text{store}() \text{ in } E_{3.1}$
 $E_{3.1} = \text{bind } x' = x \ p' \text{ in } E_{3.11}$
 $E_{3.11} = \text{let}\langle\langle fm, r \rangle\rangle = x' \text{ in } E_{3.12}$
 $E_{3.12} = \text{let}\langle\langle f, m \rangle\rangle = fm \text{ in } E_{3.2}$
 $E_{3.2} = \text{ret}\langle\langle f, E_{3.3} \rangle\rangle$
 $E_{3.3} = C5 \ (\lambda p''. E_{3.31})$
 $E_{3.4} = - = \text{release } p_o \text{ in } E_{3.41}$
 $E_{3.41} = \text{release } p'' \text{ in } E_{3.5}$
 $E_{3.5} = \text{bind } p''' = \text{store}() \text{ in } E_{3.6}$
 $E_{3.6} = \text{bind } ht = HT \ p''' \ \square \ m \text{ in ret}\langle\langle ht, r \rangle\rangle$
 $E_4 = \text{bind } p' = \text{store}() \text{ in } E_{4.1}$
 $E_{4.1} = \text{bind } x' = x \ p' \text{ in } E_{4.2}$
 $E_{4.2} = \text{let}\langle\langle f, m \rangle\rangle = x' \text{ in } E_{4.3}$
 $E_{4.3} = \text{let}\langle\langle f_1, f_2 \rangle\rangle = f \text{ in } E_{4.4}$
 $E_{4.4} = \text{ret}\langle\langle f_1, C2 \ (\lambda p''. \text{ret}\langle\langle f_2, m \rangle\rangle)\rangle$
 $E_5 = \text{bind } p' = \text{store}() \text{ in } E_{5.1}$
 $E_{5.1} = \text{bind } x' = x \ p' \text{ in } E_{5.2}$
 $E_{5.2} = \text{let}\langle\langle fm, r \rangle\rangle = x' \text{ in } E_{5.3}$
 $E_{5.3} = \text{let}\langle\langle f, m \rangle\rangle = fm \text{ in } E_{5.4}$
 $E_{5.4} = \text{let}\langle\langle f_1, f_2 \rangle\rangle = f \text{ in } E_{5.5}$
 $E_{5.5} = \text{ret}\langle\langle f_1, (C3 \ (\lambda p''. \text{ret}\langle\langle\langle f_2, m \rangle\rangle, r) \rangle\rangle)\rangle$

$T_{0.0} = [3] \ \mathbf{1} \multimap \forall \alpha. \text{Queue } \alpha \multimap \mathbb{M}0(\alpha \otimes \text{Queue } \alpha)$
 $T_{0.2} = [1] \ \mathbf{1} \multimap \mathbb{M}0(\alpha \otimes \text{Queue } (\alpha \otimes \alpha))$
 $T_{0.21} = \mathbb{M}0(\alpha \otimes \text{Queue } (\alpha \otimes \alpha))$
 $T_{0.22} = (\alpha \otimes \text{Queue } (\alpha \otimes \alpha))$

$$\begin{aligned}
T_{0.23} &= \text{Queue } (\alpha \otimes \alpha) \\
T_{0.3} &= [0] \mathbf{1} \multimap \mathbb{M}0((\alpha \otimes \text{Queue } (\alpha \otimes \alpha)) \otimes \alpha) \\
T_{0.31} &= \mathbb{M}0((\alpha \otimes \text{Queue } (\alpha \otimes \alpha)) \otimes \alpha) \\
T_{0.32} &= ((\alpha \otimes \text{Queue } (\alpha \otimes \alpha)) \otimes \alpha) \\
T_{0.33} &= (\alpha \otimes \text{Queue } (\alpha \otimes \alpha)) \\
T_{0.34} &= \text{Queue } (\alpha \otimes \alpha) \\
T_{0.4} &= [2] \mathbf{1} \multimap \mathbb{M}0((\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha)) \\
T_{0.41} &= \mathbb{M}0((\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha)) \\
T_{0.411} &= \mathbb{M}1((\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha)) \\
T_{0.413} &= \mathbb{M}3((\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha)) \\
T_{0.42} &= ((\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha)) \\
T_{0.43} &= (\alpha \otimes \alpha) \\
T_{0.44} &= \text{Queue } (\alpha \otimes \alpha) \\
T_{0.5} &= [1] \mathbf{1} \multimap \mathbb{M}0(((\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha)) \otimes \alpha) \\
T_{0.51} &= \mathbb{M}0(((\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha)) \otimes \alpha) \\
T_{0.511} &= \mathbb{M}1(((\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha)) \otimes \alpha) \\
T_{0.512} &= \mathbb{M}2(((\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha)) \otimes \alpha) \\
T_{0.513} &= \mathbb{M}3(((\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha)) \otimes \alpha) \\
T_{0.52} &= (((\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha)) \otimes \alpha) \\
T_{0.53} &= ((\alpha \otimes \alpha) \otimes \text{Queue } (\alpha \otimes \alpha)) \\
T_{0.54} &= (\alpha \otimes \alpha) \\
T_{0.55} &= \text{Queue } (\alpha \otimes \alpha) \\
T_0 &= \mathbb{M}0(\alpha \otimes \text{Queue } \alpha) \\
T_1 &= \mathbb{M}1(\alpha \otimes \text{Queue } \alpha) \\
T_2 &= \mathbb{M}2(\alpha \otimes \text{Queue } \alpha)
\end{aligned}$$

D5.51:

$$\frac{\frac{\alpha; .; .; HT : T_{0.0}; f_2 : \alpha, m : T_{0.55}, r : \alpha, p'' : [0] \mathbf{1} \vdash \text{ret}\langle\langle\langle f_2, m \rangle\rangle, r \rangle\rangle T_{0.31}}{\alpha; .; .; HT : T_{0.0}; f_2 : \alpha, m : T_{0.55}, r : \alpha \vdash (\lambda p''. \text{ret}\langle\langle\langle f_2, m \rangle\rangle, r \rangle\rangle) : T_{0.3}}}{\alpha; .; .; HT : T_{0.0}; f_2 : \alpha, m : T_{0.55}, r : \alpha \vdash (C^3 (\lambda p''. \text{ret}\langle\langle\langle f_2, m \rangle\rangle, r \rangle\rangle)) : \text{Queue } \alpha}$$

D5.5:

$$\frac{\frac{\frac{\alpha; .; .; HT : T_{0.0}; f_1 : \alpha \vdash f_1 : \alpha}{\alpha; .; .; HT : T_{0.0}; f_1 : \alpha, f_2 : \alpha, m : T_{0.55}, r : \alpha \vdash \langle\langle f_1, (C^3 (\lambda p''. \text{ret}\langle\langle\langle f_2, m \rangle\rangle, r \rangle\rangle)) \rangle\rangle : \alpha \otimes \text{Queue } \alpha}}{\alpha; .; .; HT : T_{0.0}; f_1 : \alpha, f_2 : \alpha, m : T_{0.55}, r : \alpha \vdash \text{ret}\langle\langle f_1, (C^3 (\lambda p''. \text{ret}\langle\langle\langle f_2, m \rangle\rangle, r \rangle\rangle)) \rangle\rangle : T_1}}{D5.51}{\alpha; .; .; HT : T_{0.0}; f_1 : \alpha, f_2 : \alpha, m : T_{0.55}, r : \alpha \vdash E_{5.5} : T_1}$$

D5.4:

$$\frac{\frac{\frac{\alpha; .; .; HT : T_{0.0}; f : T_{0.54} \vdash f : T_{0.54}}{\alpha; .; .; HT : T_{0.0}; f : T_{0.54}, m : T_{0.55}, r : \alpha \vdash \text{let}\langle\langle f_1, f_2 \rangle\rangle = f \text{ in } E_{5.5} : T_1}}{\alpha; .; .; HT : T_{0.0}; f : T_{0.54}, m : T_{0.55}, r : \alpha \vdash E_{5.4} : T_1}}{D5.5}$$

D5.3:

$$\frac{\frac{\alpha; .; .; HT : T_{0.0}; fm : T_{0.53} \vdash fm : T_{0.53}}{D5.4}}{\alpha; .; .; HT : T_{0.0}; fm : T_{0.53}, r : \alpha \vdash \text{let}\langle\langle f, m \rangle\rangle = fm \text{ in } E_{5.4} : T_1} \\ \alpha; .; .; HT : T_{0.0}; fm : T_{0.53}, r : \alpha \vdash E_{5.3} : T_1$$

D5.2:

$$\frac{\frac{\alpha; .; .; HT : T_{0.0}; x' : T_{0.52} \vdash x' : T_{0.52}}{D5.3}}{\alpha; .; .; HT : T_{0.0}; x' : T_{0.52} \vdash \text{let}\langle\langle fm, r \rangle\rangle = x' \text{ in } E_{5.3} : T_1} \\ \alpha; .; .; HT : T_{0.0}; x' : T_{0.52} \vdash E_{5.2} : T_1$$

D5.1:

$$\frac{\frac{\alpha; .; .; HT : T_{0.0}; x : T_{0.5}, p' : [1] \mathbf{1} \vdash x p' : T_{0.51}}{D5.2}}{\alpha; .; .; HT : T_{0.0}; x : T_{0.5}, p' : [1] \mathbf{1} \vdash \text{bind } x' = x p' \text{ in } E_{5.2} : T_1} \\ \alpha; .; .; HT : T_{0.0}; x : T_{0.5}, p' : [1] \mathbf{1} \vdash E_{5.1} : T_1$$

D5:

$$\frac{\frac{\alpha; .; .; HT : T_{0.0}; \cdot \vdash \text{store}() : \mathbb{M} 1 ([1] \mathbf{1})}{D5.1}}{\alpha; .; .; HT : T_{0.0}; x : T_{0.5} \vdash E_5 : T_2}$$

D4.41:

$$\frac{\frac{\alpha; .; .; HT : T_{0.0}; f_2 : \alpha, m : T_{0.44}, p'' : [1] \mathbf{1} \vdash \text{ret}\langle\langle f_2, m \rangle\rangle : T_{0.21}}{\alpha; .; .; HT : T_{0.0}; f_2 : \alpha, m : T_{0.44} \vdash (\lambda p''. \text{ret}\langle\langle f_2, m \rangle\rangle) : T_{0.2}}}{\alpha; .; .; HT : T_{0.0}; f_2 : \alpha, m : T_{0.44} \vdash C2 (\lambda p''. \text{ret}\langle\langle f_2, m \rangle\rangle) : Queue \alpha}$$

D4.4:

$$\frac{\frac{\frac{\alpha; .; .; HT : T_{0.0}; f_1 : \alpha \vdash f_1 : \alpha}{D4.41}}{\alpha; .; .; HT : T_{0.0}; f_1 : \alpha, f_2 : \alpha, m : T_{0.44} \vdash \langle\langle f_1, C2 (\lambda p''. \text{ret}\langle\langle f_2, m \rangle\rangle) \rangle\rangle : \alpha \otimes Queue \alpha}}{\alpha; .; .; HT : T_{0.0}; f_1 : \alpha, f_2 : \alpha, m : T_{0.44} \vdash \text{ret}\langle\langle f_1, C2 (\lambda p''. \text{ret}\langle\langle f_2, m \rangle\rangle) \rangle\rangle : T_0} \\ \alpha; .; .; HT : T_{0.0}; f_1 : \alpha, f_2 : \alpha, m : T_{0.44} \vdash E_{4.4} : T_0$$

D4.3:

$$\frac{\frac{\alpha; .; .; HT : T_{0.0}; f : T_{0.43} \vdash f : T_{0.43}}{D4.4}}{\alpha; .; .; HT : T_{0.0}; f : T_{0.43}, m : T_{0.44} \vdash \text{let}\langle\langle f_1, f_2 \rangle\rangle = f \text{ in } E_{4.4} : T_0} \\ \alpha; .; .; HT : T_{0.0}; f : T_{0.43}, m : T_{0.44} \vdash E_{4.3} : T_0$$

D4.2:

$$\frac{\frac{\alpha; .; .; HT : T_{0.0}; x' : T_{0.42} \vdash x' : T_{0.42}}{D4.3}}{\alpha; .; .; HT : T_{0.0}; x' : T_{0.42} \vdash \text{let}\langle\langle f, m \rangle\rangle = x' \text{ in } E_{4.3} : T_0} \\ \alpha; .; .; HT : T_{0.0}; x' : T_{0.42} \vdash E_{4.2} : T_0$$

D4.1:

$$\frac{\frac{\alpha; .; .; HT : T_{0.0}; x : T_{0.4}, p' : [2] \mathbf{1} \vdash x p' : T_{0.41}}{D4.2}}{\alpha; .; .; HT : T_{0.0}; x : T_{0.4}, p' : [2] \mathbf{1} \vdash \text{bind } x' = x p' \text{ in } E_{4.2} : T_0} \\ \alpha; .; .; HT : T_{0.0}; x : T_{0.4}, p' : [2] \mathbf{1} \vdash E_{4.1} : T_0$$

D4:

$$\frac{\alpha; .; .; HT : T_{0.0}; . \vdash \text{store}() : \mathbb{M} 2 [2] \mathbf{1}}{\alpha; .; .; HT : T_{0.0}; x : T_{0.4} \vdash E_4 : T_2} \quad D4.1$$

D3.61:

$$\frac{}{\alpha; .; .; HT : T_{0.0}; r : \alpha, ht : T_{0.53} \vdash \text{ret}\langle\langle ht, r \rangle\rangle : T_{0.51}}$$

D3.6:

$$\frac{\frac{\alpha; .; .; HT : T_{0.0}; m : T_{0.34}, r : \alpha, p''' : [3] \mathbf{1} \vdash HT p''' \square m : \mathbb{M} 0 T_{0.53}}{\alpha; .; .; HT : T_{0.0}; m : T_{0.34}, r : \alpha, p''' : [3] \mathbf{1} \vdash \text{bind } ht = HT p''' \square m \text{ in } \text{ret}\langle\langle ht, r \rangle\rangle : T_{0.51}}}{\alpha; .; .; HT : T_{0.0}; m : T_{0.34}, r : \alpha, p''' : [3] \mathbf{1} \vdash E_{3.6} : T_{0.51}} \quad D3.61$$

D3.5:

$$\frac{\frac{\alpha; .; .; HT : T_{0.0}; . \vdash \text{store}() : [3] [3] \mathbf{1}}{\alpha; .; .; HT : T_{0.0}; m : T_{0.34}, r : \alpha \vdash \text{bind } p''' = \text{store}() \text{ in } E_{3.6} : T_{0.511}}}{\alpha; .; .; HT : T_{0.0}; m : T_{0.34}, r : \alpha \vdash E_{3.5} : T_{0.513}} \quad D3.6$$

D3.41:

$$\frac{\alpha; .; .; HT : T_{0.0}; p'' : [1] \mathbf{1} \vdash p'' : [1] \mathbf{1}}{\alpha; .; .; HT : T_{0.0}; m : T_{0.34}, r : \alpha, p'' : [1] \mathbf{1} \vdash - = \text{release } p'' \text{ in } E_{3.5} : T_{0.512}} \quad D3.5$$

D3.4:

$$\frac{\frac{\alpha; .; .; HT : T_{0.0}; p_o : [2] \mathbf{1} \vdash p_o : [2] \mathbf{1}}{\alpha; .; .; HT : T_{0.0}; m : T_{0.34}, r : \alpha, p_o : [2] \mathbf{1}, p'' : [1] \mathbf{1} \vdash - = \text{release } p_o \text{ in } E_{3.41} : T_{0.51}}}{\alpha; .; .; HT : T_{0.0}; m : T_{0.34}, r : \alpha, p'' : [1] \mathbf{1} \vdash E_{3.4} : T_{0.51}} \quad D3.41$$

D3.3:

$$\frac{\frac{\alpha; .; .; HT : T_{0.0}; m : T_{0.34}, r : \alpha \vdash (\lambda p'' . E_{3.4}) : T_{0.5}}{\alpha; .; .; HT : T_{0.0}; m : T_{0.34}, r : \alpha \vdash C5 (\lambda p'' . E_{3.4}) : \text{Queue } \alpha}}{\alpha; .; .; HT : T_{0.0}; m : T_{0.34}, r : \alpha \vdash E_{3.3} : \text{Queue } \alpha} \quad D3.4$$

D3.2:

$$\frac{\frac{\alpha; .; .; HT : T_{0.0}; f : \alpha \vdash f : \alpha}{\alpha; .; .; HT : T_{0.0}; f : \alpha, m : T_{0.34}, r : \alpha \vdash \langle\langle f, E_{3.3} \rangle\rangle : (\alpha \otimes \text{Queue } \alpha)}}{\alpha; .; .; HT : T_{0.0}; f : \alpha, m : T_{0.34}, r : \alpha \vdash \text{ret}\langle\langle f, E_{3.3} \rangle\rangle : T_2} \quad D3.3$$

$$\frac{}{\alpha; .; .; HT : T_{0.0}; f : \alpha, m : T_{0.34}, r : \alpha \vdash E_{3.2} : T_2}$$

D3.12:

$$\frac{\frac{\alpha; .; .; HT : T_{0.0}; fm : T_{0.33} \vdash fm : T_{0.33}}{\alpha; .; .; HT : T_{0.0}; fm : T_{0.33}, r : \alpha \vdash \text{let}\langle\langle f, m \rangle\rangle = fm \text{ in } E_{3.2} : T_2}}{\alpha; .; .; HT : T_{0.0}; fm : T_{0.33}, r : \alpha \vdash E_{3.12} : T_2} \quad D3.2$$

D3.11:

$$\frac{\frac{\alpha; \cdot, \cdot; HT : T_{0.0}; x' : T_{0.32} \vdash x' : T_{0.32}}{D3.12}}{\alpha; \cdot, \cdot; HT : T_{0.0}; x' : T_{0.32} \vdash \text{let}\langle\langle fm, r \rangle\rangle = x' \text{ in } E_{3.12} : T_2} \\ \frac{}{\alpha; \cdot, \cdot; HT : T_{0.0}; x' : T_{0.32} \vdash E_{3.11} : T_2}$$

D3.1:

$$\frac{\frac{\alpha; \cdot, \cdot; HT : T_{0.0}; x : T_{0.3}, p' : [0] \mathbf{1} \vdash x p' : T_{0.31}}{D3.11}}{\alpha; \cdot, \cdot; HT : T_{0.0}; x : T_{0.3}, p' : [0] \mathbf{1} \vdash \text{bind } x' = x p' \text{ in } E_{3.11} : T_2} \\ \frac{}{\alpha; \cdot, \cdot; HT : T_{0.0}; x : T_{0.3}, p' : [0] \mathbf{1}, p_o : [2] \mathbf{1} \vdash E_{3.1} : T_2}$$

D3.0:

$$\frac{\frac{\alpha; \cdot, \cdot; HT : T_{0.0}; x : T_{0.3} \vdash \text{store}() : \mathbb{M} 2 [2] \mathbf{1}}{D3.1}}{\alpha; \cdot, \cdot; HT : T_{0.0}; x : T_{0.3}, p' : [0] \mathbf{1} \vdash \text{bind } p_o = \text{store}() \text{ in } E_{3.1} : T_2} \\ \frac{}{\alpha; \cdot, \cdot; HT : T_{0.0}; x : T_{0.3}, p' : [0] \mathbf{1} \vdash E_{3.0} : T_2}$$

D3:

$$\frac{\frac{\alpha; \cdot, \cdot; HT : T_{0.0}; \cdot \vdash \text{store}() : \mathbb{M} 0 \mathbf{1}}{D3.0}}{\alpha; \cdot, \cdot; HT : T_{0.0}; x : T_{0.3} \vdash E_3 : T_2}$$

D2.51:

$$\frac{}{\alpha; \cdot, \cdot; HT : T_{0.0}; m : T_{0.23}, p_r : [3] \mathbf{1} \vdash HT p_r \square m : T_{0.41}}$$

D2.5:

$$\frac{\frac{\alpha; \cdot, \cdot; HT : T_{0.0}; m : T_{0.23} \vdash \text{store}() : \mathbb{M} 3 [3] \mathbf{1}}{D2.51}}{\alpha; \cdot, \cdot; HT : T_{0.0}; m : T_{0.23} \vdash \text{bind } p_r = \text{store}() \text{ in } HT p_r \square m : T_{0.413}} \\ \frac{}{\alpha; \cdot, \cdot; HT : T_{0.0}; m : T_{0.23} \vdash E_{2.5} : T_{0.413}}$$

D2.4:

$$\frac{\frac{\alpha; \cdot, \cdot; HT : T_{0.0}; p'' : [2] \mathbf{1} \vdash p'' : [2] \mathbf{1}}{D2.5}}{\alpha; \cdot, \cdot; HT : T_{0.0}; m : T_{0.23}, p'' : [2] \mathbf{1} \vdash \text{release } p'' \text{ in } E_{2.5} : T_{0.411}} \\ \frac{}{\alpha; \cdot, \cdot; HT : T_{0.0}; m : T_{0.23}, p'' : [2] \mathbf{1} \vdash E_{2.4} : T_{0.411}}$$

D2.3:

$$\frac{\frac{\alpha; \cdot, \cdot; HT : T_{0.0}; p_o : [1] \mathbf{1} \vdash p_o : [1] \mathbf{1}}{D2.4}}{\alpha; \cdot, \cdot; HT : T_{0.0}; m : T_{0.23}, p_o : [1] \mathbf{1}, p'' : [2] \mathbf{1} \vdash - = \text{release } p_o \text{ in } E_{2.4} : T_{0.41}}$$

D2.21:

$$\frac{\frac{\frac{\alpha; \cdot, \cdot; HT : T_{0.0}; m : T_{0.23}, p_o : [1] \mathbf{1}, p'' : [2] \mathbf{1} \vdash E_{2.3} : T_{0.41}}{D2.3}}{\alpha; \cdot, \cdot; HT : T_{0.0}; m : T_{0.23}, p_o : [1] \mathbf{1} \vdash \lambda p'' . E_{2.3} : T_{0.4}}}{\alpha; \cdot, \cdot; HT : T_{0.0}; m : T_{0.23}, p_o : [1] \mathbf{1} \vdash C_4 (\lambda p'' . E_{2.3}) : \text{Queue } \alpha}$$

D2.2:

$$\frac{\frac{\frac{\alpha; \cdot, \cdot; HT : T_{0.0}; f : \alpha \vdash f : \alpha}{D2.21}}{\alpha; \cdot, \cdot; HT : T_{0.0}; f : \alpha, m : T_{0.23}, p_o : [1] \mathbf{1} \vdash \langle\langle f, (C_4 (\lambda p'' . E_{2.3})) \rangle\rangle : (\alpha \otimes \text{Queue } \alpha)} \\ \frac{}{\alpha; \cdot, \cdot; HT : T_{0.0}; f : \alpha, m : T_{0.23}, p_o : [1] \mathbf{1} \vdash \text{ret}\langle\langle f, (C_4 (\lambda p'' . E_{2.3})) \rangle\rangle : T_0} \\ \frac{}{\alpha; \cdot, \cdot; HT : T_{0.0}; f : \alpha, m : T_{0.23}, p_o : [1] \mathbf{1} \vdash E_{2.2} : T_0}$$

D2.11:

$$\frac{\frac{\alpha; .; .; HT : T_{0.0}; x' : T_{0.22} \vdash x' : T_{0.22}}{D2.2}}{\alpha; .; .; HT : T_{0.0}; x' : T_{0.22}, p_o : [1] \mathbf{1} \vdash \text{let}\langle\langle f, m \rangle\rangle = x' \text{ in } E_{2.2} : T_0} \\ \frac{}{\alpha; .; .; HT : T_{0.0}; x' : T_{0.22}, p_o : [1] \mathbf{1} \vdash E_{2.11} : T_0}$$

D2.1:

$$\frac{\frac{\alpha; .; .; HT : T_{0.0}; x : T_{0.2}, p' : [1] \mathbf{1} \vdash x p' : T_{0.21}}{D2.11}}{\alpha; .; .; HT : T_{0.0}; x : T_{0.2}, p_o : [1] \mathbf{1}, p' : [1] \mathbf{1} \vdash \text{bind } x' = x p' \text{ in } E_{2.11} : T_0} \\ \frac{}{\alpha; .; .; HT : T_{0.0}; x : T_{0.2}, p_o : [1] \mathbf{1}, p' : [1] \mathbf{1} \vdash E_{2.1} : T_0}$$

D2.0:

$$\frac{\frac{\alpha; .; .; HT : T_{0.0}; . \vdash \text{store}() : \mathbb{M} 1 [1] \mathbf{1}}{D2.1}}{\alpha; .; .; HT : T_{0.0}; x : T_{0.2}, p' : [1] \mathbf{1} \vdash \text{bind } p_o = \text{store}() \text{ in } E_{2.1} : T_1} \\ \frac{}{\alpha; .; .; HT : T_{0.0}; x : T_{0.2}, p' : [1] \mathbf{1} \vdash E_{2.0} : T_1}$$

D2:

$$\frac{\frac{\alpha; .; .; HT : T_{0.0}; . \vdash \text{store}() : \mathbb{M} 1 [1] \mathbf{1}}{D2.0}}{\alpha; .; .; HT : T_{0.0}; x : T_{0.2} \vdash E_2 : T_2}$$

D1:

$$\frac{\alpha; .; .; HT : T_{0.0}; x : \alpha \vdash \text{ret } \langle\langle x, C0 \rangle\rangle : T_2}{\alpha; .; .; HT : T_{0.0}; x : \alpha \vdash E_1 : T_2}$$

D0:

$$\frac{\alpha; .; .; HT : T_{0.0}; . \vdash \text{fix } x.x : T_2}{\alpha; .; .; HT : T_{0.0}; . \vdash E_0 : T_2}$$

D0.2:

$$\frac{\alpha; .; .; HT : T_{0.0}; q : \text{Queue } \alpha \vdash q : \text{Queue } \alpha}{D0 \quad D1 \quad D2 \quad D3 \quad D4 \quad D5} \\ \frac{}{\alpha; .; .; HT : T_{0.0}; q : \text{Queue } \alpha \vdash E_{0.2} : T_2}$$

D0.1:

$$\frac{\frac{\alpha; .; .; HT : T_{0.0}; . \vdash \uparrow^1 : \mathbb{M} 1 \mathbf{1}}{D0.2}}{\alpha; .; .; HT : T_{0.0}; q : \text{Queue } \alpha \vdash - = \uparrow^1; E_{0.2} : T_3}$$

Main derivation:

$$\frac{\frac{\alpha; .; .; HT : T_{0.0}; p : [3] \mathbf{1}, q : \text{Queue } \alpha \vdash p : [3] \mathbf{1}}{D0.1}}{\alpha; .; .; HT : T_{0.0}; p : [3] \mathbf{1}, q : \text{Queue } \alpha \vdash E_{0.1} : T_0} \\ \frac{}{.; .; .; . \vdash E_{0.0} : T_{0.0}}$$

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