Relational Cost Analysis (Technical Appendix)

Ezgi Çiçek
MPI-SWS, Germany

Gilles Barthe
IMDEA Software Institute, Spain

Marco Gaboardi
SUNY Buffalo, USA

Deepak Garg
MPI-SWS, Germany

Jan Hoffmann
Carnegie Mellon University, USA

1 Structure of the Appendix

This appendix considers the following additions to the main paper.

- Two additional types: trees and the constrained type $C \supset \tau$. The latter is eliminated with the $e._c$ construct.
- We generalize the unrelated type $U\,A$ to $U\,(A_1, A_2)$, allowing us to relate two expressions of two different unary types A_1 and A_2, respectively. As a result of this change, **switch** rule, $\rightarrow$ **exec** subtyping rule, and some of the asynchronous rules are also generalized. The advantage of this generalization is useful for the `2Dcount` example, explained in depth in Section 2.

We first present RelCost's syntax, typing and subtyping rules and semantic model. The remaining sections describe the necessary definitions, lemmas and theorems for proving the soundness of the RelCost's unary and binary (relational) typing with respect to the abstract cost semantics. Finally, we present three additional examples.

We use some abbreviations throughout. STS stands for "suffices to show", TS stands for "to show", and RTS stands for "remains to show".

List of Figures

List of Theorems and Lemmas

Relational types	τ	$::=$	$\mathrm{unit} \mid \mathrm{int} \mid \tau_1 \times \tau_2 \mid \tau_1 + \tau_2 \mid \tau_1 \xrightarrow{\mathrm{diff}(t)} \tau_2 \mid \mathrm{list}[n]^\alpha\,\tau \mid$ $\mathrm{tree}[n]^\alpha\,\tau \mid \forall i \overset{\mathrm{diff}(t)}{::} S.\,\tau \mid \exists i{::}S.\,\tau \mid U\,(A_1, A_2) \mid \square\,\tau \mid$ $C \,\&\, \tau \mid C \supset \tau$
Unary types	A	$::=$	$\mathrm{unit}_r \mid \mathrm{int}_r \mid A_1 \times A_2 \mid A_1 + A_2 \mid A_1 \xrightarrow{\mathrm{exec}(k,t)} A_2 \mid \mathrm{list}[n]\,A \mid$ $\mathrm{tree}[n]\,A \mid \forall i \overset{\mathrm{exec}(k,t)}{::} S.\,A \mid \exists i{::}S.\,A \mid C \,\&\, A \mid C \supset A$
Sorts	S	$::=$	$\mathbb{N} \mid \mathbb{R}$
Index terms	I, k, t, α	$::=$	$i \mid 0 \mid \infty \mid I+1 \mid I_1 + I_2 \mid I_1 - I_2 \mid \frac{I_1}{I_2} \mid I_1 \cdot I_2 \mid \lceil I \rceil \mid \lfloor I \rfloor \mid$ $\mathtt{log}_2(I) \mid I_1^{I_2} \mid \mathtt{min}(I_1, I_2) \mid \mathtt{max}(I_1, I_2) \mid \sum_{i=I_1}^{I_n} I$
Constraints	C	$::=$	$I_1 \doteq I_2 \mid I_1 < I_2 \mid \neg C \mid$
Constraint env.	Φ	$::=$	$\top \mid C \wedge \Phi$
Sort env.	Δ	$::=$	$\emptyset \mid \Delta, i :: S$
Unary type env.	Ω	$::=$	$\emptyset \mid \Omega, x : A$
Relational type env.	Γ	$::=$	$\emptyset \mid \Gamma, x : \tau$
Primitive env.	Υ	$::=$	$\emptyset \mid \Upsilon, \tau_1 \xrightarrow{\mathrm{diff}(t)} \tau_2 \mid \Upsilon, A_1 \xrightarrow{\mathrm{exec}(k,t)} A_2$
Typing judgments			$\Omega \vdash_k^t e : A$ $\Gamma \vdash e_1 \ominus e_2 \lesssim t : \tau$

Figure 1: Syntax of types and contexts

Terms	e	$::=$	$x \mid \mathtt{n} \mid \mathrm{fix}\ f(x).e \mid e_1\ e_2 \mid \zeta\ e \mid \langle e_1, e_2 \rangle \mid \pi_1(e) \mid \pi_2(e) \mid$ $\mathrm{inl}\ e \mid \mathrm{inr}\ e \mid \mathrm{case}\ (e, x.e_1, y.e_2) \mid \mathrm{nil} \mid \mathrm{cons}(e_1, e_2) \mid$ $\mathrm{case}\ e\ \mathrm{of}\ \mathrm{nil} \rightarrow e_1 \mid h :: tl \rightarrow e_2 \mid \mathrm{leaf} \mid \mathrm{node}(e_l, e, e_r) \mid$ $\mathrm{case}\ e\ \mathrm{of}\ \mathrm{leaf} \rightarrow e_1 \mid \mathrm{node}(l, x, r) \rightarrow e_2 \mid \Lambda e \mid e[\,] \mid$ $\mathrm{pack}\ e \mid \mathrm{unpack}\ e_1\ \mathrm{as}\ x\ \mathrm{in}\ e_2 \mid \mathrm{let}\ x = e_1\ \mathrm{in}\ e_2 \mid () \mid$ $\mathrm{clet}\ e_1\ \mathrm{as}\ x\ \mathrm{in}\ e_2 \mid \cdot_c\ e$
Values	v	$::=$	$\mathtt{n} \mid \mathrm{fix}\ f(x).v \mid \langle v_1, v_2 \rangle \mid \mathrm{inl}\ v \mid \mathrm{inr}\ v \mid \mathrm{nil} \mid \mathrm{cons}(v_1, v_2) \mid \mathrm{leaf} \mid$ $\mathrm{node}(v_l, v, v_r) \mid \Lambda e \mid \mathrm{pack}\ v \mid ()$

Figure 2: Syntax of values and terms

$\boxed{\Delta; \Phi \vdash \tau \ \texttt{wf}}$ Binary type τ is well-formed.

$\boxed{\Delta; \Phi \vdash^{\mathsf{A}} A \ \texttt{wf}}$ Unary type A is well-formed.

$$\frac{}{\Delta; \Phi \vdash \text{unit}_r \ \texttt{wf}} \ \textbf{wf-unit} \qquad \frac{}{\Delta; \Phi \vdash \text{int}_r \ \texttt{wf}} \ \textbf{wf-int}$$

$$\frac{\Delta; \Phi \vdash \tau_1 \ \texttt{wf} \qquad \Delta; \Phi \vdash \tau_2 \ \texttt{wf}}{\Delta; \Phi \vdash \tau_1 \times \tau_2 \ \texttt{wf}} \ \textbf{wf-prod} \qquad \frac{\Delta; \Phi \vdash \tau_1 \ \texttt{wf} \qquad \Delta; \Phi \vdash \tau_2 \ \texttt{wf}}{\Delta; \Phi \vdash \tau_1 + \tau_2 \ \texttt{wf}} \ \textbf{wf-sum}$$

$$\frac{\Delta; \Phi \vdash \tau_1 \ \texttt{wf} \qquad \Delta; \Phi \vdash \tau_2 \ \texttt{wf} \qquad \Delta; \Phi \vdash t :: \mathbb{R}}{\Delta; \Phi \vdash \tau_1 \xrightarrow{\text{diff}(t)} \tau_2 \ \texttt{wf}} \ \textbf{wf-fun}$$

$$\frac{\Delta; \Phi \vdash n :: \mathbb{N} \qquad \Delta; \Phi \vdash \alpha :: \mathbb{N} \qquad \Delta; \Phi \vdash \tau \ \texttt{wf}}{\Delta; \Phi \vdash \text{list}[n]^{\alpha} \, \tau \ \texttt{wf}} \ \textbf{wf-list}$$

$$\frac{\Delta; \Phi \vdash n :: \mathbb{N} \qquad \Delta; \Phi \vdash \alpha :: \mathbb{N} \qquad \Delta; \Phi \vdash \tau \ \texttt{wf}}{\Delta; \Phi \vdash \text{tree}[n]^{\alpha} \, \tau \ \texttt{wf}} \ \textbf{wf-tree}$$

$$\frac{i :: S, \Delta; \Phi \vdash \tau \ \texttt{wf} \qquad i :: S, \Delta; \Phi \vdash t :: \mathbb{R}}{\Delta; \Phi \vdash \forall i \overset{\text{diff}(t)}{::} S.\, \tau \ \texttt{wf}} \ \textbf{wf-}\forall \qquad \frac{i :: S, \Delta; \Phi \vdash \tau \ \texttt{wf}}{\Delta; \Phi \vdash \exists i{::}S.\, \tau \ \texttt{wf}} \ \textbf{wf-}\exists$$

$$\frac{\Delta; \Phi \vdash^{\mathsf{A}} A_1 \ \texttt{wf} \qquad \Delta; \Phi \vdash^{\mathsf{A}} A_2 \ \texttt{wf}}{\Delta; \Phi \vdash U\,(A_1, A_2) \ \texttt{wf}} \ \textbf{wf-U} \qquad \frac{\Delta; \Phi \vdash \tau \ \texttt{wf}}{\Delta; \Phi \vdash \square\, \tau \ \texttt{wf}} \ \textbf{wf-box}$$

$$\frac{\Delta; \Phi \vdash C \ \texttt{wf} \qquad \Delta; C \wedge \Phi \vdash \tau \ \texttt{wf}}{\Delta; \Phi \vdash C \supset \tau \ \texttt{wf}} \ \textbf{wf-C}\supset \qquad \frac{\Delta; \Phi \vdash C \ \texttt{wf} \qquad \Delta; C \wedge \Phi \vdash \tau \ \texttt{wf}}{\Delta; \Phi \vdash C \,\&\, \tau \ \texttt{wf}} \ \textbf{wf-C\&}$$

Figure 3: Well-formedness of relational types

$\boxed{\Delta; \Phi \vdash^{\mathsf{A}} A \ \texttt{wf}}$ Unary type A is well-formed.

$$\frac{}{\Delta; \Phi \vdash^{\mathsf{A}} \text{unit} \ \texttt{wf}} \ \textbf{wf-u-unit} \qquad \frac{}{\Delta; \Phi \vdash^{\mathsf{A}} \text{int} \ \texttt{wf}} \ \textbf{wf-u-int}$$

$$\frac{\Delta; \Phi \vdash^{\mathsf{A}} A_1 \ \texttt{wf} \qquad \Delta; \Phi \vdash^{\mathsf{A}} A_2 \ \texttt{wf}}{\Delta; \Phi \vdash^{\mathsf{A}} A_1 \times A_2 \ \texttt{wf}} \ \textbf{wf-u-prod}$$

$$\frac{\Delta; \Phi \vdash^{\mathsf{A}} A_1 \ \texttt{wf} \qquad \Delta; \Phi \vdash^{\mathsf{A}} A_2 \ \texttt{wf}}{\Delta; \Phi \vdash^{\mathsf{A}} A_1 + A_2 \ \texttt{wf}} \ \textbf{wf-u-sum}$$

$$\frac{\Delta; \Phi \vdash^{\mathsf{A}} A_1 \ \texttt{wf} \qquad \Delta; \Phi \vdash^{\mathsf{A}} A_2 \ \texttt{wf} \qquad \Delta; \Phi \vdash k :: \mathbb{R} \qquad \Delta; \Phi \vdash t :: \mathbb{R}}{\Delta; \Phi \vdash^{\mathsf{A}} A_1 \xrightarrow{\text{exec}(k,t)} A_2 \ \texttt{wf}} \ \textbf{wf-u-fun}$$

$$\frac{\Delta; \Phi \vdash n :: \mathbb{N} \qquad \Delta; \Phi \vdash^{\mathsf{A}} A \ \texttt{wf}}{\Delta; \Phi \vdash^{\mathsf{A}} \text{list}[n] \, A \ \texttt{wf}} \ \textbf{wf-u-list} \qquad \frac{\Delta; \Phi \vdash n :: \mathbb{N} \qquad \Delta; \Phi \vdash^{\mathsf{A}} A \ \texttt{wf}}{\Delta; \Phi \vdash^{\mathsf{A}} \text{tree}[n] \, A \ \texttt{wf}} \ \textbf{wf-u-tree}$$

$$\frac{i :: S, \Delta; \Phi \vdash^{\mathsf{A}} A \ \texttt{wf} \qquad i :: S, \Delta; \Phi \vdash k :: \mathbb{R} \qquad i :: S, \Delta; \Phi \vdash t :: \mathbb{R}}{\Delta; \Phi \vdash^{\mathsf{A}} \forall i \overset{\text{exec}(k,t)}{::} S. \, A \ \texttt{wf}} \ \textbf{wf-u-}\forall$$

$$\frac{i :: S, \Delta; \Phi \vdash^{\mathsf{A}} A \ \texttt{wf}}{\Delta; \Phi \vdash^{\mathsf{A}} \exists i{::}S. \, A \ \texttt{wf}} \ \textbf{wf-u-}\exists \qquad \frac{\Delta; \Phi \vdash C \ \texttt{wf} \qquad \Delta; C \wedge \Phi \vdash^{\mathsf{A}} A \ \texttt{wf}}{\Delta; \Phi \vdash^{\mathsf{A}} C \supset A \ \texttt{wf}} \ \textbf{wf-u-C}\supset$$

$$\frac{\Delta; \Phi \vdash C \ \texttt{wf} \qquad \Delta; C \wedge \Phi \vdash^{\mathsf{A}} A \ \texttt{wf}}{\Delta; \Phi \vdash^{\mathsf{A}} C \ \& \ A \ \texttt{wf}} \ \textbf{wf-u-C\&}$$

Figure 4: Well-formedness of types

$\boxed{\Delta \vdash C \ \texttt{wf}}$

$$\frac{\Delta \vdash I_1 :: S \qquad \Delta \vdash I_2 :: S \qquad S \in \{\mathbb{N}, \mathbb{R}\}}{\Delta \vdash I_1 < I_2 \ \texttt{wf}} \ \textbf{wf-cs} < \qquad \frac{\Delta \vdash I_1 :: S \qquad \Delta \vdash I_2 :: S \qquad S \in \{\mathbb{N}, \mathbb{R}\}}{\Delta \vdash I_1 \doteq I_2 \ \texttt{wf}} \ \textbf{wf-cs} \doteq$$

$$\frac{\Delta \vdash C \ \texttt{wf}}{\Delta \vdash \neg C \ \texttt{wf}} \ \textbf{wf-cs} \ \neg$$

Figure 5: Constraint well-formedness

$$
\begin{array}{lcl}
|\cdot|_{i\in\{1,2\}} & : & \text{Binary type} \to \text{Unary type} \\
\\
|\mathrm{unit}_r|_i & = & \mathrm{unit} \\
|\mathrm{int}_r|_i & = & \mathrm{int} \\
|\tau_1 \times \tau_2|_i & = & |\tau_1|_i \times |\tau_2|_i \\
|\tau_1 + \tau_2|_i & = & |\tau_1|_i + |\tau_2|_i \\
|\tau_1 \xrightarrow{\mathrm{diff}(t)} \tau_2|_i & = & |\tau_1|_i \xrightarrow{\mathrm{exec}(0,\infty)} |\tau_2|_i \\
|\mathrm{list}[n]^{\alpha}\,\tau|_i & = & \mathrm{list}[n]\,|\tau|_i \\
|\mathrm{tree}[n]^{\alpha}\,\tau|_i & = & \mathrm{tree}[n]\,|\tau|_i \\
|\forall j \overset{\mathrm{diff}(t)}{::} S.\,\tau|_i & = & \forall i = j \overset{\mathrm{exec}(0,\infty)}{::} S.\,|\tau|_i \\
|\exists j{::}S.\,\tau|_i & = & \exists j{::}S.\,|\tau|_i \\
|C \supset \tau|_i & = & C \supset |\tau|_i \\
|C \,\&\, \tau|_i & = & C \,\&\, |\tau|_i \\
\\
|U\,(A_1, A_2)|_i & = & A_i \\
|\Box\,\tau|_i & = & |\tau|_i \\
\\
\\
|\emptyset|_i & = & \emptyset \\
|\Gamma, x : \tau|_i & = & |\Gamma|_i, x : |\tau|_i
\end{array}
$$

Figure 6: Refinement removal operation

$\boxed{\Delta \vdash I :: S}$

$$
\frac{\Delta(i) = S}{\Delta \vdash i :: S}\ \textbf{inVar}
\qquad
\frac{}{\Delta \vdash 0 :: \mathbb{N}}\ \textbf{zero}
\qquad
\frac{}{\Delta \vdash \infty :: \mathbb{R}}\ \textbf{infinity}
\qquad
\frac{\Delta \vdash I :: \mathbb{N}}{\Delta \vdash (I+1) :: \mathbb{N}}\ \textbf{plus}
$$

$$
\frac{\Delta \vdash I_1 :: \mathbb{N} \qquad \Delta \vdash I_2 :: \mathbb{N} \qquad \diamond \in \{\mathtt{min}, \mathtt{max}, +, -, *, \div, \widehat{}\,\}}{\Delta \vdash (I_1 \diamond I_2) :: \mathbb{N}}\ \textbf{op-bin-N}
$$

$$
\frac{\Delta \vdash I :: \mathbb{R} \qquad \circ \in \{\lfloor\ \rfloor, \lceil\ \rceil\}}{\Delta \vdash (\circ S) :: \mathbb{N}}\ \textbf{op-un-N}
$$

$$
\frac{\Delta \vdash t_1 :: \mathbb{R} \qquad \Delta \vdash t_2 :: \mathbb{R} \qquad \star \in \{\mathtt{min}, \mathtt{max}, +, -, *, /, \widehat{}\,\}}{\Delta \vdash (t_1 \star t_2) :: \mathbb{R}}\ \textbf{op-bin-R}
\qquad
\frac{\Delta \vdash t :: \mathbb{R}}{\Delta \vdash \mathtt{log}_2(t) :: \mathbb{R}}\ \textbf{op-log}
$$

$$
\frac{\Delta \vdash I_1 :: \mathbb{N} \qquad \Delta \vdash I_n :: \mathbb{N} \qquad \Delta, i :: \mathbb{N} \vdash I :: S \qquad S \in \{\mathbb{N}, \mathbb{R}\}}{\Delta \vdash \sum_{i=I_1}^{I_n} I :: S}\ \textbf{isum}
\qquad
\frac{\Delta \vdash I :: \mathbb{N}}{\Delta \vdash I :: \mathbb{R}}\ \textbf{i}\sqsubseteq
$$

Figure 7: Sorting rules

General rules

$$\frac{\Delta; \Phi; |\Gamma|_1 \vdash_{k_1}^{t_1} e_1 : A_1 \qquad \Delta; \Phi; |\Gamma|_2 \vdash_{k_2}^{t_2} e_2 : A_2}{\Delta; \Phi; \Gamma \vdash e_1 \ominus e_2 \lesssim t_1 - k_2 : U\,(A_1, A_2)} \ \textbf{switch}$$

$$\frac{\begin{array}{c}\Delta; \Phi; \Gamma \vdash e \ominus e \lesssim t : \tau \\ \forall x \in dom(\Gamma).\ \ \Delta; \Phi \models \Gamma(x) \sqsubseteq \Box\,\Gamma(x)\end{array}}{\Delta; \Phi; \Gamma, \Gamma'; \Omega \vdash e \ominus e \lesssim 0 : \Box\,\tau} \ \textbf{nochange}$$

Constant integers and unit

$$\frac{}{\Delta; \Phi; \Omega \vdash_0^0 \mathtt{n} : \mathrm{int}} \ \textbf{const} \qquad \frac{}{\Delta; \Phi; \Gamma \vdash \mathtt{n} \ominus \mathtt{n} \lesssim 0 : \mathrm{int}_r} \ \textbf{r-const}$$

$$\frac{}{\Delta; \Phi; \Omega \vdash_0^0 () : \mathrm{unit}} \ \textbf{unit} \qquad \frac{}{\Delta; \Phi; \Gamma \vdash () \ominus () \lesssim 0 : \mathrm{unit}_r} \ \textbf{r-unit}$$

Variables x

$$\frac{\Omega(x) = A}{\Delta; \Phi; \Omega \vdash_0^0 x : A} \ \textbf{var} \qquad \frac{\Gamma(x) = \tau}{\Delta; \Phi; \Gamma \vdash x \ominus x \lesssim 0 : \tau} \ \textbf{r-var}$$

inl e

$$\frac{\Delta; \Phi; \Omega \vdash_k^t e : A_1 \qquad \Delta; \Phi \vdash^{\mathsf{A}} A_2\ \mathtt{wf}}{\Delta; \Phi; \Omega \vdash_k^t \mathrm{inl}\ e : A_1 + A_2} \ \textbf{inl} \qquad \frac{\Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t : \tau_1 \qquad \Delta; \Phi \vdash \tau_2\ \mathtt{wf}}{\Delta; \Phi; \Gamma \vdash \mathrm{inl}\ e \ominus \mathrm{inl}\ e' \lesssim t : \tau_1 + \tau_2} \ \textbf{r-inl}$$

inr e

$$\frac{\Delta; \Phi; \Omega \vdash_k^t e : A_2 \qquad \Delta; \Phi \vdash^{\mathsf{A}} A_1\ \mathtt{wf}}{\Delta; \Phi; \Omega \vdash_k^t \mathrm{inr}\ e : A_1 + A_2} \ \textbf{inr} \qquad \frac{\Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t : \tau_2 \qquad \Delta; \Phi \vdash \tau_1\ \mathtt{wf}}{\Delta; \Phi; \Gamma \vdash \mathrm{inr}\ e \ominus \mathrm{inr}\ e' \lesssim t : \tau_1 + \tau_2} \ \textbf{r-inr}$$

case $(e, x.e_1, y.e_2)$

$$\frac{\Delta; \Phi; \Omega \vdash_k^t e : A_1 + A_2 \qquad \Delta; \Phi; x : A_1, \Omega \vdash_{k'}^{t'} e_1 : A \qquad \Delta; \Phi; y : A_2, \Omega \vdash_{k'}^{t'} e_2 : A}{\Delta; \Phi; \Omega \vdash_{k+k'+c_{case}}^{t+t'+c_{case}} \mathrm{case}\ (e, x.e_1, y.e_2) : A} \ \textbf{case}$$

$$\frac{\begin{array}{c}\Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t : \tau_1 + \tau_2 \\ \Delta; \Phi; x : \tau_1, \Gamma \vdash e_1 \ominus e_1' \lesssim t' : \tau \qquad \Delta; \Phi; y : \tau_2, \Gamma \vdash e_2 \ominus e_2' \lesssim t' : \tau\end{array}}{\Delta; \Phi; \Gamma \vdash \ \mathrm{case}\ (e, x.e_1, y.e_2) \ominus \ \mathrm{case}\ (e', x.e_1', y.e_2') \lesssim t + t' : \tau} \ \textbf{r-case}$$

Figure 8: Typing rules (Part 1)

$\boxed{\text{fix } f(x).e}$

$$\frac{\Delta;\Phi \vdash^{\mathsf{A}} A_1 \xrightarrow{\text{exec}(k,t)} A_2 \ \texttt{wf} \qquad \Delta;\Phi;x : A_1, f : A_1 \xrightarrow{\text{exec}(k,t)} A_2, \Omega \vdash_k^t e : A_2}{\Delta;\Phi;\Omega \vdash_0^0 \text{fix } f(x).e : A_1 \xrightarrow{\text{exec}(k,t)} A_2} \ \textbf{fix}$$

$$\frac{\Delta;\Phi \vdash \tau_1 \xrightarrow{\text{diff}(t)} \tau_2 \ \texttt{wf} \qquad \Delta;\Phi;x : \tau_1, f : \tau_1 \xrightarrow{\text{diff}(t)} \tau_2, \Gamma \vdash e_1 \ominus e_2 \lesssim t : \tau_2}{\Delta;\Phi;\Gamma \vdash \text{fix } f(x).e_1 \ominus \text{fix } f(x).e_2 \lesssim 0 : \tau_1 \xrightarrow{\text{diff}(t)} \tau_2} \ \textbf{r-fix}$$

$$\frac{\begin{array}{c}\Delta;\Phi \vdash \tau_1 \xrightarrow{\text{diff}(t)} \tau_2 \ \texttt{wf} \qquad \Delta;\Phi;x : \tau_1, f : \square\,(\tau_1 \xrightarrow{\text{diff}(t)} \tau_2), \Gamma \vdash e \ominus e \lesssim t : \tau_2 \\ \forall x \in dom(\Gamma).\ \ \Delta;\Phi \models \Gamma(x) \sqsubseteq \square\,\Gamma(x)\end{array}}{\Delta;\Phi;\Gamma \vdash \text{fix } f(x).e \ominus \text{fix } f(x).e \lesssim 0 : \square\,(\tau_1 \xrightarrow{\text{diff}(t)} \tau_2)} \ \textbf{r-fixNC}$$

$\boxed{e_1\ e_2}$

$$\frac{\Delta;\Phi;\Omega \vdash_{k_1}^{t_1} e_1 : A_1 \xrightarrow{\text{exec}(k,t)} A_2 \qquad \Delta;\Phi;\Omega \vdash_{k_2}^{t_2} e_2 : A_1}{\Delta;\Phi;\Omega \vdash_{k_1+k_2+k+c_{app}}^{t_1+t_2+t+c_{app}} e_1\ e_2 : A_2} \ \textbf{app}$$

$$\frac{\begin{array}{c}\Delta;\Phi;\Gamma \vdash e_1 \ominus e_1' \lesssim t_1 : \tau_1 \xrightarrow{\text{diff}(t)} \tau_2 \\ \Delta;\Phi;\Gamma \vdash e_2 \ominus e_2' \lesssim t_2 : \tau_1\end{array}}{\Delta;\Phi;\Gamma \vdash e_1\ e_2 \ominus e_1'\ e_2' \lesssim t_1 + t_2 + t : \tau_2} \ \textbf{r-app}$$

$\boxed{\langle e_1, e_2\rangle}$

$$\frac{\Delta;\Phi;\Omega \vdash_{k_1}^{t_1} e_1 : A_1 \qquad \Delta;\Phi;\Omega \vdash_{k_2}^{t_2} e_2 : A_2}{\Delta;\Phi;\Omega \vdash_{k_1+k_2}^{t_1+t_2} \langle e_1, e_2\rangle : A_1 \times A_2} \ \textbf{prod}$$

$$\frac{\Delta;\Phi;\Gamma \vdash e_1 \ominus e_1' \lesssim t_1 : \tau_1 \qquad \Delta;\Phi;\Gamma \vdash e_2 \ominus e_2' \lesssim t_2 : \tau_2}{\Delta;\Phi;\Gamma \vdash \langle e_1, e_2\rangle \ominus \langle e_1', e_2'\rangle \lesssim t_1 + t_2 : \tau_1 \times \tau_2} \ \textbf{r-prod}$$

$\boxed{\pi_1(e)}$

$$\frac{\Delta;\Phi;\Omega \vdash_k^t e : A_1 \times A_2}{\Delta;\Phi;\Omega \vdash_{k+c_{proj}}^{t+c_{proj}} \pi_1(e) : A_1} \ \textbf{proj1} \qquad \frac{\Delta;\Phi;\Gamma \vdash e \ominus e' \lesssim t : \tau_1 \times \tau_2}{\Delta;\Phi;\Gamma \vdash \pi_1(e) \ominus \pi_1(e') \lesssim t : \tau_1} \ \textbf{r-proj1}$$

$\boxed{\pi_2(e)}$

Symmetric rules.

Figure 9: Typing rules (Part 2)

$\boxed{\text{nil}}$

$$\frac{\Delta; \Phi \vdash^{\mathsf{A}} A \ \mathtt{wf}}{\Delta; \Phi; \Omega \vdash_{0}^{0} \text{nil} \ : \text{list}[0]\, A} \ \textbf{nil} \qquad \frac{\Delta; \Phi \vdash \tau \ \mathtt{wf}}{\Delta; \Phi; \Gamma \vdash \text{nil} \ominus \text{nil} \ \lesssim 0 : \text{list}[0]^{\alpha}\, \tau} \ \textbf{r-nil}$$

$\boxed{\text{cons}(e_1, e_2)}$

$$\frac{\Delta; \Phi; \Omega \vdash_{k_1}^{t_1} e_1 : A \qquad \Delta; \Phi; \Omega \vdash_{k_2}^{t_2} e_2 : \text{list}[n]\, A}{\Delta; \Phi; \Omega \vdash_{k_1+k_2}^{t_1+t_2} \text{cons}(e_1, e_2) : \text{list}[n+1]\, A} \ \textbf{cons}$$

$$\frac{\Delta; \Phi; \Gamma \vdash e_1 \ominus e_1' \lesssim t_1 : \tau \qquad \Delta; \Phi; \Gamma \vdash e_2 \ominus e_2' \lesssim t_2 : \text{list}[n]^{\alpha}\, \tau}{\Delta; \Phi; \Gamma \vdash \text{cons}(e_1, e_2) \ominus \text{cons}(e_1', e_2') \lesssim t_1 + t_2 : \text{list}[n+1]^{\alpha+1}\, \tau} \ \textbf{r-cons1}$$

$$\frac{\Delta; \Phi; \Gamma \vdash e_1 \ominus e_1' \lesssim t_1 : \Box\, \tau \qquad \Delta; \Phi; \Gamma \vdash e_2 \ominus e_2' \lesssim t_2 : \text{list}[n]^{\alpha}\, \tau}{\Delta; \Phi; \Gamma \vdash \text{cons}(e_1, e_2) \ominus \text{cons}(e_1', e_2') \lesssim t_1 + t_2 : \text{list}[n+1]^{\alpha}\, \tau} \ \textbf{r-cons2}$$

$\boxed{\text{case } e \text{ of nil } \to e_1 \mid h :: tl \to e_2}$

$$\frac{\begin{array}{c}\Delta; \Phi; \Omega \vdash_{k}^{t} e : \text{list}[n]\, A \\ \Delta; \Phi \wedge n = 0; \Omega \vdash_{k'}^{t'} e_1 : A' \qquad i, \Delta; \Phi \wedge n = i+1; h : A, tl : \text{list}[i]\, A, \Omega \vdash_{k'}^{t'} e_2 : A'\end{array}}{\Delta; \Phi; \Omega \vdash_{k+k'+c_{caseL}}^{t+t'+c_{caseL}} \text{case } e \text{ of nil } \to e_1 \mid h :: tl \to e_2 : A'} \ \textbf{caseL}$$

$$\frac{\begin{array}{c}\Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t : \text{list}[n]^{\alpha}\, \tau \qquad \Delta; \Phi \wedge n = 0; \Gamma \vdash e_1 \ominus e_1' \lesssim t' : \tau' \\ i, \Delta; \Phi \wedge n = i+1; h : \Box\, \tau, tl : \text{list}[i]^{\alpha}\, \tau, \Gamma \vdash e_2 \ominus e_2' \lesssim t' : \tau' \\ i, \beta, \Delta; \Phi \wedge n = i+1 \wedge \alpha = \beta + 1; h : \tau, tl : \text{list}[i]^{\beta}\, \tau, \Gamma \vdash e_2 \ominus e_2' \lesssim t' : \tau'\end{array}}{\Delta; \Phi; \Gamma \vdash \ \text{case } e \text{ of nil } \to e_1 \mid h :: tl \to e_2 \ominus \ \text{case } e' \text{ of nil } \to e_1' \mid h :: tl \to e_2' \lesssim t + t' : \tau'} \ \textbf{r-caseL}$$

$\boxed{\text{leaf}}$

$$\frac{\Delta; \Phi \vdash^{\mathsf{A}} A \ \mathtt{wf}}{\Delta; \Phi; \Omega \vdash_{0}^{0} \text{leaf} : \text{tree}[0]\, A} \ \textbf{leaf} \qquad \frac{\Delta; \Phi \vdash \tau \ \mathtt{wf}}{\Delta; \Phi; \Gamma \vdash \text{leaf} \ominus \text{leaf} \lesssim 0 : \text{tree}[0]^{\alpha}\, \tau} \ \textbf{r-leaf}$$

$\boxed{\text{node}(e_l, e, e_r)}$

$$\frac{\Delta; \Phi; \Omega \vdash_{k}^{t} e : A \qquad \Delta; \Phi; \Omega \vdash_{k_1}^{t_1} e_l : \text{tree}[i]\, A \qquad \Delta; \Phi; \Omega \vdash_{k_2}^{t_2} e_r : \text{tree}[j]\, A}{\Delta; \Phi; \Omega \vdash_{k+k_1+k_2}^{t+t_1+t_2} \text{node}(e_l, e, e_r) : \text{tree}[i+j+1]\, A} \ \textbf{node}$$

$$\frac{\begin{array}{c}\Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t : \tau \\ \Delta; \Phi; \Gamma \vdash e_l \ominus e_l' \lesssim t_1 : \text{tree}[i]^{\alpha}\, \tau \qquad \Delta; \Phi; \Gamma \vdash e_r \ominus e_r' \lesssim t_2 : \text{tree}[j]^{\beta}\, \tau\end{array}}{\Delta; \Phi; \Gamma \vdash \text{node}(e_l, e, e_r) \ominus \text{node}(e_l', e', e_r') \lesssim t + t_1 + t_2 : \text{tree}[i+j+1]^{\alpha+\beta+1}\, \tau} \ \textbf{r-node1}$$

$$\frac{\begin{array}{c}\Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t : \Box\, \tau \\ \Delta; \Phi; \Gamma \vdash e_l \ominus e_l' \lesssim t_1 : \text{tree}[i]^{\alpha}\, \tau \qquad \Delta; \Phi; \Gamma \vdash e_r \ominus e_r' \lesssim t_2 : \text{tree}[j]^{\beta}\, \tau\end{array}}{\Delta; \Phi; \Gamma \vdash \text{node}(e_l, e, e_r) \ominus \text{node}(e_l', e', e_r') \lesssim t + t_1 + t_2 : \text{tree}[i+j+1]^{\alpha+\beta}\, \tau} \ \textbf{r-node2}$$

Figure 10: Typing rules (Part 3)

$\boxed{\text{case } e \text{ of leaf} \to e_1 \mid \text{node}(l,x,r) \to e_2}$

$$\frac{\begin{array}{c}\Delta;\Phi;\Omega \vdash_k^t e : \text{tree}[n]\, A \qquad \Delta;\Phi \wedge n = 0;\Omega \vdash_{k'}^{t'} e_1 : A' \\ i,j,\Delta;\Phi \wedge n = i+j+1; x : A, l : \text{tree}[i]\, A, r : \text{tree}[j]\, A, \Omega \vdash_{k'}^{t'} e_2 : A'\end{array}}{\Delta;\Phi;\Omega \vdash_{k+k'+c_{caseT}}^{t+t'+c_{caseT}} \text{ case } e \text{ of leaf} \to e_1 \mid \text{node}(l,x,r) \to e_2 : A'} \textbf{ caseT}$$

$$\frac{\begin{array}{c}\Delta;\Phi;\Gamma \vdash e \ominus e' \lesssim t : \text{tree}[n]^{\alpha}\, \tau \qquad \Delta;\Phi \wedge n = 0 \wedge;\Gamma \vdash e_1 \ominus e_1' \lesssim t' : \tau' \\ i,j,\beta,\theta,\Delta;\Phi \wedge n = i+j+1 \wedge \alpha = \beta+\theta; x : \Box\tau, l : \text{tree}[i]^{\beta}\, \tau, r : \text{tree}[j]^{\theta}\, \tau, \Gamma \vdash e_2 \ominus e_2' \lesssim t' : \tau' \\ i,j,\beta,\theta,\Delta;\Phi \wedge n = i+j+1 \wedge \alpha = \beta+\theta+1; x : \tau, l : \text{tree}[i]^{\beta}\, \tau, r : \text{tree}[j]^{\theta}\, \tau, \Gamma \vdash e_2 \ominus e_2' \lesssim t' : \tau'\end{array}}{\Delta;\Phi;\Gamma \vdash \text{ case } e \text{ of leaf} \to e_1 \mid \text{node}(l,x,r) \to e_2 \ominus \text{ case } e' \text{ of leaf} \to e_1' \mid \text{node}(l,x,r) \to \lesssim t+t' : e_2'\, \tau'} \textbf{ r-cas}$$

$\boxed{\Lambda e}$

$$\frac{i :: S, \Delta;\Phi;\Omega \vdash_k^t e : A \qquad i \notin \text{FIV}(\Phi;\Omega)}{\Delta;\Phi;\Omega \vdash_0^0 \Lambda e : \forall i \overset{\text{exec}(k,t)}{::} S.\, A} \textbf{ iLam}$$

$$\frac{i :: S, \Delta;\Phi;\Gamma \vdash e \ominus e' \lesssim t : \tau \qquad i \notin \text{FIV}(\Phi;\Gamma)}{\Delta;\Phi;\Gamma \vdash \Lambda e \ominus \Lambda e' \lesssim 0 : \forall i \overset{\text{diff}(t)}{::} S.\, \tau} \textbf{ r-iLam}$$

$\boxed{e[\,]}$

$$\frac{\Delta;\Phi;\Omega \vdash_k^t e : \forall i \overset{\text{exec}(k',t')}{::} S.\, A \qquad \Delta \vdash I : S}{\Delta;\Phi;\Omega \vdash_{k+k'[I/i]}^{t+t'[I/i]} e[\,] : A\{I/i\}} \textbf{ iApp}$$

$$\frac{\Delta;\Phi;\Gamma \vdash e \ominus e' \lesssim t : \forall i \overset{\text{diff}(t')}{::} S.\, \tau \qquad \Delta \vdash I : S}{\Delta;\Phi;\Gamma \vdash e[\,] \ominus e'[\,] \lesssim t + t'[I/i] : \tau\{I/i\}} \textbf{ r-iApp}$$

$\boxed{\text{pack } e}$

$$\frac{\Delta;\Phi;\Omega \vdash_k^t e : A\{I/i\} \qquad \Delta \vdash I :: S}{\Delta;\Phi;\Omega \vdash_k^t \text{pack } e : \exists i{::}S.\, A} \textbf{ pack}$$

$$\frac{\Delta;\Phi;\Gamma \vdash e \ominus e' \lesssim t : \tau\{I/i\} \qquad \Delta \vdash I :: S}{\Delta;\Phi;\Gamma \vdash \text{pack } e \ominus \text{pack } e' \lesssim t : \exists i{::}S.\, \tau} \textbf{ r-pack}$$

$\boxed{\text{unpack } e \text{ as } x \text{ in } e'}$

$$\frac{\begin{array}{c}\Delta;\Phi;\Omega \vdash_{k_1}^{t_1} e_1 : \exists i{::}S.\, A_1 \\ i :: S, \Delta;\Phi; x : A_1, \Omega \vdash_{k_2}^{t_2} e_2 : A_2 \qquad i \notin FV(\Phi;\Gamma, A_2, k_2, t_2)\end{array}}{\Delta;\Phi;\Omega \vdash_{k_1+k_2}^{t_1+t_2} \text{unpack } e_1 \text{ as } x \text{ in } e_2 : A_2} \textbf{ unpack}$$

$$\frac{\begin{array}{c}\Delta;\Phi;\Gamma \vdash e_1 \ominus e_1' \lesssim t_1 : \exists i{::}S.\, \tau_1 \\ i :: S, \Delta;\Phi; x : \tau_1, \Gamma \vdash e_2 \ominus e_2' \lesssim t_2 : \tau_2 \qquad i \notin FV(\Phi;\Gamma, \tau_2, t_2)\end{array}}{\Delta;\Phi;\Gamma \vdash \text{unpack } e_1 \text{ as } x \text{ in } e_2 \ominus \text{unpack } e_1' \text{ as } x \text{ in } e_2' \lesssim t_1 + t_2 : \tau_2} \textbf{ r-unpack1}$$

Figure 11: Typing rules (Part 4)

Primitive application

$$\frac{\Upsilon(\zeta) = A_1 \xrightarrow{\text{exec}(k,t)} A_2 \qquad \Delta;\Phi;\Omega \vdash_{k'}^{t'} e : A_1}{\Delta;\Phi;\Omega \vdash_{k+k'+c_{app}}^{t+t'+c_{app}} \zeta\ e : A_2}\ \textbf{primapp}$$

$$\frac{\Upsilon(\zeta) = \tau_1 \xrightarrow{\text{diff}(t)} \tau_2 \qquad \Delta;\Phi;\Gamma \vdash e \ominus e' \lesssim t' : \tau_1}{\Delta;\Phi;\Gamma \vdash \zeta\ e \ominus \zeta\ e' \lesssim t + t' : \tau_2}\ \textbf{r-primapp}$$

$C\ \&\ \tau$ intro. rules

$$\frac{\Delta;\Phi \models C \qquad \Delta;\Phi \wedge C;\Omega \vdash_k^t e : A}{\Delta;\Phi;\Omega \vdash_k^t e : C\ \&\ A}\ \textbf{c-andI}$$

$$\frac{\Delta;\Phi \models C \qquad \Delta;\Phi \wedge C;\Gamma \vdash e \ominus e' \lesssim t : \tau}{\Delta;\Phi;\Gamma \vdash e \ominus e' \lesssim t : C\ \&\ \tau}\ \textbf{c-andI}$$

$C\ \&\ \tau$ elim. rules

$$\frac{\Delta;\Phi;\Omega \vdash_{k_1}^{t_1} e_1 : C\ \&\ A_1 \qquad \Delta;\Phi \wedge C; x : A_1, \Omega \vdash_{k_2}^{t_2} e_2 : A_2}{\Delta;\Phi;\Omega \vdash_{k_1+k_2}^{t_1+t_2} \text{clet } e_1 \text{ as } x \text{ in } e_2 : A_2}\ \textbf{c-andE}$$

$$\frac{\Delta;\Phi;\Gamma \vdash e_1 \ominus e_1' \lesssim t_1 : C\ \&\ \tau_1 \qquad \Delta;\Phi \wedge C; x : \tau_1, \Gamma \vdash e_2 \ominus e_2' \lesssim t_2 : \tau_2}{\Delta;\Phi;\Gamma \vdash \text{clet } e_1 \text{ as } x \text{ in } e_2 \ominus \text{clet } e_1' \text{ as } x \text{ in } e_2' \lesssim t_1 + t_2 : \tau_2}\ \textbf{r-c-andE}$$

$C \supset \tau$ intro. rules

$$\frac{\Delta;\Phi \wedge C;\Omega \vdash_k^t e : A}{\Delta;\Phi;\Omega \vdash_k^t e : C \supset A}\ \textbf{c-impI}$$

$$\frac{\Delta;\Phi \wedge C;\Gamma \vdash e \ominus e' \lesssim t : \tau}{\Delta;\Phi;\Gamma \vdash e \ominus e' \lesssim t : C \supset \tau}\ \textbf{r-c-impI}$$

$C \supset \tau$ elim. rules

$$\frac{\Delta;\Phi;\Omega \vdash_k^t e : C \supset A \qquad \Delta;\Phi \models C}{\Delta;\Phi;\Omega \vdash_k^t \text{celim } e : A}\ \textbf{c-implE}$$

$$\frac{\Delta;\Phi;\Gamma \vdash e \ominus e' \lesssim t : C \supset \tau \qquad \Delta;\Phi \models C}{\Delta;\Phi;\Gamma \vdash \text{celim } e \ominus \text{celim } e' \lesssim t : \tau}\ \textbf{r-c-implE}$$

let $x = e_1$ in e_2

$$\frac{\Delta;\Phi;\Omega \vdash_{k_1}^{t_1} e_1 : A_1 \qquad \Delta;\Phi; x : A_1, \Omega \vdash_{k_2}^{t_2} e_2 : A_2}{\Delta;\Phi;\Omega \vdash_{k_1+k_2+c_{let}}^{t_1+t_2+c_{let}} \text{let } x = e_1 \text{ in } e_2 : A_2}\ \textbf{let}$$

$$\frac{\Delta;\Phi;\Gamma \vdash e_1 \ominus e_1' \lesssim t_1 : \tau_1 \qquad \Delta;\Phi; x : \tau_1, \Gamma \vdash e_2 \ominus e_2' \lesssim t_2 : \tau_2}{\Delta;\Phi;\Gamma \vdash \text{let } x = e_1 \text{ in } e_2 \ominus \text{let } x = e_1' \text{ in } e_2' \lesssim t_1 + t_2 : \tau_2}\ \textbf{r-let1}$$

Subtyping

$$\frac{\Delta; \Phi; \Omega \vdash_k^t e : A \qquad \Delta; \Phi \models A \sqsubseteq A' \qquad \Delta; \Phi \models k' \leq k \qquad \Delta; \Phi \models t \leq t'}{\Delta; \Phi; \Omega \vdash_{k'}^{t'} e : A'} \sqsubseteq \text{exec}$$

$$\frac{\Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t : \tau \qquad \Delta; \Phi \models \tau \sqsubseteq \tau' \qquad \Delta; \Phi \models t \leq t'}{\Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t' : \tau'} \textbf{ r-} \sqsubseteq$$

Constraint dependent typing

$$\frac{\Delta; \Phi \wedge C; \Gamma \vdash_k^t e : A \qquad \Delta; \Phi \wedge \neg C; \Gamma \vdash_k^t e : A}{\Delta; \Phi; \Gamma \vdash_k^t e : A} \textbf{ split}$$

$$\frac{\Delta; \Phi \wedge C; \Gamma \vdash e_1 \ominus e_2 \lesssim t : \tau \qquad \Delta; \Phi \wedge \neg C; \Gamma \vdash e_1 \ominus e_2 \lesssim t : \tau}{\Delta; \Phi; \Gamma \vdash e_1 \ominus e_2 \lesssim t : \tau} \textbf{ r-split}$$

$$\frac{\Delta; \Phi \models \bot \qquad \Delta; \Phi \vdash \Gamma, \Omega \; \texttt{wf}}{\Delta; \Phi; \Gamma \vdash_k^t e : A} \textbf{ contra} \qquad \frac{\Delta; \Phi \models \bot \qquad \Delta; \Phi \vdash \Gamma, \Omega \; \texttt{wf}}{\Delta; \Phi; \Gamma \vdash e_1 \ominus e_2 \lesssim t : \tau} \textbf{ r-contra}$$

Heuristic typing

$$\frac{\Delta; \Phi; |\Gamma|_1 \vdash_{k_1}^{t_1} e_1 : A_1 \qquad \Delta; \Phi; x : U\,(A_1, A_1), \Gamma \vdash e_2 \ominus e \lesssim t_2 : \tau_2}{\Delta; \Phi; \Gamma \vdash \text{let } x = e_1 \text{ in } e_2 \ominus e \lesssim t_1 + t_2 + c_{let} : \tau_2} \textbf{ r-let-e}$$

$$\frac{\Delta; \Phi; |\Gamma|_2 \vdash_{k_1}^{t_1} e_1 : A_1 \qquad \Delta; \Phi; x : U\,(A_1, A_1), \Gamma \vdash e \ominus e_2 \lesssim t_2 : \tau_2}{\Delta; \Phi; \Gamma \vdash e \ominus \text{let } x = e_1 \text{ in } e_2 \lesssim t_2 - k_1 - c_{let} : \tau_2} \textbf{ r-e-let}$$

$$\frac{\Delta; \Phi; |\Gamma|_1 \vdash_{k_1}^{t_1} e_1 : A_1 \xrightarrow{\text{exec}(k,t)} A_2 \qquad \Delta; \Phi; \Gamma \vdash e_2 \ominus e_2' \lesssim t_2 : U\,(A_1, A_2')}{\Delta; \Phi; \Gamma \vdash e_1 \; e_2 \ominus e_2' \lesssim t_1 + t_2 + t + c_{app} : U\,(A_2, A_2')} \textbf{ r-app-e}$$

$$\frac{\Delta; \Phi; |\Gamma|_2 \vdash_{k_1}^{t_1} e_1' : A_1' \xrightarrow{\text{exec}(k,t)} A_2' \qquad \Delta; \Phi; \Gamma \vdash e_2 \ominus e_2' \lesssim t_2 : U\,(A_2, A_1')}{\Delta; \Phi; \Gamma \vdash e_2 \ominus e_1' \; e_2' \lesssim t_2 - k_1 - k - c_{app} : U\,(A_2, A_2')} \textbf{ r-e-app}$$

$$\frac{\begin{array}{c}\Delta; \Phi; |\Gamma|_1 \vdash_{-}^{t} e : A_1 + A_2 \\ \Delta; \Phi; x : U\,(A_1, A_1), \Gamma \vdash e_1 \ominus e' \lesssim t' : \tau \qquad \Delta; \Phi; y : U\,(A_2, A_2), \Gamma \vdash e_2 \ominus e' \lesssim t' : \tau\end{array}}{\Delta; \Phi; \Gamma \vdash \text{ case } (e, x.e_1, y.e_2) \ominus e' \lesssim t' + t + c_{case} : \tau} \textbf{ r-case-e}$$

$$\frac{\begin{array}{c}\Delta; \Phi; |\Gamma|_2 \vdash_{k'}^{-} e' : A_1 + A_2 \\ \Delta; \Phi; x : U\,(A_1, A_1), \Gamma \vdash e \ominus e_1' \lesssim t : \tau \qquad \Delta; \Phi; y : U\,(A_2, A_2), \Gamma \vdash e \ominus e_2' \lesssim t : \tau\end{array}}{\Delta; \Phi; \Gamma \vdash e \ominus \text{ case } (e', x.e_1', y.e_2') \lesssim t - k' - c_{case} : \tau} \textbf{ r-e-case}$$

Figure 12: Typing rules (Part 6)

$$
\begin{array}{lcl}
c_{case} & = & 1 \\
c_{app} & = & 1 \\
c_{caseL} & = & 1 \\
c_{caseT} & = & 1 \\
c_{proj} & = & 1 \\
c_{let} & = & 1
\end{array}
$$

Figure 13: Evaluation costs

$\boxed{\Delta; \Phi \models \tau_1 \sqsubseteq \tau_2}$ Binary type τ_1 is a subtype of relational type τ_2

$\boxed{\Delta; \Phi \models^{\mathsf{A}} A_1 \sqsubseteq A_2}$ Unary type A_1 is a subtype of type A_2

$$\frac{}{\Delta; \Phi \models \mathrm{int}_r \sqsubseteq \square\, \mathrm{int}_r}\ \textbf{int-}\square \qquad \frac{}{\Delta; \Phi \models \square\, U\,(\mathrm{int}, \mathrm{int}) \sqsubseteq \mathrm{int}_r}\ \square\textbf{U-int}$$

$$\frac{}{\Delta; \Phi \models \mathrm{unit}_r \sqsubseteq \square\, \mathrm{unit}_r}\ \textbf{unit}$$

$$\frac{\Delta; \Phi \models \tau_1' \sqsubseteq \tau_1 \qquad \Delta; \Phi \models \tau_2 \sqsubseteq \tau_2' \qquad \Delta; \Phi \models t \leq t'}{\Delta; \Phi \models \tau_1 \xrightarrow{\mathrm{diff}(t)} \tau_2 \sqsubseteq \tau_1' \xrightarrow{\mathrm{diff}(t')} \tau_2'}\ \rightarrow \textbf{diff}$$

$$\frac{}{\Delta; \Phi \models \square\,(\tau_1 \xrightarrow{\mathrm{diff}(t)} \tau_2) \sqsubseteq \square\, \tau_1 \xrightarrow{\mathrm{diff}(0)} \square\, \tau_2}\ \rightarrow \square\textbf{diff}$$

$$\frac{}{\Delta; \Phi \models U\,(A_1 \xrightarrow{\mathrm{exec}(k,t)} A_2, A_1' \xrightarrow{\mathrm{exec}(k',t')} A_2') \sqsubseteq U\,(A_1, A_1') \xrightarrow{\mathrm{diff}(t-k')} U\,(A_2, A_2')}\ \rightarrow \textbf{execdiff}$$

$$\frac{i :: S, \Delta; \Phi \models \tau \sqsubseteq \tau' \qquad i :: S, \Delta; \Phi \models t \leq t' \qquad i \notin FV(\Phi)}{\Delta; \Phi \models \forall i \overset{\mathrm{diff}(t)}{::} S.\, \tau \sqsubseteq \forall i \overset{\mathrm{diff}(t')}{::} S.\, \tau'}\ \forall\textbf{diff}$$

$$\frac{}{\Delta; \Phi \models \square\,(\forall i \overset{\mathrm{diff}(t)}{::} S.\, \tau) \sqsubseteq \forall i \overset{\mathrm{diff}(0)}{::} S.\, \square\, \tau}\ \forall\ \square$$

$$\frac{}{\Delta; \Phi \models U\,(\forall i \overset{\mathrm{exec}(k,t)}{::} S.\, A, \forall i \overset{\mathrm{exec}('k,t')}{::} S.\, A') \sqsubseteq \forall i \overset{\mathrm{diff}(t-k')}{::} S.\, U\,(A, A')}\ \forall\textbf{U}$$

$$\frac{\Delta; \Phi \models \tau_1 \sqsubseteq \tau_1' \qquad \Delta; \Phi \models \tau_2 \sqsubseteq \tau_2'}{\Delta; \Phi \models \tau_1 \times \tau_2 \sqsubseteq \tau_1' \times \tau_2'}\ \times \qquad \frac{}{\Delta; \Phi \models \square\, \tau_1 \times \square\, \tau_2 \equiv \square\,(\tau_1 \times \tau_2)}\ \times\square$$

$$\frac{}{\Delta; \Phi \models U\,(A_1 \times A_2, A_1' \times A_2') \sqsubseteq U\,(A_1, A_1') \times U\,(A_2, A_2')}\ \times\textbf{U}$$

$$\frac{\Delta; \Phi \models \tau_1 \sqsubseteq \tau_1' \qquad \Delta; \Phi \models \tau_2 \sqsubseteq \tau_2'}{\Delta; \Phi \models \tau_1 + \tau_2 \sqsubseteq \tau_1' + \tau_2'}\ + \qquad \frac{}{\Delta; \Phi \models \square\, \tau_1 + \square\, \tau_2 \sqsubseteq \square\,(\tau_1 + \tau_2)}\ +\square$$

$$\frac{\Delta; \Phi \models n \doteq n' \qquad \Delta; \Phi \models \alpha \leq \alpha' \qquad \Delta; \Phi \models \tau \sqsubseteq \tau'}{\Delta; \Phi \models \mathrm{list}[n]^{\alpha}\, \tau \sqsubseteq \mathrm{list}[n']^{\alpha'}\, \tau'}\ \textbf{l1}$$

$$\frac{\Delta; \Phi \models \alpha \doteq 0}{\Delta; \Phi \models \mathrm{list}[n]^{\alpha}\, \tau \sqsubseteq \mathrm{list}[n]^{\alpha}\, \square\, \tau}\ \textbf{l2} \qquad \frac{}{\Delta; \Phi \models \mathrm{list}[n]^{\alpha}\, \square\, \tau \sqsubseteq \square\,(\mathrm{list}[n]^{\alpha}\, \tau)}\ \textbf{l}\square$$

Figure 14: Subtyping rules (part 1)

$\boxed{\Delta; \Phi \models \tau_1 \sqsubseteq \tau_2}$ Binary type τ_1 is a subtype of type τ_2

$$\frac{\Delta; \Phi \models n \doteq n' \qquad \Delta; \Phi \models \alpha \leq \alpha' \qquad \Delta; \Phi \models \tau \sqsubseteq \tau'}{\Delta; \Phi \models \mathrm{tree}[n]^{\alpha}\, \tau \sqsubseteq \mathrm{tree}[n']^{\alpha'}\, \tau'}\ \mathbf{t1}$$

$$\frac{\Delta; \Phi \models \alpha \doteq 0}{\Delta; \Phi \models \mathrm{tree}[n]^{\alpha}\, \tau \sqsubseteq \mathrm{tree}[n]^{\alpha}\, \square\, \tau}\ \mathbf{t2} \qquad \frac{}{\Delta; \Phi \models \mathrm{tree}[n]^{\alpha}\, \square\, \tau \sqsubseteq \square\, (\mathrm{tree}[n]^{\alpha}\, \tau)}\ \mathbf{t\square}$$

$$\frac{i :: S, \Delta; \Phi \models \tau \sqsubseteq \tau' \qquad i \notin FV(\Phi)}{\Delta; \Phi \models \exists i{::}S.\, \tau \sqsubseteq \exists i{::}S.\, \tau'}\ \exists \qquad \frac{}{\Delta; \Phi \models \exists i{::}S.\, \square\, \tau \sqsubseteq \square\, (\exists i{::}S.\, \tau)}\ \exists\square$$

$$\frac{\Delta; \Phi \wedge C \models C' \qquad \Delta; \Phi \models \tau \sqsubseteq \tau'}{\Delta; \Phi \models C \,\&\, \tau \sqsubseteq C' \,\&\, \tau'}\,\mathbf{c\text{-}and} \qquad \frac{}{\Delta; \Phi \models C \,\&\, \square\, \tau \sqsubseteq \square\, (C \,\&\, \tau)}\ \mathbf{c\text{-}and\text{-}\square}$$

$$\frac{\Delta; \Phi \wedge C' \models C \qquad \Delta; \Phi \models \tau \sqsubseteq \tau'}{\Delta; \Phi \models C \supset \tau \sqsubseteq C' \supset \tau'}\,\mathbf{c\text{-}impl} \qquad \frac{}{\Delta; \Phi \models \square\, (C \supset \tau) \sqsubseteq C \supset \square\, \tau}\ \mathbf{c\text{-}impl\text{-}\square}$$

$$\frac{}{\Delta; \Phi \models \square\, \tau \sqsubseteq \tau}\ \mathbf{T} \qquad \frac{}{\Delta; \Phi \models \square\, \tau \sqsubseteq \square\, \square\, \tau}\ \mathbf{D} \qquad \frac{\Delta; \Phi \models \tau_1 \sqsubseteq \tau_2}{\Delta; \Phi \models \square\, \tau_1 \sqsubseteq \square\, \tau_2}\ \mathbf{B\text{-}\square}$$

$$\frac{}{\Delta; \Phi \models \tau \sqsubseteq U\, (|\tau|_1, |\tau|_2)}\ \mathbf{W} \qquad \frac{}{\Delta; \Phi \models \tau \sqsubseteq \tau}\,\mathbf{refl}$$

$$\frac{\Delta; \Phi \models \tau_1 \sqsubseteq \tau_2 \qquad \Delta; \Phi \models \tau_2 \sqsubseteq \tau_3}{\Delta; \Phi \models \tau_1 \sqsubseteq \tau_3}\,\mathbf{tran}$$

Figure 15: Subtyping rules (Part 2)

$\boxed{\Delta; \Phi \models^{\mathsf{A}} A_1 \sqsubseteq A_2}$ Unary type A_1 is a subtype of type A_2

$$\frac{\Delta; \Phi \models^{\mathsf{A}} A_1' \sqsubseteq A_1 \qquad \Delta; \Phi \models^{\mathsf{A}} A_2 \sqsubseteq A_2' \qquad \Delta; \Phi \models k' \leq k \qquad \Delta; \Phi \models t \leq t'}{\Delta; \Phi \models^{\mathsf{A}} A_1 \xrightarrow{\text{exec}(k,t)} A_2 \sqsubseteq A_1' \xrightarrow{\text{exec}(k',t')} A_2'} \rightarrow \textbf{exec}$$

$$\frac{i :: S, \Delta; \Phi \models^{\mathsf{A}} A \sqsubseteq A' \qquad i :: S, \Delta; \Phi \models k' \leq k \qquad i :: S, \Delta; \Phi \models t \leq t' \qquad i \notin FV(\Phi)}{\Delta; \Phi \models^{\mathsf{A}} \forall i \overset{\text{exec}(k,t)}{::} S.\, A \sqsubseteq \forall i \overset{\text{exec}(k',t')}{::} S.\, A} \textbf{u-}\forall\textbf{exec}$$

$$\frac{\Delta; \Phi \models^{\mathsf{A}} A_1 \sqsubseteq A_1' \qquad \Delta; \Phi \models^{\mathsf{A}} A_2 \sqsubseteq A_2'}{\Delta; \Phi \models^{\mathsf{A}} A_1 \times A_2 \sqsubseteq A_1' \times A_2'} \textbf{u-}\times$$

$$\frac{\Delta; \Phi \models^{\mathsf{A}} A_1 \sqsubseteq A_1' \qquad \Delta; \Phi \models^{\mathsf{A}} A_2 \sqsubseteq A_2'}{\Delta; \Phi \models^{\mathsf{A}} A_1 + A_2 \sqsubseteq A_1' + A_2'} \textbf{u-}+ \qquad \frac{\Delta; \Phi \models n \doteq n' \qquad \Delta; \Phi \models^{\mathsf{A}} A \sqsubseteq A'}{\Delta; \Phi \models^{\mathsf{A}} \text{list}[n]\, A \sqsubseteq \text{list}[n']\, A'} \textbf{u-l}$$

$$\frac{\Delta; \Phi \models n \doteq n' \qquad \Delta; \Phi \models^{\mathsf{A}} A \sqsubseteq A'}{\Delta; \Phi \models^{\mathsf{A}} \text{tree}[n]\, A \sqsubseteq \text{tree}[n']\, A'} \textbf{u-t} \qquad \frac{i :: S, \Delta; \Phi \models^{\mathsf{A}} A \sqsubseteq A' \qquad i \notin FV(\Phi)}{\Delta; \Phi \models^{\mathsf{A}} \exists i{::}S.\, A \sqsubseteq \exists i{::}S.\, A'} \textbf{u-}\exists$$

$$\frac{\Delta; \Phi \wedge C \models C' \qquad \Delta; \Phi \models^{\mathsf{A}} A \sqsubseteq A'}{\Delta; \Phi \models^{\mathsf{A}} C \,\&\, A \sqsubseteq C' \,\&\, A'} \textbf{u-c-and}$$

$$\frac{\Delta; \Phi \wedge C' \models C \qquad \Delta; \Phi \models^{\mathsf{A}} A \sqsubseteq A'}{\Delta; \Phi \models^{\mathsf{A}} C \supset A \sqsubseteq C' \supset A'} \textbf{u-c-impl} \qquad \frac{}{\Delta; \Phi \models^{\mathsf{A}} A \sqsubseteq A} \textbf{u-refl}$$

$$\frac{\Delta; \Phi \models^{\mathsf{A}} A_1 \sqsubseteq A_2 \qquad \Delta; \Phi \models^{\mathsf{A}} A_2 \sqsubseteq A_3}{\Delta; \Phi \models^{\mathsf{A}} A_1 \sqsubseteq A_3} \textbf{u-tran}$$

Figure 16: Unary subtyping rules

$\boxed{e \Downarrow^c v}$ Expression e evaluates to value v with cost c

$$\frac{}{\texttt{n} \Downarrow^0 \texttt{n}}\ \textbf{const} \qquad \frac{e \Downarrow^c v}{\text{inl } e \Downarrow^c \text{inl } v}\ \textbf{inl} \qquad \frac{e \Downarrow^c v}{\text{inr } e \Downarrow^c \text{inr } v}\ \textbf{inr}$$

$$\frac{e \Downarrow^c \text{inl } v \qquad e_1[v/x] \Downarrow^{c_r} v_r}{\text{case } (e, x.e_1, y.e_2) \Downarrow^{c+c_r+c_{case}} v_r}\ \textbf{case-inl} \qquad \frac{e \Downarrow^c \text{inr } v \qquad e_2[v/y] \Downarrow^{c_r} v_r}{\text{case } (e, x.e_1, y.e_2) \Downarrow^{c+c_r+c_{case}} v_r}\ \textbf{case-inr}$$

$$\frac{}{\text{fix } f(x).e \Downarrow^0 \text{fix } f(x).e}\ \textbf{fix}$$

$$\frac{e_1 \Downarrow^{c_1} \text{fix } f(x).e \qquad e_2 \Downarrow^{c_2} v_2 \qquad e[v_2/x, (\text{fix } f(x).e)/f] \Downarrow^{c_r} v_r}{e_1\ e_2 \Downarrow^{c_1+c_2+c_r+c_{app}} v_r}\ \textbf{app}$$

$$\frac{e \Downarrow^c v \qquad \hat{\zeta}(v) = (c_r, v_r)}{\zeta\ e \Downarrow^{c+c_r+c_{app}} v_r}\ \textbf{primapp} \qquad \frac{}{\Lambda e \Downarrow^0 \Lambda e}\ \textbf{Lam}$$

$$\frac{e \Downarrow^c \Lambda e_b \qquad e_b \Downarrow^{c_r} v_r}{e[\,] \Downarrow^{c+c_r} v_r}\ \textbf{iApp} \qquad \frac{e \Downarrow^c v}{\text{pack } e \Downarrow^c \text{pack } v}\ \textbf{pack}$$

$$\frac{e_1 \Downarrow^{c_1} \text{pack } v \qquad e_2[v/x] \Downarrow^{c_2} v_r}{\text{unpack } e_1 \text{ as } x \text{ in } e_2 \Downarrow^{c_1+c_2} v_r}\ \textbf{unpack} \qquad \frac{}{\text{nil } \Downarrow^0 \text{nil}}\ \textbf{nil}$$

$$\frac{e_1 \Downarrow^{c_1} v_1 \qquad e_2 \Downarrow^{c_2} v_2}{\text{cons}(e_1, e_2) \Downarrow^{c_1+c_2} \text{cons}(v_1, v_2)}\ \textbf{cons}$$

$$\frac{e \Downarrow^c \text{nil} \qquad e_1 \Downarrow^{c_r} v_r}{\text{case } e \text{ of nil } \to e_1 \mid h :: tl \to e_2 \Downarrow^{c+c_r+c_{caseL}} v_r}\ \textbf{caseL-nil}$$

$$\frac{e \Downarrow^c \text{cons}(v_1, v_2) \qquad e_2[v_1/h, v_2/tl] \Downarrow^{c_r} v_r}{\text{case } e \text{ of nil } \to e_1 \mid h :: tl \to e_2 \Downarrow^{c+c_r+c_{caseL}} v_r}\ \textbf{caseL-cons}$$

$$\frac{e_l \Downarrow^{c_l} v_l \qquad e \Downarrow^c v \qquad e_r \Downarrow^{c_r} v_r}{\text{node}(e_l, e, e_r) \Downarrow^{c+c_l+c_r} \text{node}(v_l, v, v_r)}\ \textbf{node}$$

$$\frac{e \Downarrow^c \text{leaf} \qquad e_1 \Downarrow^{c_r} v_r}{\text{case } e \text{ of leaf} \to e_1 \mid \text{node}(l, x, r) \to e_2 \Downarrow^{c+c_r+c_{caseT}} v_r}\ \textbf{caseT-leaf}$$

$$\frac{e \Downarrow^c \text{node}(v_l, v, v_r) \qquad e_2[v_l/l, v/x, v_r/r] \Downarrow^{c_r} v_r}{\text{case } e \text{ of nil } \to e_1 \mid \text{node}(l, x, r) \to e_2 \Downarrow^{c+c_r+c_{caseT}} v_r}\ \textbf{caseT-node}$$

$$\frac{e_1 \Downarrow^{c_1} v_1 \qquad e_2 \Downarrow^{c_2} v_2}{\langle e_1, e_2 \rangle \Downarrow^{c_1+c_2} \langle v_1, v_2 \rangle}\ \textbf{prod} \qquad \frac{e \Downarrow^c \langle v_1, v_2 \rangle}{\pi_1(e) \Downarrow^{c+c_{proj}} v_1}\ \textbf{proj1} \qquad \frac{e \Downarrow^c \langle v_1, v_2 \rangle}{\pi_2(e) \Downarrow^{c+c_{proj}} v_2}\ \textbf{proj2}$$

$$\frac{e_1 \Downarrow^{c_1} v_1 \qquad e_2[v_1/x] \Downarrow^{c_r} v_r}{\text{let } x = e_1 \text{ in } e_2 \Downarrow^{c_1+c_r+c_{let}} v_r}\ \textbf{let} \qquad \frac{e_1 \Downarrow^{c_1} v_1 \qquad e_2[v_1/x] \Downarrow^{c_r} v_r}{\text{clet } e_1 \text{ as } x \text{ in } e_2 \Downarrow^{c_1+c_r} v_r}\ \textbf{clet}$$

$$\frac{e \Downarrow^c v}{\text{celim } e \Downarrow^c v}\ \textbf{celim}$$

Figure 17: Evaluation semantics

$$
\begin{aligned}
(\!|\tau|\!)_v &\subseteq \text{Step index} \times \text{Value} \times \text{Value}\\
(\!|\tau|\!)^t_\varepsilon &\subseteq \text{Step index} \times \text{Expression} \times \text{Expression}
\end{aligned}
$$

$$
\begin{aligned}
(\!|\square\,\tau|\!)_v &= \{(m, v, v) \mid (m, v, v) \in (\!|\tau|\!)_v\}\\
(\!|U\,(A_1, A_2)|\!)_v &= \{(m, v_1, v_2) \mid \forall j.\ (j, v_1) \in [\![A_1]\!]_v \wedge (j, v_2) \in [\![A_2]\!]_v\}\\
(\!|\text{int}_r|\!)_v &= \{(m, \mathtt{n}, \mathtt{n})\}\\
(\!|\text{unit}_r|\!)_v &= \{(m, (), ())\}\\
(\!|\tau_1 \times \tau_2|\!)_v &= \{(m, \langle v_1, v_2\rangle, \langle v_1', v_2'\rangle) \mid (m, v_1, v_1') \in (\!|\tau_1|\!)_v \ \wedge\ (m, v_2, v_2') \in (\!|\tau_2|\!)_v\}\\
(\!|\tau_1 + \tau_2|\!)_v &= \{(m, \text{inl } v, \text{inl } v') \mid (m, v, v') \in (\!|\tau_1|\!)_v\} \cup \{(m, \text{inr } v, \text{inr } v') \mid (m, v, v') \in (\!|\tau_2|\!)_v\}\\
(\!|\tau_1 \wedge \tau_2|\!)_v &= \{(m, v, v') \mid (m, v, v') \in (\!|\tau_1|\!)_v \ \wedge\ (m, v, v') \in (\!|\tau_2|\!)_v\}\\
(\!|\text{list}[0]^\alpha\,\tau|\!)_v &= \{(m, \text{nil}\ , \text{nil}\ )\}\\
(\!|\text{list}[n{+}1]^\alpha\,\tau|\!)_v &= \{(m, \text{cons}(e_1, e_2), \text{cons}(e_1', e_2')) \mid ((m, e_1, e_1') \in (\!|\square\,\tau|\!)_v \wedge (m, e_2, e_2') \in (\!|\text{list}[n]^\alpha\,\tau|\!)_v) \vee\\
&\qquad ((m, e_1, e_1') \in (\!|\tau|\!)_v \wedge (m, e_2, e_2') \in (\!|\text{list}[n]^{\alpha-1}\,\tau|\!)_v \wedge \alpha > 0)\}\\
(\!|\text{tree}[0]^\alpha\,\tau|\!)_v &= \{(m, \text{leaf}, \text{leaf})\}\\
(\!|\text{tree}[i{+}j{+}1]^\alpha\,\tau|\!)_v &= \{(m, \text{node}(e_l, e, e_r), \text{node}(e_l', e', e_r')) \mid\\
&\quad ((m, e_l, e_l') \in (\!|\text{tree}[i]^\beta\,\tau|\!)_v \ \wedge\ (m, e_r, e_r') \in (\!|\text{tree}[j]^\gamma\,\tau|\!)_v \ \wedge\ (m, e, e') \in (\!|\square\,\tau|\!)_v \ \wedge\ \alpha{=}\beta{+}\gamma) \vee\\
&\quad ((m, e_l, e_l') \in (\!|\text{tree}[i]^\beta\,\tau|\!)_v \ \wedge\ (m, e_r, e_r') \in (\!|\text{tree}[j]^\gamma\,\tau|\!)_v \ \wedge\ (m, e, e') \in (\!|\tau|\!)_v \ \wedge\ \alpha{=}\beta{+}\gamma{+}1)\}\\
(\!|\tau_1 \xrightarrow{\text{diff}(t)} \tau_2|\!)_v &= \{(m, \text{fix } f(x).e_1, \text{fix } f(x).e_2) \mid (\forall j < m.\ \forall v_1, v_2.\ (j, v_1, v_2) \in (\!|\tau_1|\!)_v. \implies\\
&\qquad (j, e_1[v_1/x, \text{fix } f(x).e_1/f], e_2[v_2/x, \text{fix } f(x).e_2/f]) \in (\!|\tau_2|\!)^t_\varepsilon) \ \wedge\\
&\quad (\forall j.(j, \text{fix } f(x).e_1) \in [\![|\tau_1|_1 \xrightarrow{\text{exec}(0,\infty)} |\tau_2|_1]\!]_v \ \wedge\ (j, \text{fix } f(x).e_2) \in [\![|\tau_1|_2 \xrightarrow{\text{exec}(0,\infty)} |\tau_2|_2]\!]_v)\}\\
(\!|\forall i \overset{\text{diff}(t)}{::} S.\,\tau|\!)_v &= \{(m, \Lambda e, \Lambda e') \mid \forall I.\ \vdash I :: S.\ ((m, e, e') \in (\!|\tau\{I/i\}|\!)^{t[I/i]}_\varepsilon) \ \wedge\\
&\qquad (\forall j.(j, e) \in [\![|\tau\{I/i\}|_1]\!]^{0,\infty}_\varepsilon \ \wedge\ (j, e') \in [\![|\tau\{I/i\}|_2]\!]^{0,\infty}_\varepsilon)\}\\
(\!|\exists i{::}S.\,\tau|\!)_v &= \{(m, \text{pack } v, \text{pack } v') \mid \exists I. \vdash I :: S \ \wedge (m, v, v') \in (\!|\tau\{I/t\}|\!)_v\}\\
(\!|C \supset \tau|\!)_v &= \{(m, v_1, v_2) \mid \ \not\models C \ \vee\ (m, v_1, v_2) \in (\!|\tau|\!)_v\}\\
(\!|C \,\&\, \tau|\!)_v &= \{(m, v_1, v_2) \mid \ \models C \ \wedge\ (m, v_1, v_2) \in (\!|\tau|\!)_v\}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{G}(\!|\cdot|\!) &= \{(m, \emptyset, \emptyset)\}\\
\mathcal{G}(\!|\Gamma, x : \tau|\!) &= \{(m, \theta[x \mapsto v_1], \theta'[x \mapsto v_2]) \mid (m, \theta, \theta') \in \mathcal{G}(\!|\Gamma|\!) \ \wedge (m, v_1, v_2) \in (\!|\tau|\!)_v\}
\end{aligned}
$$

$$
\begin{aligned}
(\!|\tau|\!)^t_\varepsilon &= \{(m, e_1, e_2) \mid (e_1 \Downarrow^{c_1} v_1 \ \wedge\ e_2 \Downarrow^{c_2} v_2 \ \wedge\ c_1 < m) \implies\\
&\qquad\qquad 1.\ c_1 - c_2 \leq t\\
&\qquad\qquad 2.\ (m - c_1, v_1, v_2) \in (\!|\tau|\!)_v\\
&\quad\}
\end{aligned}
$$

Figure 18: Relational interpretation of types

$$
\begin{aligned}
&[\![A]\!]_v \;\;\subseteq \text{Step index} \times \text{Value}\\
&[\![A]\!]_\varepsilon^{k,t} \subseteq \text{Step index} \times \text{Expression}
\end{aligned}
$$

$$
\begin{aligned}
[\![\text{int}]\!]_v &= \{(m, \mathtt{n})\}\\
[\![\text{unit}]\!]_v &= \{(m, ())\}\\
[\![A_1 \times A_2]\!]_v &= \{(m, \langle v_1, v_2\rangle) \mid (m, v_1) \in [\![A_1]\!]_v \wedge (m, v_2) \in [\![A_2]\!]_v\}\\
[\![A_1 + A_2]\!]_v &= \{(m, \text{inl } v) \mid (m, v) \in [\![A_1]\!]_v\} \cup \{(m, \text{inr } v) \mid (m, v) \in [\![A_2]\!]_v\}\\
[\![A_1 \wedge A_2]\!]_v &= \{(m, v) \mid (m, v) \in [\![A_1]\!]_v \wedge (m, v) \in [\![A_2]\!]_v\}\\
[\![\text{list}[0]\, A]\!]_v &= \{(m, \text{nil })\}\\
[\![\text{list}[n{+}1]\, A]\!]_v &= \{(m, \text{cons}(e_1, e_2)) \mid (m, e_1) \in [\![A]\!]_v \wedge (m, e_2) \in [\![\text{list}[n]\, A]\!]_v\}\\
[\![\text{tree}[0]\, A]\!]_v &= \{(m, \text{leaf})\}\\
[\![\text{tree}[i{+}j{+}1]\, A]\!]_v &= \{(m, \text{node}(e_l, e, e_r)) \mid (m, e_l) \in [\![\text{tree}[i]\, A]\!]_v \wedge (m, e_r) \in [\![\text{tree}[j]\, A]\!]_v \wedge (m, e) \in [\![A]\!]_v\}\\
[\![A_1 \xrightarrow{\text{exec}(k,t)} A_2]\!]_v &= \{(m, \text{fix } f(x).e) \mid \forall j < m.\ \forall v.\ (j, v) \in [\![A_1]\!]_v \implies (j, e[v/x, \text{fix } f(x).e]) \in [\![A_2]\!]_\varepsilon^{k,t}\}\\
[\![\forall i \overset{\text{exec}(k,t)}{::} S.\, A]\!]_v &= \{(m, \Lambda e) \mid \forall I.\ \vdash I :: S.\ (m, e) \in [\![A\{I/i\}]\!]_\varepsilon^{k[I/i], t[I/i]}\}\\
[\![\exists i{::}S.\, A]\!]_v &= \{(m, \text{pack } v) \mid \exists I. \vdash I :: S\ \wedge (m, v) \in [\![A\{I/t\}]\!]_v\}\\
[\![C \supset A]\!]_v &= \{(m, v) \mid\ \not\models C\ \vee\ (m, v) \in [\![A]\!]_v\}\\
[\![C\ \&\ A]\!]_v &= \{(m, v) \mid\ \models C\ \wedge\ (m, v) \in [\![A]\!]_v\}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{G}[\![\cdot]\!] &= \{(m, \emptyset)\}\\
\mathcal{G}[\![\Omega, x : A]\!] &= \{(m, \gamma[x \mapsto v]) \mid (m, \gamma) \in \mathcal{G}[\![\Omega]\!]\ \wedge (m, v) \in [\![A]\!]_v\}
\end{aligned}
$$

$$
\begin{aligned}
[\![A]\!]_\varepsilon^{k,t} &= \{(m, e) \mid (t < m \implies \begin{array}{l} 1.\ e \Downarrow^c v\\ 2.\ c \leq t\\ 3.\ (m\ -\ c, v) \in [\![A]\!]_v \end{array}) \ \wedge\\
&\qquad ((e \Downarrow^c m\ \wedge\ c < m) \implies \begin{array}{l} 1.\ k \leq c\\ 2.\ (m\ -\ c, v) \in [\![A]\!]_v \end{array}\}
\end{aligned}
$$

Figure 19: Non-relational interpretation of types

Lemma 1 (Value evaluation)
$v \Downarrow^0 v$

Proof. Proof is by induction on the value term v. $\square$

Lemma 2 (Value interpretation containment)
The following hold.

1. $(m, v_1, v_2) \in (\!|\tau|\!)_v$ then $(m, v_1, v_2) \in (\!|\tau|\!)^0_\varepsilon$.
2. $(m, v) \in [\![A]\!]_v$ then $(m, v) \in [\![A]\!]^{0,t}_\varepsilon$.

Proof of (1). Assume that $(m, v_1, v_2) \in (\!|\tau|\!)_v$ $(\star)$.
TS: $(m, v_1, v_2) \in (\!|\tau|\!)^0_\varepsilon$.
Following the definition of $(\!|\tau|\!)^0_\varepsilon$, and assume that $(v_1 \Downarrow^0 v_1 \ \wedge\ 0 < m)$ (cost and resulting value obtained by Lemma 1).
Then, we can immediately show

1. $v_2 \Downarrow^0 v_2$ by Lemma 1
2. $0 - 0 \leq 0$ is trivially true.
3. $(m - 0, v_1, v_2) \in (\!|\tau|\!)_v$ follows from the main assumption $(\star)$.

$\square$

Proof of (2). Assume that $(m, v) \in [\![A]\!]_v$ $(\star)$.
TS: $(m, v) \in [\![A]\!]^{0,t}_\varepsilon$.
Following the definition of $[\![A]\!]^{0,t}_\varepsilon$, there are two parts:

- Assume that $t < m$. Then we can immediately show
  1. $v \Downarrow^0 v$ (by Lemma 1)
  2. $0 \leq t$
  3. $(m - 0, v) \in [\![A]\!]_v$ which follows from the assumption $(\star)$.
- Assume that $v \Downarrow^0 v$ (cost and the resulting value obtained by Lemma 1) and $0 < m$. Then, we can immediately show
  1. $0 \leq 0$
  2. $(m - 0, v) \in [\![A]\!]_v$ which follows from the assumption $(\star)$.

$\square$

Lemma 3 (Value Projection)
The following holds.

1. If $(m, v_1, v_2) \in (\!|\tau|\!)_v$ then $\forall m.(m, v_1) \in [\![|\tau|_1]\!]_v$ and $(m, v_2) \in [\![|\tau|_2]\!]_v$.
2. If $(m, \delta_1, \delta_2) \in \mathcal{G}(\!|\Gamma|\!)$ then $\forall m.(m, \delta_1) \in \mathcal{G}[\![|\Gamma|_1]\!]$ and $(m, \delta_2) \in \mathcal{G}[\![|\Gamma|_2]\!]$.

Proof. Proof of statement (1) is by induction on $(\!|\tau|\!)_v$. Proof of statement (2) follows by proof of (1). $\square$

Lemma 4 (Downward Closure)
The following hold.

1. If $(m, v_1, v_2) \in (\!|\tau|\!)_v$ and $m' \leq m$, then $(m', v_1, v_2) \in (\!|\tau|\!)_v$
2. If $(m, v) \in [\![A]\!]_v$ and $m' \leq m$, then $(m', v) \in [\![A]\!]_v$
3. If $(m, e_1, e_2) \in (\!|\tau|\!)^t_\varepsilon$ and $m' \leq m$, then $(m', e_1, e_2) \in (\!|\tau|\!)^t_\varepsilon$
4. If $(m, e) \in [\![A]\!]^{k,t}_\varepsilon$ and $m \leq m'$, then $(m', e) \in [\![A]\!]^{k,t}_\varepsilon$
5. If $(m, \delta_1, \delta_2) \in \mathcal{G}(\!|\Gamma|\!)$ and $m' \leq m$, then $(m', \delta_1, \delta_2) \in \mathcal{G}(\!|\Gamma|\!)$
6. If $(m, \gamma) \in \mathcal{G}[\![\Omega]\!]$ and $m' \leq m$, then $(m', \gamma) \in \mathcal{G}[\![\Omega]\!]$

Proof. (1,3) and (2,4) are proved simultaneously by induction on τ. (5,6) follows from (1,2). $\square$

Lemma 5 (Subtyping Soundness)
The following hold.

1. If $\Delta; \Phi \models \tau \sqsubseteq \tau'$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, v, v') \in (\!|\sigma\tau|\!)_v$, then $(m, v, v') \in (\!|\sigma\tau'|\!)_v$.
2. If $\Delta; \Phi \models^{\mathsf{A}} A \sqsubseteq A'$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, v) \in [\![\sigma A]\!]_v$, then $(m, v) \in [\![\sigma A']\!]_v$.
3. If $\Delta; \Phi \models \tau \sqsubseteq \tau'$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, e, e') \in (\!|\sigma\tau|\!)^t_\varepsilon$ and $t \leq t'$, then $(m, e, e') \in (\!|\sigma\tau'|\!)^{t'}_\varepsilon$.
4. If $\Delta; \Phi \models^{\mathsf{A}} A \sqsubseteq A'$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, e) \in [\![\sigma A]\!]^{k,t}_\varepsilon$ and $k' \leq k$ and $t \leq t'$, then $(m, e) \in [\![\sigma A']\!]^{k',t'}_\varepsilon$.

Proof. Statements (1) and (2) are by proven simultaneously by induction on the subtyping derivation. We first show the proof of statements (3) and (4). $\square$

Proof of statement (3). Assume that $\Delta; \Phi \models \tau \sqsubseteq \tau'$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, e, e') \in (\!|\sigma\tau|\!)^t_\varepsilon$ and $t \leq t'$.
TS: $(m, e, e') \in (\!|\sigma\tau'|\!)^{t'}_\varepsilon$
Assume that

a) $e \Downarrow^c v$
b) $e' \Downarrow^{c'} v'$
c) $c < m$

By unfolding the assumption $(m, e, e') \in (\!|\sigma\tau|\!)^t_\varepsilon$ using (a-c), we obtain

d) $c - c' \leq t$
e) $(m - c, v, v') \in (\!|\sigma\tau|\!)_v$

We can conclude as follows:

1. Since $c - c' \leq t$ from d) and $t \leq t'$ from the assumption, we get $c - c' \leq t'$.
2. By IH 1 on e), we get $(m - c, v, v') \in (\!|\sigma\tau'|\!)_v$.

$\square$

Proof of statement (4). Assume that $\Delta; \Phi \models A \sqsubseteq A'$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $(m, e) \in [\![\sigma A]\!]_\varepsilon^{k,t}$ and $k' \leq k$ and $t \leq t'$.
TS: $(m, e) \in [\![\sigma A']\!]_\varepsilon^{k',t'}$
There are two parts to show:

- Assume that $t' < m$.
  By unfolding the main assumption $(m, e) \in [\![\sigma A]\!]_\varepsilon^{k,t}$ with $t \leq t' < m$, we get

  a) $e \Downarrow^c v$
  b) $c \leq t$
  c) $(m - c, v) \in [\![\sigma A]\!]_v$

  We can conclude as follows:

  1. By a), $e \Downarrow^c v$
  2. Since $c \leq t$ from a) and $t \leq t'$ from the assumption, we get $c \leq t'$.
  3. By IH 2 on the main assumption using c), we get $(m - c, v) \in [\![\sigma A']\!]_v$.

- Assume that $e \Downarrow^c v$ and $c < m$.
  By unfolding the main assumption $(m, e) \in [\![\sigma A]\!]_\varepsilon^{k,t}$ with $e \Downarrow^c v$ and $c < m$, we get

  d) $k \leq c \leq t$
  e) $(m - c, v) \in [\![\sigma A]\!]_v$

  We can conclude as follows:

  1. Since $k' \leq k$ and $t \leq t'$ (from the assumption) and $k \leq c \leq t$ (from a), we get $k' \leq c \leq t'$.

  2. By IH 2 on the main assumption using b).

$\square$

Proof of statement (1). Proof is by induction on the subtyping derivation.

Case $$\dfrac{\Delta; \Phi \models \tau_1' \sqsubseteq \tau_1 \qquad \Delta; \Phi \models \tau_2 \sqsubseteq \tau_2' \qquad \Delta; \Phi \models t \leq t'}{\Delta; \Phi \models \tau_1 \xrightarrow{\mathbf{diff}(t)} \tau_2 \sqsubseteq \tau_1' \xrightarrow{\mathbf{diff}(t')} \tau_2'} \rightarrow \mathbf{diff}$$

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$.
We have

$$(m, \text{fix } f(x).e, \text{fix } f(x).e') \in (\!|\sigma\tau_1 \xrightarrow{\text{diff}(\sigma t)} \sigma\tau_2|\!)_v \tag{1}$$

TS: $(m, \text{fix } f(x).e, \text{fix } f(x).e') \in (\!|\sigma\tau_1' \xrightarrow{\text{diff}(\sigma t')} \sigma\tau_2'|\!)_v$.
There are two cases to show.

subcase 1: Assume that $j < m$ and $(j, v, v') \in (\!|\sigma\tau_1'|\!)_v$.
STS: $(j, e[v/x, (\text{fix } f(x).e)/f], e'[v'/x, (\text{fix } f(x).e')/f]) \in (\!|\sigma\tau_2'|\!)_\varepsilon^{\sigma t'}$.
By IH 1 on $(j, v, v') \in (\!|\sigma\tau_1'|\!)_v$ using the first premise, we get

$$(j, v, v') \in (\!|\sigma\tau_1|\!)_v \tag{2}$$

By unrolling (1) with (2) using $j < m$, we get

$$(j, e[v/x, (\text{fix } f(x).e)/f], e'[v'/x, (\text{fix } f(x).e')/f]) \in (\!|\sigma\tau_2|\!)_\varepsilon^{\sigma t} \tag{3}$$

By Assumption 13 on the third premise, we get $\sigma t \leq \sigma t'$.
We conclude by applying IH 3 to (3) using the second premise and $\sigma t \leq \sigma t'$.

subcase 2: STS: $\forall j.(j, \text{fix } f(x).e) \in [\![|\sigma\tau_1'|_1 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2'|_1]\!]_v \ \wedge\ (j, \text{fix } f(x).e') \in [\![|\sigma\tau_1'|_2 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2'|_2]\!]_v$ These follow by unrolling the second part of (1)'s definition.

Case

$$\frac{}{\Delta; \Phi \models U\,(A_1 \xrightarrow{\mathbf{exec}(k,t)} A_2, A_1' \xrightarrow{\mathbf{exec}(k',t')} A_2') \sqsubseteq U\,(A_1, A_1') \xrightarrow{\mathbf{diff}(t-k')} U\,(A_2, A_2')} \rightarrow \mathbf{execdiff}$$

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$.
We have

$$(m, \text{fix } f(x).e, \text{fix } f(x).e') \in (\!|U\,(\sigma A_1 \xrightarrow{\text{exec}(\sigma k,\sigma t)} \sigma A_2, \sigma A_1' \xrightarrow{\text{exec}(\sigma k',\sigma t')} \sigma A_2')|\!)_v \tag{1}$$

TS: $(m, \text{fix } f(x).e, \text{fix } f(x).e') \in (\!|U\,(\sigma A_1, \sigma A_1') \xrightarrow{\text{diff}(\sigma t - \sigma k')} U\,(\sigma A_2, \sigma A_2')|\!)_v$.
There are two cases to show.

subcase 1: Assume that

a) $j < m$
b) $(j, v, v') \in (\!|U\,(\sigma A_1, \sigma A_1')|\!)_v$

STS: $(j, e[v/x, (\text{fix } f(x).e)/f], e'[v'/x, (\text{fix } f(x).e')/f]) \in (\!|U\,(\sigma A_2, \sigma A_2')|\!)_\varepsilon^{\sigma t - \sigma k'}$.
Assume that

c) $e[v/x, (\text{fix } f(x).e)/f] \Downarrow^{c_r} v_r$
d) $e'[v'/x, (\text{fix } f(x).e')/f] \Downarrow^{c_r'} v_r'$
e) $c_r < j$

STS 1: $c_r - c_r' \leq \sigma t - \sigma k'$
STS 2: $(m - c_r, v_r, v_r') \in (\!|U\,(\sigma A_2, \sigma A_2')|\!)_v$.
We first show the second statement, the first one shown later.
Then, it suffices to show $\forall j.(j, v_r) \in [\![\sigma A_2]\!]_v \ \wedge\ (j, v_r')[\![\sigma A_2']\!]_v$. Pick j.
RTS1 : $(j, v_r) \in [\![\sigma A_2]\!]_v$
RTS2 : $(j, v_r')[\![\sigma A_2']\!]_v$
By (1), we know that

$$\forall j'.(j', \text{fix } f(x).e) \in (\!|A_1 \xrightarrow{\text{exec}(k,t)} A_2|\!)_v \ \wedge\ (j', \text{fix } f(x).e') \in [\![A_1' \xrightarrow{\text{exec}(k',t')} A_2']\!]_v \tag{2}$$

By instantiating j' in the first part of (2) with $j + \sigma t + 2$, we get

$$(j + \sigma t + 2, \text{fix } f(x).e) \in (\!|\sigma A_1 \xrightarrow{\text{exec}(\sigma k, \sigma t)} \sigma A_2|\!)_v \tag{3}$$

By unrolling the definition of b) and instantiating the universal quantifier with $j+\sigma t+1$, we get

$$(j + \sigma t + 1, v) \in [\![\sigma A_1]\!]_v \tag{4}$$

Then, unrolling the definition of (3) with (4) using $j + \sigma t + 1 < j + \sigma t + 2$, we get

$$(j + \sigma t + 1, e[v/x, \text{fix } f(x).e/f]) \in [\![\sigma A_2]\!]_v \sigma k \sigma t \tag{5}$$

By unrolling the definition of (5) using $\sigma t < j + \sigma t + 1$, we get

f) $c_r \leq st$

g) $(j + \sigma t + 1 - c_r, v_r) \in [\![\sigma A_2]\!]_v$

Next, we instantiate j' in the second part of (2) with $j + c'_r + 2$, we get

$$(j + c'_r + 2, \text{fix } f(x).e) \in (\!|\sigma A'_1 \xrightarrow{\text{exec}(\sigma k', \sigma t')} \sigma A'_2|\!)_v \tag{6}$$

By unrolling the definition of b) and instantiating the universal quantifier with $j+c'_r+1$, we get

$$(j + c'_r + 1, v') \in [\![\sigma A'_1]\!]_v \tag{7}$$

Then, unrolling the definition of (6) with (7) using $j + c'_r + 1 < j + c'_r + 2$, we get

$$(j + c'_r + 1, e'[v'/x, \text{fix } f(x).e'/f]) \in [\![\sigma A'_2]\!]_v \sigma k' \sigma t' \tag{8}$$

By unrolling the definition of (8) using d), $c'_r < j + c'_r + 1$, we get

h) $\sigma k' \leq c'_r$

i) $(j + 1, v'_r) \in [\![\sigma A'_2]\!]_v$

Now, we can conclude as follows

1. By f) and h), we get $c_r - c'_r \leq \sigma t - \sigma k'$
2. By downward closure (Lemma 4) on g) using

$$j \leq j + \sigma t - c_r + 1 \qquad \text{by f)}, c_r \leq \sigma t$$

We get $(j, v_r) \in [\![\sigma A_2]\!]_v$

By downward closure (Lemma 4) on i) using

$$j \leq j + 1$$

We get $(j, v'_r) \in [\![\sigma A'_2]\!]_v$ These conclude this subcase.

subcase 2: STS: $\forall j.(j, \text{fix } f(x).e) \in [\![A_1 \xrightarrow{\text{exec}(0,\infty)} grt_2]\!]_v \ \wedge \ (j, \text{fix } f(x).e') \in [\![A'_1 \xrightarrow{\text{exec}(0,\infty)} A'_2]\!]_v$

Pick j.

STS1: $(j, \text{fix } f(x).e) \in [\![A_1 \xrightarrow{\text{exec}(0,\infty)} A_2]\!]_v$

STS2: $(j, \text{fix } f(x).e') \in [\![A_1' \xrightarrow{\text{exec}(0,\infty)} A_2']\!]_v$
We will only show the first statement above, the second one is similar.
Assume $j' < j$ and $(j', v) \in [\![A_1]\!]_v$.
RTS1: $(j, e[v/x, (\text{fix } f(x).e)/f]) \in [\![A_2]\!]_\varepsilon^{0,\infty}$.
By unrolling the first part of (1)'s definition, we get

$$\forall m.(m, \text{fix } f(x).e) \in [\![A_1 \xrightarrow{\text{exec}(0,\infty)} A_2]\!]_v \tag{9}$$

By instantiating first part of (9) with j, we get

$$(j, \text{fix } f(x).e) \in [\![A_1 \xrightarrow{\text{exec}(0,\infty)} A_2]\!]_v \tag{10}$$

Then, the conclusion follows by unrolling (10) with $(j', v) \in [\![A_1]\!]_v$ and $j' < j$.

Case $\dfrac{}{\Delta; \Phi \models \square\,(\tau_1 \xrightarrow{\textbf{diff}(t)} \tau_2) \sqsubseteq \square\,\tau_1 \xrightarrow{\textbf{diff}(0)} \square\,\tau_2} \rightarrow \square\textbf{diff}$

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$.
We have

$$(m, \text{fix } f(x).e, \text{fix } f(x).e) \in (\!|\square\,(\sigma\tau_1 \xrightarrow{\text{diff}(\sigma t)} \sigma\tau_2)|\!)_v \tag{1}$$

TS: $(m, \text{fix } f(x).e, \text{fix } f(x).e) \in (\!|\square\,\sigma\tau_1 \xrightarrow{\text{diff}(0)} \square\,\sigma\tau_2|\!)_v$.
There are two cases:

subcase 1: Assume that $j < m$ and $(j, v, v) \in (\!|\square\,\sigma\tau_1|\!)_v$ (we have the same values due to box).
STS: $(j, e[v/x, (\text{fix } f(x).e)/f], e[v/x, (\text{fix } f(x).e)/f]) \in (\!|\square\,\sigma\tau_2|\!)_\varepsilon^0$.
Assume that

a) $e[v/x, (\text{fix } f(x).e)/f] \Downarrow^{c_r} v_r$
b) $e[v/x, (\text{fix } f(x).e)/f] \Downarrow^{c_r} v_r$
c) $c_r < m$

By unrolling first part of the definition of (1) with $j < m$ and $(j, v, v) \in (\!|\sigma\tau_1|\!)_v$, we get

$$(j, e[v/x, (\text{fix } f(x).e)/f], e[v/x, (\text{fix } f(x).e)/f]) \in (\!|\sigma\tau_2|\!)_\varepsilon^{\sigma t} \tag{2}$$

Unrolling the definition of (2) with (a-c), we get

d) $c_r - c_r \leq \sigma t$
e) $(m - c_r, v_r, v_r) \in (\!|\sigma\tau_2|\!)_v$

We can conclude as follows

1. Trivially $c_r - c_r \leq 0$
2. By e), we get $(m - c_r, v_r, v_r) \in (\!|\square\,\sigma\tau_2|\!)_v$

subcase 2: STS: $\forall j.(j, \text{fix } f(x).e) \in [\![|\square\,\sigma\tau_1 \xrightarrow{\text{diff}(0)} \square\,\sigma\tau_2|_1]\!]_v \equiv [\![|\sigma\tau_1|_1 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_1]\!]_v$.
Follows by unrolling the second part of the definition of (1) as shown in the previous case.

Case $\dfrac{}{\Delta; \Phi \models \tau \sqsubseteq U\,(|\tau|_1, |\tau|_2)}$ **W**

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$.
We have

$$(m, v_1, v_2) \in (\!|\sigma\tau|\!)_v \tag{1}$$

TS: $(m, v_1, v_2) \in (\!|U\,(|\sigma\tau|_1, |\sigma\tau|_2)|\!)_v$.
Proof is by induction on τ.
We show a few representative cases below.

subcase 1: $(m, v_1, v_2) \in (\!|U\,(A_1, A_2)|\!)_v$ $(\star)$
Since $\sigma\tau = U\,(A_1, A_2) = U\,(|\sigma\tau|_1, |\sigma\tau|_2)$, we immediately conclude by $(\star)$.

subcase 2: $(m, \text{inl } v_1, \text{inl } v_2) \in (\!|\sigma\tau_1 + \sigma\tau_2|\!)_v$ $(\star)$
TS: $(m, \text{inl } v_1, \text{inl } v_2) \in (\!|U\,(|\sigma\tau_1 + \sigma\tau_2|_1, |\sigma\tau_1 + \sigma\tau_2|_2)|\!)_v$.
STS: $\forall j.(j, \text{inl } v_1) \in [\![|\sigma\tau_1 + \sigma\tau_2|_1]\!]_v \ \wedge\ (j, \text{inl } v_2) \in [\![|\sigma\tau_1 + \sigma\tau_2|_2]\!]_v$.
By unrolling their definition and noting that $|\sigma\tau_1 + \sigma\tau_2|_i = |\sigma\tau_1|_i + |\sigma\tau_2|_i \ \forall i \in \{1, 2\}$, RTS:

$$\forall j.(j, v_1) \in [\![|\sigma\tau_1|_1]\!]_v \ \wedge\ (j, v_2) \in [\![|\sigma\tau_1|_2]\!]_v \tag{2}$$

By unrolling the definition of $(\star)$, we have $(m, v_1, v_2) \in (\!|\sigma\tau_1|\!)_v$.
By IH, we get $(m, v_1, v_2) \in (\!|U\,(|\sigma\tau_1|_1, |\sigma\tau_1|_2)|\!)_v$ which is equivalent to (2).

subcase 3: $(m, \text{fix } f(x).e_1, \text{fix } f(x).e_2) \in (\!|\sigma\tau_1 \xrightarrow{\text{diff}(k)} \sigma\tau_2|\!)_v$ $(\star)$
TS: $(m, \text{fix } f(x).e_1, \text{fix } f(x).e_2) \in (\!|U\,(|\sigma\tau_1 \xrightarrow{\text{diff}(k)} \sigma\tau_2|_1, |\sigma\tau_1 \xrightarrow{\text{diff}(k)} \sigma\tau_2|_2)|\!)_v$
STS: $\forall j.(j, \text{fix } f(x).e_1) \in [\![|\sigma\tau_1|_1 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_1]\!]_v \ \wedge\ (j, \text{fix } f(x).e_2) \in [\![|\sigma\tau_1|_2 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_2]\!]_v$.
Follows by unrolling the second part of the definition of $(\star)$.

Case $\dfrac{\Delta; \Phi \models n \doteq n' \qquad \Delta; \Phi \models \alpha \leq \alpha' \qquad \Delta; \Phi \models \tau \sqsubseteq \tau'}{\Delta; \Phi \models \textbf{list}[n]^{\alpha}\,\tau \sqsubseteq \textbf{list}[n']^{\alpha'}\,\tau'}$ **l1**

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$ and $(m, v, v') \in (\!|\text{list}[n]^{\alpha}\,\tau|\!)_v$.
TS: $(m, v, v') \in (\!|\text{list}[\sigma n']^{\sigma\alpha'}\,\sigma\tau'|\!)_v$
From Assumption 13 applied to the first premise, $\sigma n = \sigma n'$. Therefore,
STS: $(m, v, v') \in (\!|\text{list}[\sigma n]^{\sigma\alpha'}\,\sigma\tau'|\!)_v$
From Assumption 13 applied to the second premise, $\sigma\alpha \leq \sigma\alpha'$. Therefore,
We prove the following more general statement
$\forall\, m, v, v', n, \alpha, \alpha'.$ if $\alpha \leq \alpha'$ and $(m, v, v') \in (\!|\text{list}[\sigma n]^{\sigma\alpha}\,\sigma\tau|\!)_v$, then $(m, v, v') \in (\!|\text{list}[\sigma n]^{\sigma\alpha'}\,\sigma\tau'|\!)_v$.
We establish this statement by subinduction on v and v'.

subcase 1: $v = v' = \text{nil}$
We can immediately conclude that $(m, \text{nil}\,, \text{nil}\,) \in (\!|\text{list}[0]^{\sigma\alpha'}\,\sigma\tau'|\!)_v$ by definition.

subcase 2: $v = \text{cons}(v_1, v_2)$ and $v' = \text{cons}(v_1', v_2')$
TS: $(m, \text{cons}(v_1, v_2), \text{cons}(v_1', v_2')) \in (\!|\text{list}[I+1]^{\sigma\alpha'}\,\sigma\tau'|\!)_v$ for some $I + 1 = \sigma n$.
We have two possible cases:

- $(m, v_1, v_1') \in (\!|\square\,\sigma\tau|\!)_v$ $(\dagger)$ and $(m, v_2, v_2') \in (\!|\text{list}[I]^{\sigma\alpha}\,\sigma\tau|\!)_v$ $(\dagger\dagger)$.
  By subIH on $(\dagger\dagger)$, we get

$$(m, v_2, v_2') \in (\!|\text{list}[I]^{\sigma\alpha'}\,\sigma\tau'|\!)_v \tag{1}$$

By IH on ($\dagger$), we get

$$(m, v_1, v_1') \in (\!|\square\, \sigma\tau'|\!)_v \tag{2}$$

Combining (2) with (1), we get $(m, \mathrm{cons}(v_1, v_2), \mathrm{cons}(v_1', v_2')) \in (\!|\mathrm{list}[I+1]^{\sigma\alpha'}\, \sigma\tau|\!)_v$.

- $(m, v_1, v_1') \in (\!|\sigma\tau|\!)_v$ ($\diamond$) and $(m, v_2, v_2') \in (\!|\mathrm{list}[I]^{\sigma\alpha-1}\, \sigma\tau|\!)_v$ ($\diamond\diamond$).
  By subIH on ($\diamond\diamond$), we get

$$(m, v_2, v_2') \in (\!|\mathrm{list}[I]^{\sigma\alpha'-1}\, \sigma\tau'|\!)_v \tag{3}$$

  By IH on ($\diamond$), we get

$$(m, v_1, v_1') \in (\!|\sigma\tau'|\!)_v \tag{4}$$

  Combining (4) with (3), we get $(m, \mathrm{cons}(v_1, v_2), \mathrm{cons}(v_1', v_2')) \in (\!|\mathrm{list}[I+1]^{\sigma\alpha'}\, \sigma\tau|\!)_v$.

subcase 3: $v = \mathrm{nil}$ and $v' = \mathrm{cons}(v_1', v_2')$
This case is impossible since they can't be related.

subcase 4: $v = \mathrm{cons}(v_1, v_2)$ and $v' = \mathrm{nil}$
This case is impossible since they can't be related.

Case $\dfrac{\Delta; \Phi \models \alpha \doteq 0}{\Delta; \Phi \models \mathbf{list}[n]^{\alpha}\, \tau \sqsubseteq \mathbf{list}[n]^{\alpha}\, \square\, \tau}$ **l2**

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$ and $(m, v, v') \in (\!|\mathrm{list}[n]^{\alpha}\, \tau|\!)_v$.
TS: $(m, v, v') \in (\!|\mathrm{list}[\sigma n]^{\sigma\alpha}\, \square\, \sigma\tau|\!)_v$
We prove the following more general statement by subinduction on n.

subcase 1: $n = 0$
Then, we know that $v = v' = \mathrm{nil}$
We can immediately conclude that $(m, \mathrm{nil}\ , \mathrm{nil}\ ) \in (\!|\mathrm{list}[0]^0\, \square\, \sigma\tau|\!)_v$ by definition.

subcase 2: $n = I + 1$
Then, we know that $v = \mathrm{cons}(v_1, v_2)$ and $v' = \mathrm{cons}(v_1', v_2')$
TS: $(m, \mathrm{cons}(v_1, v_2), \mathrm{cons}(v_1', v_2')) \in (\!|\mathrm{list}[I+1]^0\, \square\, \sigma\tau|\!)_v$.
We have two possible cases:

- $(m, v_1, v_1') \in (\!|\square\, \sigma\tau|\!)_v$ ($\dagger$) and $(m, v_2, v_2') \in (\!|\mathrm{list}[I]^0\, \sigma\tau|\!)_v$ ($\dagger\dagger$).
  By subIH on ($\dagger\dagger$), we get $(m, v_2, v_2') \in (\!|\mathrm{list}[I]^0\, \square\, \sigma\tau|\!)_v$.
  Combining the ($\dagger$) with the previous statement, we get $(m, \mathrm{cons}(v_1, v_2), \mathrm{cons}(v_1', v_2')) \in (\!|\mathrm{list}[I+1]^0\, \square\, \sigma\tau|\!)_v$.
- $(m, v_1, v_1') \in (\!|\sigma\tau|\!)_v$ and $(m, v_2, v_2') \in (\!|\mathrm{list}[I]^{0-1}\, \sigma\tau|\!)_v$.
  This case is impossible since $0 - 1 \not\geq 0$.

Case $\dfrac{}{\Delta; \Phi \models \mathbf{list}[n]^{\alpha}\, \square\, \tau \sqsubseteq \square\, (\mathbf{list}[n]^{\alpha}\, \tau)}$ **l$\square$**

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$ and $(m, v, v') \in (\!|\mathrm{list}[\sigma n]^{\sigma\alpha}\, \square\, \sigma\tau|\!)_v$.
TS: $(m, v, v') \in (\!|\square\, (\mathrm{list}[n]^{\alpha}\, \tau)|\!)_v$
We prove the following more general statement
$\forall i, \beta, \tau.$ if $(m, v, v) \in (\!|\mathrm{list}[i]^{\beta}\, \square\, \tau|\!)_v$, then $(m, v, v) \in (\!|\square\, (\mathrm{list}[i]^{\beta}\, \tau)|\!)_v$ by subinduction on i.

subcase 1: $n = 0$
Then, we know that $v = v' = \mathrm{nil}$
We can immediately conclude that $(m, \mathrm{nil}\ , \mathrm{nil}\ ) \in (\!|\square\, \mathrm{list}[0]^{\sigma\alpha}\, \sigma\tau|\!)_v$ by definition.

subcase 2: $n = I + 1$

TS: $(m, \text{cons}(v_1, v_2), \text{cons}(v_1', v_2')) \in (\!|\square\, \text{list}[I+1]^{\sigma\alpha}\, s\tau|\!)_v$.

We have two possible cases:

- $(m, v_1, v_1') \in (\!|\square\,\square\,\sigma\tau|\!)_v$ ($\dagger$) and $(m, v_2, v_2) \in (\!|\text{list}[I]^{\sigma\alpha}\,\square\,\sigma\tau|\!)_v$ ($\dagger\dagger$).
Instantiating subIH on ($\dagger\dagger$), we get

$$(m, v_2, v_2') \in (\!|\square\,\text{list}[I]^{\sigma\alpha}\,\sigma\tau|\!)_v \text{ and } v_2 = v_2' \tag{1}$$

By ($\dagger$), we also know that

$$(m, v_1, v_1) \in (\!|\square\,\sigma\tau|\!)_v \tag{2}$$

Combining (2) with (1), we get $(m, \text{cons}(v_1, v_2), \text{cons}(v_1, v_2)) \in (\!|\square\,\text{list}[I+1]^{\sigma\alpha}\,\sigma\tau|\!)_v$.

- $(m, v_1, v_1) \in (\!|\square\,\sigma\tau|\!)_v$ ($\diamond$) and $(m, v_2, v_2) \in (\!|\text{list}[I]^{\sigma\alpha-1}\,\square\,\sigma\tau|\!)_v$ ($\diamond\diamond$).
Instantiating subIH on ($\diamond\diamond$), we get

$$(m, v_2, v_2') \in (\!|\square\,\text{list}[I]^{\sigma\alpha-1}\,\sigma\tau|\!)_v \text{ and } v_2 = v_2' \tag{3}$$

Combining ($\diamond$) with (3), we get $(m, \text{cons}(v_1, v_2), \text{cons}(v_1, v_2)) \in (\!|\square\,\text{list}[I+1]^{\sigma\alpha}\,\sigma\tau|\!)_v$.

□

Proof of statement (2). Proof is by induction on the subtyping derivation.

Case $$\frac{\Delta;\Phi \models^{\mathsf{A}} A_1' \sqsubseteq A_1 \qquad \Delta;\Phi \models^{\mathsf{A}} A_2 \sqsubseteq A_2' \qquad \Delta;\Phi \models k' \leq k \qquad \Delta;\Phi \models t \leq t'}{\Delta;\Phi \models^{\mathsf{A}} A_1 \xrightarrow{\textbf{exec}(k,t)} A_2 \sqsubseteq A_1' \xrightarrow{\textbf{exec}(k',t')} A_2'} \rightarrow \textbf{exec}$$

Assume that $\sigma \in \mathcal{D}[\![\Delta]\!]$.

We have

$$(m, \text{fix } f(x).e) \in (\!|\sigma A_1 \xrightarrow{\text{exec}(\sigma k, \sigma t)} \sigma A_2|\!)_v \tag{1}$$

TS: $(m, \text{fix } f(x).e) \in (\!|\sigma A_1' \xrightarrow{\text{exec}(\sigma k', \sigma t')} \sigma A_2'|\!)_v$.

STS: $(m, \text{fix } f(x).e) \in [\![\sigma A_1' \xrightarrow{\text{exec}(\sigma k', \sigma t')} \sigma A_2']\!]_v$.

Pick j and assume that

$$j < m \tag{2}$$

$$(j, v) \in [\![\sigma A_1']\!]_v \tag{3}$$

STS: $(j, e[v/x, (\text{fix } f(x).e)/f]) \in [\![\sigma A_2']\!]_\varepsilon^{\sigma k', \sigma t'}$.

By IH 2 on (3) using the first premise, we get

$$(j, v) \in [\![\sigma A_1]\!]_v \tag{4}$$

By unrolling the definition of (1) with (4) and $j < m$, we get

$$(j, e[v/x, (\text{fix } f(x).e)/f]) \in [\![\sigma A_2]\!]_\varepsilon^{\sigma k, \sigma t} \tag{5}$$

By Assumption 13 on the third and fourth premises, we get $\sigma k' \leq \sigma k$ and $\sigma t \leq \sigma t'$.

We conclude by applying IH 4 to (5) using σ, i.e $\sigma t \leq \sigma t'$ and $\sigma k' \leq \sigma k$.

$\square$

Lemma 6 (Sort Substitution)
The following hold.

1. If $\Delta \vdash I :: S$ and $\Delta, i :: S \vdash I' :: S'$, then $\Delta \vdash I'[I/i] :: S'$.
2. If $\Delta \vdash I :: S$ and $\Delta, \vdash i :: S \vdash C$ $\texttt{wf}$, then $\Delta \vdash C[I/i]$ $\texttt{wf}$.
3. If $\Delta \vdash I :: S$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$, then $\vdash \sigma I :: S$.

Proof. (1) and (2) are established by simultaneous induction on the second given derivations. (3) follows from (1). $\square$

Assumption 7 (Constraint Well-formedness)
If $\Delta; \Phi \models C$ then $\Delta \vdash C$ $\texttt{wf}$

Lemma 8 (Well-formedness)
The following hold.

1. If $\Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t : \tau$ and $\Delta; \Phi \vdash \Gamma$ $\texttt{wf}$ and $\text{FIV}(\Gamma) \subseteq \text{dom}(\Delta)$, then $\Phi; \Delta \vdash \tau$ $\texttt{wf}$ and $\text{FIV}(t, \tau) \subseteq \text{dom}(\Delta)$.
2. If $\Delta; \Phi; \Omega \vdash_k^t e : A$ and $\Delta; \Phi \vdash^{\mathsf{A}} \Omega$ $\texttt{wf}$ and $\text{FIV}(\Omega) \subseteq \text{dom}(\Delta)$, then $\Phi; \Delta \vdash^{\mathsf{A}} A$ $\texttt{wf}$ and $\text{FIV}(k, t, A) \subseteq \text{dom}(\Delta)$.
3. If $\Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t : \tau$, then $\text{FV}(e) \subseteq \text{dom}(\Gamma)$ and $\text{FV}(e) \subseteq \text{dom}(\Gamma)$.
4. If $\Delta; \Phi; \Omega \vdash_k^t e : A$, then $\text{FV}(e) \subseteq \text{dom}(\Omega)$.

Proof. The proof is by induction on the typing derivations. $\square$

Lemma 9 (Refinement Removal Well-formedness)
If $\Phi; \Delta \vdash \tau$ $\texttt{wf}$, then $\Phi; \Delta \vdash^{\mathsf{A}} |\tau|_i$ $\texttt{wf}$ for $i \in \{1, 2\}$.

Lemma 10 (Subtyping well-formedness)
The following hold.

- If $\Delta; \Phi \models \tau \sqsubseteq \tau'$ and $\Delta; \Phi \vdash \tau$ $\texttt{wf}$ and $FIV(\tau) \subseteq \Delta$, then $\Phi; \Delta \vdash \tau'$ $\texttt{wf}$ and $FIV(\tau') \subseteq \Delta$.
- If $\Delta; \Phi \models^{\mathsf{A}} A \sqsubseteq A'$ and $\Delta; \Phi \vdash^{\mathsf{A}} A$ $\texttt{wf}$ and $FIV(A) \subseteq \Delta$, then $\Phi; \Delta \vdash A'$ $\texttt{wf}$ and $FIV(A') \subseteq \Delta$.

Proof. The proof is by induction on the subtyping derivations. $\square$

Both of our fundamental theorems rely on the assumption that the semantic interpretation of every primitive function lies in the interpretation of the function's type. This is explained below.

Assumption 11 (Soundness of primitive functions (relational))
Suppose that $\zeta : \tau_1 \xrightarrow{\text{diff}(t)} \tau_2$ and $(m, v, v') \in (\!|\tau_1|\!)_v$ and $\hat{\zeta}\ v = (c_r, v_r)$ and $\hat{\zeta}\ v' = (c'_r, v'_r)$, then

- $(m, v_r, v'_r) \in (\!|\tau_2|\!)_v$
- $c_r - c'_r \leq t$

Assumption 12 (Soundness of primitive functions (non-relational))
Suppose that $\zeta : A_1 \xrightarrow{\text{exec}(k,t)} A_2$ and $(m, v) \in [\![A_1]\!]_v$, then

- $\hat{\zeta}\ v = (c_r, v_r)$
- $(m, v_r) \in [\![A_2]\!]_v$
- $k \leq c_r \leq t$

We assume that the constraint judgment $\Delta; \Phi \models C$ satisfies some standard properties.

Assumption 13 (Constraint conditions)
The following hold.

1. [Subst1] If $\Delta, i :: S; \Phi \models C$ and $\Delta \vdash I :: S$, then $\Delta; \Phi[I/i] \models C[I/i]$.
2. [Subst2] If $\Delta; \Phi \models C$ and $\Delta; \Phi \wedge C \models C'$, then $\Delta; \Phi \models C'$.
3. [Neg] $\Delta; \Phi \models \neg C$ iff $\Delta; \Phi \not\models C$.
4. [Corr1] If $\models n_1 \leq n_2$, then $n_1 \leq n_2$.
5. [Corr2] If $\models I \doteq I'$, then $I = I'$.

Theorem 14 (Fundamental theorem)
The following holds.

1. Assume that $\Delta; \Phi; \Gamma \vdash e_1 \ominus e_2 \lesssim t : \tau$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$ and $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$. Then, $(m, \delta e_1, \delta' e_2) \in (\!|\sigma\tau|\!)_{\varepsilon}^{\sigma t}$.
2. Assume that $\Delta; \Phi; \Omega \vdash_k^t e : A$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$ and there exists Ω' s.t. $\text{FV}(e) \subseteq \text{dom}(\Omega')$ and $\Omega' \subseteq \Omega$ and $(m, \gamma) \in \mathcal{G}[\![\sigma\Omega']\!]$. Then, $(m, \gamma e) \in [\![\sigma A]\!]_{\varepsilon}^{\sigma k, \sigma t}$.
3. Assume that $\Delta; \Phi; \Gamma \vdash e_1 \ominus e_2 \lesssim t : \tau$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$. Then for $i \in \{1, 2\}$, if there exists Γ'_i s.t. $\text{FV}(e_i) \subseteq \text{dom}(\Gamma'_i)$ and $\Gamma'_i \subseteq \Gamma$ and $(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'_i|_i]\!]$, then $(m, \delta e_i) \in [\![|\sigma\tau|_i]\!]_{\varepsilon}^{0,\infty}$.

Proof. Proofs are by induction on typing derivations. We show each statement separately.

Proof of Statement (1). We proceed by induction on the typing derivation. We show the most important cases below.

Case $\dfrac{\Gamma(x) = \tau}{\Delta; \Phi; \Gamma \vdash x \ominus x \lesssim 0 : \tau}$ **r-var**

Assume that $\models \sigma\Phi$ and $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$.

TS: $(m, \delta(x), \delta'(x)) \in (\!|\sigma\tau|\!)^{0}_{\varepsilon}$.

By Value Lemma (Lemma 2), STS: $(m, \delta(x), \delta'(x)) \in (\!|\sigma\tau|\!)_{v}$.

This follows by $\Gamma(x) = \tau$ and $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$.

Case $\dfrac{\Delta; \Phi; \Gamma \vdash e_1 \ominus e_1' \lesssim t_1 : \tau \qquad \Delta; \Phi; \Gamma \vdash e_2 \ominus e_2' \lesssim t_2 : \mathbf{list}[n]^{\alpha}\,\tau}{\Delta; \Phi; \Gamma \vdash \mathbf{cons}(e_1, e_2) \ominus \mathbf{cons}(e_1', e_2') \lesssim t_1 + t_2 : \mathbf{list}[n+1]^{\alpha+1}\,\tau}$ **r-cons1**

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS: $(m, \mathrm{cons}(\delta e_1, \delta e_2), \mathrm{cons}(\delta' e_1', \delta' e_2')) \in (\!|\mathrm{list}[\sigma n + 1]^{\sigma\alpha+1}\,\sigma\tau|\!)^{\sigma t_1 + \sigma t_2}_{\varepsilon}$.

Following the definition of $(\!|\cdot|\!)^{\cdot}_{\varepsilon}$, assume that

$\dfrac{\delta e_1 \Downarrow^{c_1} v_1 \;\; (\star) \qquad \delta e_2 \Downarrow^{c_2} v_2 \;\; (\diamond)}{\mathrm{cons}(\delta e_1, \delta e_2) \Downarrow^{c_1+c_2} \mathrm{cons}(v_1, v_2)}$ **cons** and $\dfrac{\delta' e_1' \Downarrow^{c_1'} v_1' \;\; (\star\star) \qquad \delta' e_2' \Downarrow^{c_2'} v_2' \;\; (\diamond\diamond)}{\mathrm{cons}(\delta' e_1', \delta' e_2') \Downarrow^{c_1'+c_2'} \mathrm{cons}(v_1', v_2')}$ **cons**

and

$c_1 + c_2 < m$.

By IH 1 on the first premise, we get $(m, \delta e_1, \delta' e_1') \in (\!|\sigma\tau|\!)^{\sigma t_1}_{\varepsilon}$. Unrolling its definition with $(\star)$ and $(\star\star)$ and $c_1 < m$, we get

a) $c_1 - c_1' \leq \sigma t_1$

b) $(m - c_1, v_1, v_1') \in (\!|\sigma\tau|\!)_{v}$

By IH 1 on the second premise, we get $(m, \delta e_2, \delta' e_2') \in (\!|\mathrm{list}[\sigma n]^{\sigma\alpha}\,\sigma\tau|\!)^{\sigma t_2}_{\varepsilon}$. Unrolling its definition with $(\diamond)$ and $(\diamond\diamond)$, and $c_2 < m$, we get

c) $c_2 - c_2' \leq \sigma t_2$

d) $(m - c_2, v_2, v_2') \in (\!|\mathrm{list}[\sigma n]^{\sigma\alpha}\,\sigma\tau|\!)_{v}$

Now, we can conclude as follows:

1. Using a) and c), we get $(c_1 + c_2) - (c_1' + c_2') \leq \sigma t_1 + \sigma t_2$
2. By downward closure (Lemma 4) on b) and d) using

$$m - (c_1 + c_2) \leq m - c_1$$

$$m - (c_1 + c_2) \leq m - c_2$$

we get $(m - (c_1 + c_2), v_1, v_1') \in (\!|\sigma\tau|\!)_{v}$ and $(m - (c_1 + c_2), v_2, v_2') \in (\!|\mathrm{list}[\sigma n]^{\sigma\alpha}\,\sigma\tau|\!)_{v}$, when combined, gives us $(m-(c_1+c_2), \text{cons}(v_1,v_2), \text{cons}(v_1',v_2')) \in (\!|\text{list}[\sigma n+1]^{\sigma\alpha+1}\,\sigma\tau|\!)_v$

$$\textbf{Case}\quad \frac{\Delta;\Phi;\Gamma\vdash e_1\ominus e_1'\lesssim t_1:\square\,\tau \qquad \Delta;\Phi;\Gamma\vdash e_2\ominus e_2'\lesssim t_2:\textbf{list}[n]^{\alpha}\,\tau}{\Delta;\Phi;\Gamma\vdash \textbf{cons}(e_1,e_2)\ominus\textbf{cons}(e_1',e_2')\lesssim t_1+t_2:\textbf{list}[n+1]^{\alpha}\,\tau}\ \textbf{r-cons2}$$

Assume that $(m,\delta,\delta')\in\mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models\sigma\Phi$.

TS: $(m,\text{cons}(\delta e_1,\delta e_2),\text{cons}(\delta' e_1',\delta' e_2'))\in(\!|\text{list}[\sigma n+1]^{\sigma\alpha}\,\sigma\tau|\!)_{\varepsilon}^{\sigma t_1+\sigma t_2}$.

Following the definition of $(\!|\cdot|\!)_{\varepsilon}^{\cdot}$, assume that

$$\frac{\delta e_1\Downarrow^{c_1} v_1\quad(\star)\qquad \delta e_2\Downarrow^{c_2} v_2\quad(\diamond)}{\text{cons}(\delta e_1,\delta e_2)\Downarrow^{c_1+c_2}\text{cons}(v_1,v_2)}\ \textbf{cons}\ \text{and}\ \frac{\delta' e_1'\Downarrow^{c_1'} v_1'\quad(\star\star)\qquad \delta' e_2'\Downarrow^{c_2'} v_2'\quad(\diamond\diamond)}{\text{cons}(\delta' e_1',\delta' e_2')\Downarrow^{c_1'+c_2'}\text{cons}(v_1',v_2')}\ \textbf{cons}$$

and

$c_1+c_2<m$.

By IH 1 on the first premise, we get $(m,\delta e_1,\delta' e_1')\in(\!|\square\,\sigma\tau|\!)_{\varepsilon}^{\sigma t_1}$. Unrolling its definition with $(\star)$ and $(\star\star)$, and $c_1<m$, we get

a) $c_1-c_1'\le\sigma t_1$
b) $(m-c_1,v_1,v_1')\in(\!|\square\,\sigma\tau|\!)_v$

By IH 1 on the second premise, we get $(m,\delta e_2,\delta' e_2')\in(\!|\text{list}[\sigma n]^{\sigma\alpha}\,\sigma\tau|\!)_{\varepsilon}^{\sigma t_2}$. Unrolling its definition with $(\diamond)$ and $(\diamond\diamond)$, and $c_2<m$, we get

c) $c_2-c_2'\le\sigma t_2$
d) $(m-c_2,v_2,v_2')\in(\!|\text{list}[\sigma n]^{\sigma\alpha}\,\sigma\tau|\!)_v$

Now, we can conclude as follows:

1. Using a) and c), we get $(c_1+c_2)-(c_1'+c_2')\le\sigma t_1+\sigma t_2$
2. By downward-closure (Lemma 4) on b) and d) using

$$m-(c_1+c_2)\le m-c_1$$

$$m-(c_1+c_2)\le m-c_2$$

we get $(m-(c_1+c_2),v_1,v_1')\in(\!|\square\,\sigma\tau|\!)_v$ and $(m-(c_1+c_2),v_2,v_2')\in(\!|\text{list}[\sigma n]^{\sigma\alpha}\,\sigma\tau|\!)_v$, when combined, gives us $(m-(c_1+c_2),\text{cons}(v_1,v_2),\text{cons}(v_1',v_2'))\in(\!|\text{list}[\sigma n+1]^{\sigma\alpha}\,\sigma\tau|\!)_v$

$$\textbf{Case}\quad \frac{\Delta;\Phi;\Gamma\vdash e\ominus e'\lesssim t:\tau \qquad \Delta;\Phi;\Gamma\vdash e_l\ominus e_l'\lesssim t_1:\textbf{tree}[i]^{\alpha}\,\tau \qquad \Delta;\Phi;\Gamma\vdash e_r\ominus e_r'\lesssim t_2:\textbf{tree}[j]^{\beta}\,\tau}{\Delta;\Phi;\Gamma\vdash \textbf{node}(e_l,e,e_r)\ominus\textbf{node}(e_l',e',e_r')\lesssim t+t_1+t_2:\textbf{tree}[i+j+1]^{\alpha+\beta+1}\,\tau}\ \textbf{r-node1}$$

Assume that $(m,\delta,\delta')\in\mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models\sigma\Phi$.

TS: $(m, \text{node}(\delta e_l, \delta e, \delta e_r), \text{node}(\delta' e_l', \delta' e', \delta' e_r')) \in (\!|\text{tree}[\sigma i + \sigma j + 1]^{\sigma\alpha+\sigma\beta+1}\, \sigma\tau|\!)_\varepsilon^{\sigma t+\sigma t_1+\sigma t_2}$.
Following the definition of $(\!|\cdot|\!)_\varepsilon^{\cdot}$, assume that

$$\frac{\delta e_l \Downarrow^{c_l} v_l \quad (\star) \qquad \delta e \Downarrow^{c} v \quad (\diamond) \qquad \delta e_r \Downarrow^{c_r} v_r \quad (\dagger)}{\text{node}(\delta e_l, \delta e, \delta e_r) \Downarrow^{c+c_l+c_r} \text{node}(v_l, v, v_r)} \textbf{ node} \text{ and}$$

$$\frac{\delta' e_l \Downarrow^{c_l'} v_l' \quad (\star\star) \qquad \delta' e \Downarrow^{c'} v' \quad (\diamond\diamond) \qquad \delta' e_r \Downarrow^{c_r'} v_r' \quad (\dagger\dagger)}{\text{node}(\delta' e_l, \delta' e, \delta' e_r) \Downarrow^{c'+c_l'+c_r'} \text{node}(v_l', v', v_r')} \textbf{ node} \text{ and}$$

$(c_l + c + c_r) < m$.
By IH 1 on the first premise, we get $(m, \delta e, \delta' e') \in (\!|\sigma\tau|\!)_\varepsilon^{\sigma t}$. Unrolling its definition with $(\diamond)$ and $(\diamond\diamond)$ and $c < m$, we get

a) $c - c' \leq \sigma t$
b) $(m - c, v, v') \in (\!|\sigma\tau|\!)_v$

By IH 1 on the second premise, we get $(m, \delta e_l, \delta' e_l') \in (\!|\text{tree}[\sigma i]^{\sigma\alpha}\, \sigma\tau|\!)_\varepsilon^{\sigma t_l}$. Unrolling its definition with $(\star)$ and $(\star\star)$, and $c_l < m$, we get

c) $c_l - c_l' \leq \sigma t_1$
d) $(m - c_l, v_l, v_l') \in (\!|\text{tree}[\sigma i]^{\sigma\alpha}\, \sigma\tau|\!)_v$

By IH 1 on the second premise, we get $(m, \delta e_r, \delta' e_r') \in (\!|\text{tree}[\sigma j]^{\sigma\beta}\, \sigma\tau|\!)_\varepsilon^{\sigma t_r}$. Unrolling its definition with $(\dagger)$ and $(\dagger\dagger)$, and $c_r < m$, we get

e) $c_r - c_r' \leq \sigma t_2$
f) $(m - c_r, v_r, v_r') \in (\!|\text{tree}[\sigma j]^{\sigma\beta}\, \sigma\tau|\!)_v$

Now, we can conclude as follows:

1. Using a), c) and e), we get $(c_l + c + c_r) - (c_l' + c' + c_r') \leq \sigma t + \sigma t_1 + \sigma t_2$
2. By downward-closure (Lemma 4) on b), d) and e) using

$$m - (c_l + c + c_r) \leq m - c$$

$$m - (c_l + c + c_r) \leq m - c_l$$

$$m - (c_l + c + c_r) \leq m - c_r$$

we get $(m - (c_l + c + c_r), v, v') \in (\!|\sigma\tau|\!)_v$ and $(m - (c_l + c + c_r), v_l, v_l') \in (\!|\text{tree}[\sigma i]^{\sigma\alpha}\, \sigma\tau|\!)_v$ and $(m - (c_l + c + c_r), v_r, v_r') \in (\!|\text{tree}[\sigma j]^{\sigma\beta}\, \sigma\tau|\!)_v$, when combined, gives us $(m - (c_l + c + c_r), \text{node}(v_l, v, v_r), \text{node}(v_l', v', v_r')) \in (\!|\text{tree}[\sigma i + \sigma j + 1]^{\sigma\alpha+\sigma\beta+1}\, \sigma\tau|\!)_v$

Case $$\dfrac{\Delta;\Phi;\Gamma\vdash e\ominus e'\lesssim t:\Box\,\tau \qquad \Delta;\Phi;\Gamma\vdash e_l\ominus e_l'\lesssim t_1:\mathbf{tree}[i]^{\alpha}\,\tau \qquad \Delta;\Phi;\Gamma\vdash e_r\ominus e_r'\lesssim t_2:\mathbf{tree}[j]^{\beta}\,\tau}{\Delta;\Phi;\Gamma\vdash \mathbf{node}(e_l,e,e_r)\ominus\mathbf{node}(e_l',e',e_r')\lesssim t+t_1+t_2:\mathbf{tree}[i+j+1]^{\alpha+\beta}\,\tau}\ \textbf{r-node2}$$

Assume that $(m,\delta,\delta')\in\mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models\sigma\Phi$.

TS: $(m,\mathrm{node}(\delta e_l,\delta e,\delta e_r),\mathrm{node}(\delta' e_l',\delta' e',\delta' e_r'))\in(\!|\mathrm{tree}[\sigma i+\sigma j+1]^{\sigma\alpha+\sigma\beta}\,\sigma\tau|\!)_{\varepsilon}^{\sigma t+\sigma t_1+\sigma t_2}$.

Following the definition of $(\!|\cdot|\!)_{\varepsilon}^{\cdot}$, assume that

$$\dfrac{\delta e_l\Downarrow^{c_l}v_l\ \ (\star)\qquad \delta e\Downarrow^{c}v\ \ (\diamond)\qquad \delta e_r\Downarrow^{c_r}v_r}{\mathrm{node}(\delta e_l,\delta e,\delta e_r)\Downarrow^{c+c_l+c_r}\mathrm{node}(v_l,v,v_r)}\ \textbf{node}\ \text{and}$$

$$\dfrac{\delta' e_l\Downarrow^{c_l}v_l\ \ (\star\star)\qquad \delta' e\Downarrow^{c}v\ \ (\diamond\diamond)\qquad \delta' e_r\Downarrow^{c_r}v_r\ \ '}{\mathrm{node}(\delta' e_l,\delta' e,\delta' e_r)\Downarrow^{c+c_l+c_r}\mathrm{node}(v_l,v,v_r)}\ \textbf{node}\ \ \text{and}$$

$(c_l+c+c_r)<m$.

By IH 1 on the first premise, we get $(m,\delta e,\delta' e')\in(\!|\Box\,\sigma\tau|\!)_{\varepsilon}^{\sigma t}$. Unrolling its definition with $(\star)$ and $(\star\star)$, and $c_1<m$, we get

a) $c-c'\leq\sigma t$

b) $(m-c,v,v')\in(\!|\Box\,\sigma\tau|\!)_v$

By IH 1 on the second premise, we get $(m,\delta e_l,\delta' e_l')\in(\!|\mathrm{tree}[\sigma i]^{\sigma\alpha}\,\sigma\tau|\!)_{\varepsilon}^{\sigma t_l}$. Unrolling its definition with $(\star)$ and $(\star\star)$, and $c_l<m$, we get

c) $c_l-c_l'\leq\sigma t_1$

d) $(m-c_l,v_l,v_l')\in(\!|\mathrm{tree}[\sigma i]^{\sigma\alpha}\,\sigma\tau|\!)_v$

By IH 1 on the second premise, we get $(m,\delta e_r,\delta' e_r')\in(\!|\mathrm{tree}[\sigma j]^{\sigma\beta}\,\sigma\tau|\!)_{\varepsilon}^{\sigma t_r}$. Unrolling its definition with $(\dagger)$ and $(\dagger\dagger)$, and $c_r<m$, we get

e) $c_r-c_r'\leq\sigma t_2$

f) $(m-c_r,v_r,v_r')\in(\!|\mathrm{tree}[\sigma j]^{\sigma\beta}\,\sigma\tau|\!)_v$

Now, we can conclude as follows:

1. Using a), c) and e), we get $(c_l+c+c_r)-(c_l'+c'+c_r')\leq\sigma t+\sigma t_1+\sigma t_2$
2. By downward-closure (Lemma 4) on b), d) and e) using

$$m-(c_l+c+c_r)\leq m-c$$

$$m-(c_l+c+c_r)\leq m-c_l$$

$$m-(c_l+c+c_r)\leq m-c_r$$

we get $(m-(c_l+c+c_r), v, v') \in (\!|\Box\,\sigma\tau|\!)_v$ and $(m-(c_l+c+c_r), v_l, v_l') \in (\!|\mathrm{tree}[\sigma i]^{\sigma\alpha}\,\sigma\tau|\!)_v$ and $(m-(c_l+c+c_r), v_r, v_r') \in (\!|\mathrm{tree}[\sigma j]^{\sigma\beta}\,\sigma\tau|\!)_v$, when combined, gives us $(m-(c_l+c+c_r), \mathrm{node}(v_l,v,v_r), \mathrm{node}(v_l',v',v_r')) \in (\!|\mathrm{tree}[\sigma i+\sigma j+1]^{\sigma\alpha+\sigma\beta}\,\sigma\tau|\!)_v$

Case $$\frac{\begin{array}{c}\Delta;\Phi;\Gamma \vdash e \ominus e' \lesssim t : \mathbf{list}[n]^{\alpha}\,\tau \qquad \Delta;\Phi\wedge n=0;\Gamma \vdash e_1 \ominus e_1' \lesssim t' : \tau' \\ i,\Delta;\Phi\wedge n=i+1; h:\Box\,\tau, tl:\mathbf{list}[i]^{\alpha}\,\tau,\Gamma \vdash e_2 \ominus e_2' \lesssim t' : \tau' \\ i,\beta,\Delta;\Phi\wedge n=i+1\wedge\alpha=\beta+1; h:\tau, tl:\mathbf{list}[i]^{\beta}\,\tau,\Gamma \vdash e_2 \ominus e_2' \lesssim t' : \tau'\end{array}}{\Delta;\Phi;\Gamma \vdash \ \mathbf{case}\ e\ \mathbf{of\ nil}\ \to e_1 \mid h :: tl \to e_2 \ominus\ \mathbf{case}\ e'\ \mathbf{of\ nil}\ \to e_1' \mid h :: tl \to e_2' \lesssim t+t' : \tau'}\ \textbf{r-caseL}$$

Assume that $(m,\delta,\delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS: $(m,\ \mathrm{case}\ \delta e\ \mathrm{of\ nil}\ \to \delta e_1 \mid h :: tl \to \delta e_2,\ \mathrm{case}\ \delta' e'\ \mathrm{of\ nil}\ \to \delta' e_1' \mid h :: tl \to \delta' e_2') \in (\!|\sigma\tau'|\!)_{\varepsilon}^{\sigma t+\sigma t'}$.

Following the definition of $(\!|\cdot|\!)_{\varepsilon}^{\cdot}$, assume that

$$\mathrm{case}\ \delta e\ \mathrm{of\ nil}\ \to \delta e_1 \mid h :: tl \to \delta e_2 \Downarrow^{C} v_r \qquad (1)$$

$$\mathrm{case}\ \delta' e'\ \mathrm{of\ nil}\ \to \delta' e_1' \mid h :: tl \to \delta' e_2' \Downarrow^{C'} v_r' \qquad (2)$$

and $C < m$.

Depending on what δe and $\delta' e'$ evaluate to, there are four cases.

subcase 1: $$\frac{\delta e \Downarrow^{c} \mathrm{nil} \quad (\star) \qquad \delta e_1 \Downarrow^{c_r} v_r \quad ({\color{red}\diamond})}{\mathrm{case}\ \delta e\ \mathrm{of\ nil}\ \to \delta e_1 \mid h :: tl \to \delta e_2 \Downarrow^{c+c_r+c_{caseL}} v_r}\ \textbf{caseL-nil}\ \text{and}$$

$$\frac{\delta' e' \Downarrow^{c'} \mathrm{nil} \quad (\star\star) \qquad \delta' e_1' \Downarrow^{c_r'} v_r' \quad ({\color{red}\diamond\diamond})}{\mathrm{case}\ \delta' e'\ \mathrm{of\ nil}\ \to \delta' e_1' \mid h :: tl \to \delta' e_2' \Downarrow^{c'+c_r'+c_{caseL}} v_r'}\ \textbf{caseL-nil}\ \text{and}$$

$C = c + c_r + c_{caseL} < m$

By IH 1 on the first premise, we get $(m,\delta e,\delta' e') \in (\!|\mathrm{list}[\sigma n]^{\sigma\alpha}\,\sigma\tau|\!)_{\varepsilon}^{\sigma t}$. Unrolling its definition with $(\star)$, $(\star\star)$ and $c < m$, we get

a) $c - c' \leq \sigma t$

b) $(m-c, \mathrm{nil}\ , \mathrm{nil}\ ) \in (\!|\mathrm{list}[\sigma n]^{\sigma\alpha}\,\sigma\tau|\!)_v$

By b), $\sigma n = 0$.

Then, we can instantiate IH 1 on the second premise using $\models \sigma\Phi \wedge \sigma n \doteq 0$, to obtain $(m,\delta e_1,\delta' e_1') \in (\!|\sigma\tau'|\!)_{\varepsilon}^{\sigma t'}$.

Unrolling its definition using $({\color{red}\diamond})$ and $({\color{red}\diamond\diamond})$ and $c_r < m$, we get

c) $c_r - c_r' \leq \sigma t'$

d) $(m - c_r, v_r, v'_r) \in (\!|\sigma\tau'|\!)_v$

We conclude with

1. By a) and c), we get $(c + c_r + c_{caseL}) - (c' + c'_r + c_{caseL}) \leq \sigma t + \sigma t'$
2. By downward closure (Lemma 4) on d) using

$$m - (c + c_r + c_{caseL}) \leq m - c_r$$

we get $(m - (c + c_r + c_{caseL}), v_r, v'_r) \in (\!|\sigma\tau'|\!)_v$.

subcase 2: $\dfrac{\delta e \Downarrow^{c} \text{nil} \quad (\star) \qquad \delta e_1 \Downarrow^{c_r} v_r \quad ({\color{red}\diamond})}{\text{case } \delta e \text{ of nil } \to \delta e_1 \mid h :: tl \to \delta e_2 \Downarrow^{c+c_r+c_{caseL}} v_r}$ **caseL-nil** and

$\dfrac{\delta' e' \Downarrow^{c'} \text{cons}(v'_1, v'_2) \quad (\star\star) \qquad \delta' e'_2[v'_1/h, v'_2/tl] \Downarrow^{c'_r} v'_r \quad ({\color{red}\diamond\diamond})}{\text{case } \delta' e' \text{ of nil } \to \delta' e'_1 \mid h :: tl \to \delta' e'_2 \Downarrow^{c'+c'_r+c_{caseL}} v'_r}$ **caseL-cons** and

$C = c + c_r + c_{caseL} < m$

By IH 1 on the first premise, we get $(m, \delta e, \delta' e') \in (\!|\text{list}[\sigma n]^{\sigma\alpha}\, \sigma\tau|\!)_{\varepsilon}^{\sigma t}$. Unrolling its definition with $(\star)$, $(\star\star)$ and $c < m$, we get

a) $c - c' \leq \sigma t$

b) $(m - c, \text{nil}\,, \text{cons}(v'_1, v'_2)) \in (\!|\text{list}[\sigma n]^{\sigma\alpha}\, \sigma\tau|\!)_v$

However, b) is false since two lists of different length are not related, therefore this case is vacuously true.

subcase 3: $\dfrac{\delta e \Downarrow^{c} \text{cons}(v_1, v_2) \quad (\star) \qquad \delta e_2[v_1/h, v_2/tl] \Downarrow^{c_r} v_r \quad ({\color{red}\diamond})}{\text{case } \delta e \text{ of nil } \to \delta e_1 \mid h :: tl \to \delta e_2 \Downarrow^{c+c_r+c_{caseL}} v_r}$ **caseL-cons** and

$\dfrac{\delta' e' \Downarrow^{c'} \text{cons}(v'_1, v'_2) \quad (\star\star) \qquad \delta' e'_2[v'_1/h, v'_2/tl] \Downarrow^{c'_r} v'_r \quad ({\color{red}\diamond\diamond})}{\text{case } \delta' e' \text{ of nil } \to \delta' e'_1 \mid h :: tl \to \delta' e'_2 \Downarrow^{c'+c'_r+c_{caseL}} v'_r}$ **caseL-cons**

By IH 1 on the first premise, we get $(m, \delta e) \in (\!|\text{list}[\sigma n]^{\sigma\alpha}\, \sigma\tau|\!)_{\varepsilon}^{\sigma t}$. Unrolling its definition with $(\star)$ and $(\star\star)$ and $c < m$, we get

a) $c - c' \leq \sigma t$

b) $(m - c, \text{cons}(v_1, v_2), \text{cons}(v'_1, v'_2)) \in (\!|\text{list}[\sigma n]^{\sigma\alpha}\, \sigma\tau|\!)_v$

For b), there are two cases:

subsubcase 1: $\sigma n = I + 1$ such that we have

$$(m - c, v_1, v'_1) \in (\!|\square\, \sigma\tau|\!)_v \tag{3}$$

$$(m-c, v_2, v_2') \in (\!|\text{list}[I]^{\sigma\alpha}\, \sigma\tau|\!)_v \tag{4}$$

In addition, by downward closure (Lemma 4) on $(m, \delta, \delta') \in \mathcal{G}(\!|\Gamma|\!)$, we have

$$(m-c, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!) \tag{5}$$

Then, we can instantiate IH 1 on the third premise using

- $\sigma[i \mapsto I] \in \mathcal{D}[\![i :: \mathbb{N}, \Delta]\!]$
- $\models \sigma[i \mapsto I](\Phi \wedge n \doteq i+1)$ obtained by
  - $\models \sigma\Phi$ by main assumption
  - $\models \sigma n \doteq I+1$ by sub-assumption
- $(m-c, \delta[h \mapsto v_1, tl \mapsto v_2], \delta'[h \mapsto v_1', tl \mapsto v_2']) \in \mathcal{G}(\!|\sigma[i \mapsto I](\Gamma, x : \square\,\tau, tl : \text{list}[i]^{\alpha}\, \tau)|\!)$ using (3) and (4) and (5).

we get $(m-c, \delta e_2[v_1/h, v_2/tl], \delta' e_2'[v_1'/h, v_2'/tl]) \in (\!|\sigma[i \mapsto I]\tau'|\!)_{\varepsilon}^{\sigma[i \mapsto I]t'}$.
Since, $i \notin FV(t', \tau, \tau')$, we have $(m-c, \delta e_2[v_1/h, v_2/tl], \delta' e_2'[v_1'/h, v_2'/tl]) \in (\!|\sigma\tau'|\!)_{\varepsilon}^{\sigma t'}$.
Unrolling its definition using ($\diamond$), ($\diamond\diamond$) and $c_r < m-c$, we get

c) $c_r - c_r' \leq \sigma t'$

d) $(m-(c+c_r), v_r, v_r') \in (\!|\sigma\tau'|\!)_v$

We conclude with

1. By a) and c), we get $(c + c_r + c_{caseL}) - (c' + c_r' + c_{caseL}) \leq \sigma t + \sigma t' + c_{caseL}$
2. By downward closure (Lemma 4) on f) using

$$m - (c + c_r + c_{caseL}) \leq m - (c + c_r)$$

   we get $(m - (c + c_r + c_{caseL}), v_r, v_r') \in (\!|\sigma\tau'|\!)_v$.

subsubcase 2: $\sigma n = I+1$ and $\sigma\alpha = J+1$ such that we have

$$(m-c, v_1, v_1') \in (\!|\sigma\tau|\!)_v \tag{6}$$

$$(m-c, v_2, v_2') \in (\!|\text{list}[I]^{J}\, \sigma\tau|\!)_v \tag{7}$$

In addition, by downward closure (Lemma 4) on $(m, \delta, \delta') \in \mathcal{G}(\!|\Gamma|\!)$, we have

$$(m-c, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!) \tag{8}$$

Then, we can instantiate IH 1 on the fourth premise using

- $\sigma[i \mapsto I, \beta \mapsto J] \in \mathcal{D}[\![i :: \mathbb{N}, \beta :: \mathbb{N}, \Delta]\!]$
- $\models \sigma[i \mapsto I, \beta \mapsto J](\Phi \wedge n \doteq i + 1 \wedge \alpha \doteq \beta + 1)$ obtained
  - $\models \sigma\Phi$ by main assumption
  - $\models \sigma n \doteq I + 1$ by sub-assumption
  - $\models \sigma\alpha \doteq J + 1$ by sub-assumption
- $(m - c, \delta[h \mapsto v_1, tl \mapsto v_2], \delta'[h \mapsto v_1', tl \mapsto v_2']) \in \mathcal{G}(\!|\sigma[i \mapsto I, \beta \mapsto J](\Gamma, x : \tau, tl : \text{list}[i]^{\beta}\,\tau)|\!)$ using (6) and (7) and (8)

we get $(m - c, \delta e_2[v_1/h, v_2/tl], \delta' e_2'[v_1'/h, v_2'/tl]) \in (\!|\sigma[i \mapsto I, \beta \mapsto J]\tau'|\!)_{\varepsilon}^{\sigma[i \mapsto I, \beta \mapsto J]t'}$.
Since, $i, \beta \notin FV(t', \tau, \tau')$, we have
$(m - c, \delta e_2[v_1/h, v_2/tl], \delta' e_2'[v_1'/h, v_2'/tl]) \in (\!|\sigma\tau'|\!)_{\varepsilon}^{\sigma t'}$.
Unrolling its definition using ($\diamond$), ($\diamond\diamond$) and $c_r < m - c$, we get

e) $c_r - c_r' \leq \sigma t'$

f) $(m - (c + c_r), v_r, v_r') \in (\!|\sigma\tau'|\!)_v$

We conclude with

1. By a) and e), we get $(c + c_r + c_{caseL}) - (c' + c_r' + c_{caseL}) \leq \sigma t + \sigma t' + c_{caseL}$
2. By downward closure (Lemma 4) on d) using

$$m - (c + c_r + c_{caseL}) \leq m - (c + c_r)$$

   we get $(m - (c + c_r + c_{caseL}), v_r, v_r') \in (\!|\sigma\tau'|\!)_v$.

subcase 4: $\dfrac{\delta e \Downarrow^{c} \text{cons}(v_1, v_2) \quad (\star) \qquad \delta e_2[v_1/h, v_2/tl] \Downarrow^{c_r} v_r \quad (\diamond)}{\text{case } \delta e \text{ of nil } \to \delta e_1 \mid h :: tl \to \delta e_2 \Downarrow^{c + c_r + c_{caseL}} v_r}$ **caseL-cons** and
$\dfrac{\delta' e' \Downarrow^{c'} \text{nil} \quad (\star\star) \qquad \delta' e_1' \Downarrow^{c_r'} v_r' \quad (\diamond\diamond)}{\text{case } \delta' e' \text{ of nil } \to \delta' e_1' \mid h :: tl \to \delta' e_2' \Downarrow^{c' + c_r' + c_{caseL}} v_r'}$ **caseL-nil** and
$C = c + c_r + c_{caseL} < m$

By IH 1 on the first premise, we get $(m, \delta e, \delta' e') \in (\!|\text{list}[\sigma n]^{\sigma\alpha}\,\sigma\tau|\!)_{\varepsilon}^{\sigma t}$. Unrolling its definition with $(\star)$, $(\star\star)$ and $c < m$, we get

a) $c - c' \leq \sigma t$

b) $(m - c, \text{cons}(v_1, v_2), \text{nil}\ ) \in (\!|\text{list}[\sigma n]^{\sigma\alpha}\,\sigma\tau|\!)_v$

However, b) is false since two lists of different length are not related, therefore this case is vacuously true.

$=$

$$\textbf{Case}\ \frac{\begin{array}{c}\Delta;\Phi;\Gamma\vdash e\ominus e'\lesssim t:\mathbf{tree}[n]^{\alpha}\,\tau \qquad \Delta;\Phi\wedge n=0\wedge;\Gamma\vdash e_1\ominus e_1'\lesssim t':\tau' \\ i,j,\beta,\theta,\Delta;\Phi\wedge n=i+j+1\wedge\alpha=\beta+\theta;x:\square\,\tau,l:\mathbf{tree}[i]^{\beta}\,\tau,r:\mathbf{tree}[j]^{\theta}\,\tau,\Gamma\vdash e_2\ominus e_2'\lesssim t':\tau' \\ i,j,\beta,\theta,\Delta;\Phi\wedge n=i+j+1\wedge\alpha=\beta+\theta+1;x:\tau,l:\mathbf{tree}[i]^{\beta}\,\tau,r:\mathbf{tree}[j]^{\theta}\,\tau,\Gamma\vdash e_2\ominus e_2'\lesssim t':\tau'\end{array}}{\Delta;\Phi;\Gamma\vdash\ \textbf{case}\ e\ \textbf{of leaf}\to e_1\mid\textbf{node}(l,x,r)\to e_2\ominus\ \textbf{case}\ e'\ \textbf{of leaf}\to e_1'\mid\textbf{node}(l,x,r)\to\lesssim t+t':e_2'\ \tau'}$$

caseT

Assume that $(m,\delta,\delta')\in\mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models\sigma\Phi$.

TS: $(m,\ \text{case}\ \delta e\ \text{of leaf}\to\delta e_1\mid\text{node}(l,x,r)\to\delta e_2,\ \text{case}\ \delta'e'\ \text{of leaf}\to\delta'e_1'\mid\text{node}(l,x,r)\to\delta'e_2')\in(\!|\sigma\tau'|\!)_{\varepsilon}^{\sigma t+\sigma t'}$.

Following the definition of $(\!|\cdot|\!)_{\varepsilon}^{\cdot}$, assume that

$$\text{case}\ \delta e\ \text{of leaf}\to\delta e_1\mid\text{node}(l,x,r)\to\delta e_2\Downarrow^{C}v_r \tag{1}$$

$$\text{case}\ \delta'e'\ \text{of leaf}\to\delta'e_1'\mid\text{node}(l,x,r)\to\delta'e_2'\Downarrow^{C'}v_r' \tag{2}$$

and $C<m$.

Depending on what δe and $\delta'e'$ evaluate to, there are four cases.

subcase 1: $\dfrac{\delta e\Downarrow^{c}\text{leaf}\ \ (\star)\qquad \delta e_1\Downarrow^{c_r}v_r\ \ (\diamond)}{\text{case}\ \delta e\ \text{of leaf}\to\delta e_1\mid\text{node}(l,x,r)\to\delta e_2\Downarrow^{c+c_r+c_{caseT}}v_r}$ **caseT-leaf** and

$\dfrac{\delta'e'\Downarrow^{c'}\text{leaf}\ \ (\star\star)\qquad \delta'e_1'\Downarrow^{c_r'}v_r'\ \ (\diamond\diamond)}{\text{case}\ \delta'e'\ \text{of leaf}\to\delta'e_1'\mid\text{node}(l,x,r)\to\delta'e_2'\Downarrow^{c'+c_r'+c_{caseT}}v_r'}$ **caseT-leaf** and

$C=c+c_r+c_{caseT}<m$

By IH 1 on the first premise, we get $(m,\delta e,\delta'e')\in(\!|\text{tree}[\sigma n]^{\sigma\alpha}\,\sigma\tau|\!)_{\varepsilon}^{\sigma t}$. Unrolling its definition with $(\star)$, $(\star\star)$ and $c<m$, we get

a) $c-c'\leq\sigma t$

b) $(m-c,\text{leaf},\text{leaf})\in(\!|\text{tree}[\sigma n]^{\sigma\alpha}\,\sigma\tau|\!)_v$

By b), $\sigma n=0$.

Then, we can instantiate IH 1 on the second premise using $\models\sigma\Phi\wedge\sigma n\doteq 0$, to obtain $(m,\delta e_1,\delta'e_1')\in(\!|\sigma\tau'|\!)_{\varepsilon}^{\sigma t'}$.

Unrolling its definition using $(\diamond)$ and $(\diamond\diamond)$ and $c_r<m$, we get

c) $c_r-c_r'\leq\sigma t'$

d) $(m-c_r,v_r,v_r')\in(\!|\sigma\tau'|\!)_v$

We conclude with

1. By a) and c), we get $(c + c_r + c_{caseT}) - (c' + c'_r + c_{caseT}) \leq \sigma t + \sigma t'$
2. By downward closure (Lemma 4) on d) using

$$m - (c + c_r + c_{caseT}) \leq m - c_r$$

we get $(m - (c + c_r + c_{caseT}), v_r, v'_r) \in (\!|\sigma\tau'|\!)_v$.

subcase 2: $\dfrac{\delta e \Downarrow^{c} \text{leaf} \quad (\star) \qquad \delta e_1 \Downarrow^{c_r} v_r \quad (\diamond)}{\text{case } \delta e \text{ of leaf} \to \delta e_1 \mid \text{node}(l, x, r) \to \delta e_2 \Downarrow^{c + c_r + c_{caseT}} v_r}$ **caseT-leaf** and

$\dfrac{\delta' e' \Downarrow^{c'} \text{node}(v'_l, v', v'_r) \quad (\star\star) \qquad \delta' e'_2[v'_l/l, v'/x, v'_r/r] \Downarrow^{c'_r} v'_r \quad (\diamond\diamond)}{\text{case } \delta' e' \text{ of nil } \to \delta' e'_1 \mid \text{node}(l, x, r) \to \delta' e'_2 \Downarrow^{c' + c'_r + c_{caseT}} v'_r}$ **caseT-node** and

$C = c + c_r + c_{caseT} < m$

By IH 1 on the first premise, we get $(m, \delta e, \delta' e') \in (\!|\text{tree}[\sigma n]^{\sigma\alpha}\, \sigma\tau|\!)^{\sigma t}_{\varepsilon}$. Unrolling its definition with $(\star)$, $(\star\star)$ and $c < m$, we get

a) $c - c' \leq \sigma t$

b) $(m - c, \text{leaf}, \text{node}(v'_1, v'_2)) \in (\!|\text{tree}[\sigma n]^{\sigma\alpha}\, \sigma\tau|\!)_v$

However, b) is false since two trees of different length are not related, therefore this case is vacuously true.

subcase 3: $\dfrac{\delta e \Downarrow^{c} \text{node}(v_l, v, v_r) \quad (\star) \qquad \delta e_2[v_l/l, v/x, v_r/r] \Downarrow^{c_r} v_r \quad (\diamond)}{\text{case } \delta e \text{ of nil } \to \delta e_1 \mid \text{node}(l, x, r) \to \delta e_2 \Downarrow^{c + c_r + c_{caseT}} v_r}$ **caseT-node** and

$\dfrac{\delta' e' \Downarrow^{c'} \text{node}(v'_l, v', v'_r) \quad (\star\star) \qquad \delta' e'_2[v'_l/l, v'/x, v'_r/r] \Downarrow^{c'_r} v'_r \quad (\diamond\diamond)}{\text{case } \delta' e' \text{ of nil } \to \delta' e'_1 \mid \text{node}(l, x, r) \to \delta' e'_2 \Downarrow^{c' + c'_r + c_{caseT}} v'_r}$ **caseT-node** and

$C = c + c_r + c_{caseT} < m$

By IH 1 on the first premise, we get $(m, \delta e) \in (\!|\text{tree}[\sigma n]^{\sigma\alpha}\, \sigma\tau|\!)^{\sigma t}_{\varepsilon}$. Unrolling its definition with $(\star)$ and $(\star\star)$ and $c < m$, we get

a) $c - c' \leq \sigma t$

b) $(m - c, \text{node}(v_1, v_2), \text{node}(v'_1, v'_2)) \in (\!|\text{tree}[\sigma n]^{\sigma\alpha}\, \sigma\tau|\!)_v$

For b), there are two cases:

subsubcase 1: $\sigma n = I + J + 1$ and $\sigma\alpha = M + N$ such that we have

$$(m-c, v, v') \in (\!|\Box\, \sigma\tau|\!)_v \tag{3}$$

$$(m-c, v_l, v_l') \in (\!|\mathrm{tree}[I]^M\, \sigma\tau|\!)_v \tag{4}$$

$$(m-c, v_r, v_r') \in (\!|\mathrm{tree}[J]^N\, \sigma\tau|\!)_v \tag{5}$$

In addition, by downward closure (Lemma 4) on $(m, \delta, \delta') \in \mathcal{G}(\!|\Gamma|\!)$, we have

$$(m-c, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!) \tag{6}$$

Then, we can instantiate IH 1 on the third premise using

- $\sigma[i \mapsto I, j \mapsto J, \beta \mapsto M, \theta \mapsto N] \in \mathcal{D}[\![i :: \mathbb{N}, j :: \mathbb{N}, \beta :: \mathbb{N}, \theta :: \mathbb{N}, \Delta]\!]$
- $\models \sigma[i \mapsto I, j \mapsto J, \beta \mapsto M, \theta \mapsto N](\Phi \wedge n \doteq i + j + 1 \wedge \alpha \doteq \beta + \theta)$ obtained by
  - $\models \sigma\Phi$ by main assumption
  - $\models \sigma n \doteq I + J + 1$ by sub-assumption
  - $\models \sigma\alpha \doteq M + N$ by sub-assumption
- $(m-c, \delta[x \mapsto v, l \mapsto v_l, r \mapsto v_r], \delta'[x \mapsto v', l \mapsto v_l', r \mapsto v_r']) \in \mathcal{G}(\!|\sigma[i \mapsto I, j \mapsto J, \beta \mapsto M, \theta \mapsto N](\Gamma, x : \Box\,\tau, l : \mathrm{tree}[i]^\beta\,\tau, r : \mathrm{tree}[j]^\theta\,\tau)|\!)$ using (3), (4), (5) and (6)

we get $(m-c, \delta e_2[v/x, v_l/l, v_r/r], \delta' e_2'[v'/x, v_l'/l, v_r'/r]) \in (\!|\sigma[i \mapsto I, j \mapsto J, \beta \mapsto M, \theta \mapsto N]\tau'|\!)_\varepsilon^{\sigma[i \mapsto I, j \mapsto J, \beta \mapsto M, \theta \mapsto N]t'}$.
Since, $i, j, \beta, \theta \notin FV(t', \tau, \tau')$, we have
$(m-c, \delta e_2[v/x, v_l/l, v_r/r], \delta' e_2'[v'/x, v_l'/l, v_r'/r]) \in (\!|\sigma\tau'|\!)_\varepsilon^{\sigma t'}$.
Unrolling its definition using ($\diamond$), ($\diamond\diamond$) and $c_r < m - c$, we get

c) $c_r - c_r' \leq \sigma t'$
d) $(m - (c + c_r), v_r, v_r') \in (\!|\sigma\tau'|\!)_v$

We conclude with

1. By a) and c), we get $(c + c_r + c_{caseT}) - (c' + c_r' + c_{caseT}) \leq \sigma t + \sigma t' + c_{caseT}$
2. By downward closure (Lemma 4) on d) using

$$m - (c + c_r + c_{caseT}) \leq m - (c + c_r)$$

   we get $(m - (c + c_r + c_{caseT}), v_r, v_r') \in (\!|\sigma\tau'|\!)_v$.

subsubcase 2: $\sigma n = I + J + 1$ and $\sigma\alpha = M + N + 1$ such that we have

$$(m-c, v, v') \in (\!|\sigma\tau|\!)_v \tag{7}$$

$$(m-c, v_l, v_l') \in (\!|\text{tree}[I]^M\ \sigma\tau|\!)_v \tag{8}$$

$$(m-c, v_r, v_r') \in (\!|\text{tree}[J]^N\ \sigma\tau|\!)_v \tag{9}$$

In addition, by downward closure (Lemma 4) on $(m, \delta, \delta') \in \mathcal{G}(\!|\Gamma|\!)$, we have

$$(m-c, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!) \tag{10}$$

Then, we can instantiate IH 1 on the fourth premise using

- $\sigma[i \mapsto I, j \mapsto J, \beta \mapsto M, \theta \mapsto N] \in \mathcal{D}[\![i :: \mathbb{N}, j :: \mathbb{N}, \beta :: \mathbb{N}, \theta :: \mathbb{N}, \Delta]\!]$
- $\models \sigma[i \mapsto I, j \mapsto J, \beta \mapsto M, \theta \mapsto N](\Phi \wedge n \doteq i+j+1 \wedge \alpha \doteq \beta+\theta+1)$ obtained by
  - $\models \sigma\Phi$ by main assumption
  - $\models \sigma n \doteq I+J+1$ by sub-assumption
  - $\models \sigma\alpha \doteq M+N+1$ by sub-assumption
- $(m-c, \delta[x \mapsto v, l \mapsto v_l, r \mapsto v_r], \delta'[x \mapsto v', l \mapsto v_l', r \mapsto v_r']) \in \mathcal{G}(\!|\sigma[i \mapsto I, j \mapsto J, \beta \mapsto M, \theta \mapsto N](\Gamma, x : \tau, l : \text{tree}[i]^\beta\ \tau, r : \text{tree}[j]^\theta\ \tau)|\!)$ using (7), (8), (9) and (10)

we get $(m-c, \delta e_2[v/x, v_l/l, v_r/r], \delta' e_2'[v'/x, v_l'/l, v_r'/r]) \in (\!|\sigma[i \mapsto I, j \mapsto J, \beta \mapsto M, \theta \mapsto N]\tau'|\!)_\varepsilon^{\sigma[i \mapsto I, j \mapsto J, \beta \mapsto M, \theta \mapsto N]t'}$.
Since, $i, j, \beta, \theta \notin FV(t', \tau, \tau')$, we have
$(m-c, \delta e_2[v/x, v_l/l, v_r/r], \delta' e_2'[v'/x, v_l'/l, v_r'/r]) \in (\!|\sigma\tau'|\!)_\varepsilon^{\sigma t'}$.
Unrolling its definition using $(\diamond)$, $(\diamond\diamond)$ and $c_r < m-c$, we get

e) $c_r - c_r' \leq \sigma t'$

f) $(m-(c+c_r), v_r, v_r') \in (\!|\sigma\tau'|\!)_v$

We conclude with

1. By a) and e), we get $(c + c_r + c_{caseT}) - (c' + c_r' + c_{caseT}) \leq \sigma t + \sigma t' + c_{caseT}$
2. By downward closure (Lemma 4) on d) using

$$m - (c + c_r + c_{caseT}) \leq m - (c + c_r)$$

we get $(m - (c + c_r + c_{caseT}), v_r, v_r') \in (\!|\sigma\tau'|\!)_v$.

subcase 4: $\dfrac{\delta e \Downarrow^{c} \text{node}(v_l, v, v_r) \quad (\star) \qquad \delta e_2[v_l/l, v/x, v_r/r] \Downarrow^{c_r} v_r \quad (\diamond)}{\text{case } \delta e \text{ of nil } \to \delta e_1 \mid \text{node}(l, x, r) \to \delta e_2 \Downarrow^{c+c_r+c_{caseT}} v_r}$ **caseT-node** and

$$\dfrac{\delta' e' \Downarrow^{c'} \text{leaf} \quad (\star\star) \qquad \delta' e_1' \Downarrow^{c_r'} v_r' \quad (\diamond\diamond)}{\text{case } \delta' e' \text{ of leaf} \rightarrow \delta' e_1' \mid \text{node}(l, x, r) \rightarrow \delta' e_2' \Downarrow^{c' + c_r' + c_{caseT}} v_r'} \ \textbf{caseT-leaf}$$

By IH 1 on the first premise, we get $(m, \delta e, \delta' e') \in (\!|\text{tree}[\sigma n]^{\sigma\alpha}\, \sigma\tau|\!)_{\varepsilon}^{\sigma t}$. Unrolling its definition with $(\star)$, $(\star\star)$ and $c < m$, we get

a) $c - c' \leq \sigma t$

b) $(m - c, \text{node}(v_1, v_2), \text{leaf}) \in (\!|\text{tree}[\sigma n]^{\sigma\alpha}\, \sigma\tau|\!)_v$

However, b) is false since two trees of different length are not related, therefore this case is vacuously true.

Case $$\dfrac{\Delta; \Phi \vdash \tau_1 \xrightarrow{\textbf{diff}(t)} \tau_2 \ \texttt{wf} \qquad \Delta; \Phi; x : \tau_1, f : \tau_1 \xrightarrow{\textbf{diff}(t)} \tau_2, \Gamma \vdash e_1 \ominus e_2 \lesssim t : \tau_2}{\Delta; \Phi; \Gamma \vdash \textbf{fix}\ f(x).e_1 \ominus \textbf{fix}\ f(x).e_2 \lesssim 0 : \tau_1 \xrightarrow{\textbf{diff}(t)} \tau_2} \ \textbf{r-fix}$$

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.
TS: $(m, \text{fix } f(x).\delta e_1, \text{fix } f(x).\delta' e_2) \in (\!|\sigma\tau_1 \xrightarrow{\text{diff}(\sigma t)} \sigma\tau_2|\!)_{\varepsilon}^{0}$.
By Lemma 2, STS: $(m, \text{fix } f(x).\delta e_1, \text{fix } f(x).\delta' e_2) \in (\!|\sigma\tau_1 \xrightarrow{\text{diff}(\sigma t)} \sigma\tau_2|\!)_v$.
Let $F = \text{fix } f(x).\delta e_1$ and $F' = \text{fix } f(x).\delta' e_2$.
We prove the more general statement

$$\forall\, m' \leq m.\ (m', F, F') \in (\!|\sigma\tau_1 \xrightarrow{\text{diff}(\sigma t)} \sigma\tau_2|\!)_v$$

by subinduction on m'.

There are two parts to show:

subcase 1: $m' = 0$

By the definition of function types, there are two parts to show:

subsubcase 1: $\forall j < m' = 0 \cdots$

Since there is no non-negative j such that $j < 0$, the goal is vacuously true.

subsubcase 2: STS: $\forall j.(j, F) \in [\![|\sigma\tau_1|_1 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_1]\!]_v \ \wedge\ (j, F')[\![|\sigma\tau_1|_2 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_2]\!]_v$.

Pick j.

- STS 1: $(j, F) \in [\![|\sigma\tau_1|_1 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_1]\!]_v$

  We prove the more general statement

$$\forall m' \leq j.\ (m', F) \in [\![|\sigma\tau_1|_1 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_1]\!]_v$$

by subinduction on m'.

There are two cases:

– $m' = 0$

Since there is no non-negative j such that $j < 0$, the goal is vacuously true.

– $m' = m'' + 1 \leq m$

By sub-IH

$$(m'', \text{fix } f(x).\delta e_1) \in [\![|\sigma\tau_1|_1 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_1]\!]_v \tag{1}$$

STS: $(m'' + 1, \text{fix } f(x).\delta e_1) \in [\![|\sigma\tau_1|_1 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_1]\!]_v$.

Pick $j'' < m'' + 1$ and assume that $(j'', v) \in [\![|\sigma\tau_1|_1]\!]_v$.

STS: $(j'', \delta e_1[v/x, F/f]) \in [\![|\sigma\tau_2|_1]\!]_\varepsilon^{0,\infty}$.

This follows by IH 3 on the premise instantiated with $(j'', \delta[x \mapsto v, f \mapsto F]) \in \mathcal{G}[\![x : |\sigma\tau_1|_1, f : |\sigma\tau_1|_1 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_1, |\sigma\Gamma|_1]\!]$ which holds because

* $\text{FV}(e_1) \subseteq \text{dom}(x : \tau_1, f : \tau_1 \xrightarrow{\text{diff}(t)} \tau_2, \Gamma)$ using Lemma 8 on the second premise
* $(j'', \delta) \in \mathcal{G}[\![|\sigma\Gamma|_1]\!]$ using Lemma 3 on $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$
* $(j'', v) \in [\![|\sigma\tau_1|_1]\!]_v$, from the assumption above
* $(j'', \text{fix } f(x).\delta e_1) \in [\![|\sigma\tau_1|_1 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_1]\!]_v$, obtained by downward closure (Lemma 4) on (1) using $j'' \leq m''$

• STS 2: $(j, F') \in [\![|\sigma\tau_1|_2 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_2]\!]_v$

We prove the more general statement

$$\forall m' \leq j.\ (m', F') \in [\![|\sigma\tau_1|_2 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_2]\!]_v$$

by subinduction on m'.

There are two cases:

– $m' = 0$

Since there is no non-negative j such that $j < 0$, the goal is vacuously true.

– $m' = m'' + 1 \leq j$

By sub-IH

$$(m'', F') \in [\![|\sigma\tau_1|_2 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_2]\!]_v \tag{2}$$

STS: $(m'' + 1, \text{fix } f(x).\delta' e_2) \in [\![|\sigma\tau_1|_2 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_2]\!]_v$.

Pick $j'' < m'' + 1$ and assume that $(j'', v) \in [\![|\sigma\tau_1|_2]\!]_v$.

STS: $(j'', \delta' e_2[v/x, F'/f]) \in [\![|\sigma\tau_2|_2]\!]_\varepsilon^{0,\infty}$.

This follows by IH 3 on the premise instantiated with $(j'', \delta[x \mapsto v, f \mapsto$

(fix $f(x).\delta' e_2)]) \in \mathcal{G}[\![x : |\sigma\tau_1|_2, f : |\sigma\tau_1|_2 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_2, |\sigma\Gamma|_2]\!]$ which holds because

* $\text{FV}(e_2) \subseteq \text{dom}(x : \tau_1, f : \tau_1 \xrightarrow{\text{diff}(t)} \tau_2, \Gamma)$ using Lemma 8 on the second premise
* $(j'', v) \in [\![|\sigma\tau_1|_2]\!]_v$, from the assumption above
* $(j'', \delta) \in \mathcal{G}[\![|\sigma\Gamma|_2]\!]$ using Lemma 3 on $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$
* $(j'', F') \in [\![|\sigma\tau_1|_2 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_2]\!]_v$, obtained by downward closure (Lemma 4) on (2) using $j'' \leq m''$

subcase 2: $m' = m'' + 1 \leq m$

By sub-IH

$$(m'', F, F') \in (\!|\sigma\tau_1 \xrightarrow{\text{diff}(\sigma t)} \sigma\tau_2|\!)_v \tag{3}$$

TS: $(m'' + 1, \text{fix } f(x).\delta e_1, \text{fix } f(x).\delta' e_2) \in (\!|\sigma\tau_1 \xrightarrow{\text{diff}(\sigma t)} \sigma\tau_2|\!)_v$

Pick $j < m'' + 1$ and assume that $(j, v_1, v_2) \in (\!|\sigma\tau_1|\!)_v$.

STS: $(j, \delta e_1[v_1/x, F/f], \delta' e_2[v_2/x, F'/f]) \in (\!|\sigma\tau_2|\!)_\varepsilon^{\sigma t}$.

This follows by IH on the premise instantiated with

$(j, \delta[x \mapsto v_1, f \mapsto F], \delta'[x \mapsto v_2, f \mapsto F']) \in \mathcal{G}(\!|\sigma\Gamma, x : \sigma\tau_1, f : \sigma\tau_1 \xrightarrow{\text{diff}(\sigma t)} \sigma\tau_2|\!)$ which holds because

- $(j, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ obtained by downward closure (Lemma 4) using $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $j < m' \leq m$.
- $(j, v_1, v_2) \in (\!|\sigma\tau_1|\!)_v$, from the assumption above
- $(j, F, F') \in (\!|\sigma\tau_1 \xrightarrow{\text{diff}(\sigma t)} \sigma\tau_2|\!)_v$, obtained by downward closure (Lemma 4) on (3) using $j \leq m''$

This completes the proof of this case.

$$\textbf{Case}\quad \frac{\Delta; \Phi \vdash \tau_1 \xrightarrow{\textbf{diff}(t)} \tau_2\ \texttt{wf} \qquad \Delta; \Phi; x : \tau_1, f : \square\,(\tau_1 \xrightarrow{\textbf{diff}(t)} \tau_2), \Gamma \vdash e \ominus e \lesssim t : \tau_2 \qquad \forall x \in dom(\Gamma).\ \ \Delta; \Phi \models \Gamma(x) \sqsubseteq \square\,\Gamma(x)}{\Delta; \Phi; \Gamma \vdash \textbf{fix}\ f(x).e \ominus \textbf{fix}\ f(x).e \lesssim 0 : \square\,(\tau_1 \xrightarrow{\textbf{diff}(t)} \tau_2)}\ \textbf{r-fixNC}$$

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS: $(m, \text{fix } f(x).\delta e, \text{fix } f(x).\delta' e) \in (\!|\square\,(\sigma\tau_1 \xrightarrow{\text{diff}(\sigma t)} \sigma\tau_2)|\!)_\varepsilon^0$.

By Lemma 2, STS: $(m, \text{fix } f(x).\delta e, \text{fix } f(x).\delta' e) \in (\!|\square\,(\sigma\tau_1 \xrightarrow{\text{diff}(\sigma t)} \sigma\tau_2)|\!)_v$.

By Lemma 5 using $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and the third premise, we get $(m, \delta, \delta') \in \mathcal{G}(\!|\square\,\sigma\Gamma|\!)$, i.e. $\delta = \delta'$.

Therefore, STS: $(m, \text{fix } f(x).\delta e, \text{fix } f(x).\delta e) \in (\!|\sigma\tau_1 \xrightarrow{\text{diff}(\sigma t)} \sigma\tau_2|\!)_v$.

Let $F = \text{fix } f(x).\delta e$.

We prove the more general statement

$$\forall\, m' \leq m.\ (m', F, F) \in (\!|\sigma\tau_1 \xrightarrow{\text{diff}(\sigma t)} \sigma\tau_2|\!)_v$$

by subinduction on m'.

There are two parts to show:

subcase 1: $m' = 0$

By the definition of function types, there are two parts to show:

subsubcase 1: $\forall j < m' = 0 \cdots$

Since there is no non-negative j such that $j < 0$, the goal is vacuously true.

subsubcase 2: STS: $\forall j.(j, F) \in [\![|\sigma\tau_1|_1 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_1]\!]_v$.

Pick j. STS: $(j, F) \in [\![|\sigma\tau_1|_1 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_1]\!]_v$

We prove the more general statement

$$\forall m' \leq j.\ (m', F) \in [\![|\sigma\tau_1|_1 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_1]\!]_v$$

by subinduction on m'.

There are two cases:

- $m' = 0$

  Since there is no non-negative j such that $j < 0$, the goal is vacuously true.

- $m' = m'' + 1 \leq j$

  By sub-IH
  $$(m'', \text{fix } f(x).\delta e_1) \in [\![|\sigma\tau_1|_1 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_1]\!]_v \tag{1}$$

  STS: $(m'' + 1, \text{fix } f(x).\delta e_1) \in [\![|\sigma\tau_1|_1 \xrightarrow{\text{exec}(0,t)} |\sigma\tau_2|_1]\!]_v$.

  Pick $j'' < m'' + 1$ and assume that $(j'', v) \in [\![|\sigma\tau_1|_1]\!]_v$.

  STS: $(j'', \delta e_1[v/x, F/f]) \in [\![|\sigma\tau_2|_1]\!]_\varepsilon^{0,\infty}$.

  This follows by IH 3 on the premise instantiated with

  - $\text{FV}(e) \subseteq \text{dom}(x : \tau_1, f : \tau_1 \xrightarrow{\text{diff}(t)} \tau_2, \Gamma)$ using Lemma 8 on the second premise
  - $(j'', \delta[x \mapsto v, f \mapsto F]) \in \mathcal{G}[\![x : |\sigma\tau_1|_1, f : |\sigma\tau_1|_1 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_1, |\sigma\Gamma|_1]\!]$ which holds because
    - $(j'', \delta) \in \mathcal{G}[\![|\sigma\Gamma|_1]\!]$ using Lemma 3 on $(m, \delta, \delta) \in \mathcal{G}(\!|\sigma\Gamma|\!)$
    - $(j'', v) \in [\![|\sigma\tau_1|_1]\!]_v$, from the assumption above

* $(j'', \text{fix } f(x).\delta e_1) \in [\![|\sigma\tau_1|_1 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_1]\!]_v$, obtained by downward closure (Lemma 4) on (1) using $j'' \leq m''$

subcase 2: $m' = m'' + 1 \leq m$

By sub-IH

$$(m'', F, F) \in (\!|\sigma\tau_1 \xrightarrow{\text{diff}(\sigma t)} \sigma\tau_2|\!)_v \tag{2}$$

TS: $(m'' + 1, \text{fix } f(x).\delta e_1, \text{fix } f(x).\delta e_2) \in (\!|\sigma\tau_1 \xrightarrow{\text{diff}(\sigma t)} \sigma\tau_2|\!)_v$

Pick $j < m'' + 1$ and assume that $(j, v_1, v_2) \in (\!|\sigma\tau_1|\!)_v$.

STS: $(j, \delta e_1[v_1/x, F/f], \delta e_2[v_2/x, F/f]) \in (\!|\sigma\tau_2|\!)_\varepsilon^{\sigma t}$.

This follows by IH on the premise instantiated with $(j, \delta[x \mapsto v_1, f \mapsto F], \delta[x \mapsto v_2, f \mapsto F]) \in \mathcal{G}(\!|\sigma\Gamma, x : \sigma\tau_1, f : \square\,(\sigma\tau_1 \xrightarrow{\text{diff}(\sigma t)} \sigma\tau_2)|\!)$ which holds because

- $(j, \delta, \delta) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ obtained by downward closure (Lemma 4) using $(m, \delta, \delta) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $j < m' \leq m$.
- $(j, v_1, v_2) \in (\!|\sigma\tau_1|\!)_v$, from the assumption above
- $(j, F, F) \in (\!|\square\,(\sigma\tau_1 \xrightarrow{\text{diff}(\sigma t)} \sigma\tau_2)|\!)_v$, obtained by downward closure (Lemma 4) on (2) using $j \leq m''$

This completes the proof of this case.

Case $$\frac{\Delta; \Phi; \Gamma \vdash e_1 \ominus e_1' \lesssim t_1 : \tau_1 \xrightarrow{\textbf{diff}(t)} \tau_2 \qquad \Delta; \Phi; \Gamma \vdash e_2 \ominus e_2' \lesssim t_2 : \tau_1}{\Delta; \Phi; \Gamma \vdash e_1\ e_2 \ominus e_1'\ e_2' \lesssim t_1 + t_2 + t : \tau_2}\ \textbf{r-app}$$

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS: $(m, \delta e_1\ \delta e_2, \delta' e_1'\ \delta' e_2') \in (\!|\sigma\tau_2|\!)_\varepsilon^{\sigma t_1 + \sigma t_2 + \sigma t}$.

Following the definition of $(\!|\cdot|\!)_\varepsilon^{\cdot}$, assume that

$$\frac{\delta e_1 \Downarrow^{c_1} \text{fix } f(x).e\ \ (\star) \qquad \delta e_2 \Downarrow^{c_2} v_2\ \ (\diamond) \qquad e[v_2/x, (\text{fix } f(x).e)/f] \Downarrow^{c_r} v_r\ \ (\dagger)}{\delta e_1\ \delta e_2 \Downarrow^{c_1 + c_2 + c_r + c_{app}} v_r}\ \textbf{app} \text{ and}$$

$$\frac{\delta' e_1' \Downarrow^{c_1'} \text{fix } f(x).e'\ \ (\star\star) \qquad \delta' e_2' \Downarrow^{c_2'} v_2'\ \ (\diamond\diamond) \qquad e'[v_2'/x, (\text{fix } f(x).e')/f] \Downarrow^{c_r'} v_r'\ \ (\dagger\dagger)}{\delta' e_1'\ \delta' e_2' \Downarrow^{c_1' + c_2' + c_r' + c_{app}} v_r'}\ \textbf{app}$$

and

$(c_1 + c_2 + c_r + c_{app}) < m$.

By IH 1 on the first premise, we get $(m, \delta e_1, \delta' e_1') \in (\!|\sigma\tau_1 \xrightarrow{\text{diff}(\sigma t)} \sigma\tau_2|\!)_\varepsilon^{\sigma t_1}$. Unrolling its definition with $(\star)$ and $(\star\star)$, and $c_1 < m$, we get

a) $c_1 - c_1' \leq \sigma t_1$

b) $(m - c_1, \text{fix } f(x).e, \text{fix } f(x).e') \in (\!|\sigma\tau_1 \xrightarrow{\text{diff}(\sigma t)} \sigma\tau_2|\!)_v$

By IH 1 on the second premise, we get $(m, \delta e_2, \delta' e_2') \in (\!|\sigma\tau_1|\!)_\varepsilon^{\sigma t_2}$. Unrolling its definition with $(\diamond)$ and $(\diamond\diamond)$, and $c_2 < m$, we get

c) $c_2 - c_2' \leq \sigma t_2$

d) $(m - c_2, v_2, v_2') \in (\!|\sigma\tau_1|\!)_v$

Next, we apply downward-closure (Lemma 4) to d) using

$$m - (c_1 + c_2 + c_{app}) \leq m - c_2$$

and we get

$$(m - (c_1 + c_2 + c_{app}), v_2, v_2') \in (\!|\sigma\tau_1|\!)_v \tag{1}$$

We unroll b) using (1) since

$$m - (c_1 + c_2 + c_{app}) < m - c_1 \qquad \text{note that } 0 < c_{app}$$

and get

$$(m - (c_1 + c_2 + c_{app}), e[v_2/x, \text{fix } f(x).e/f], e'[v_2'/x, \text{fix } f(x).e'/f]) \in (\!|\sigma\tau_2|\!)_\varepsilon^{\sigma t} \tag{2}$$

Next, we unroll (2) using $(\dagger)$ and $(\dagger\dagger)$ and $c_r < m - (c_1 + c_2 + c_{app})$
to obtain

e) $c_r - c_r' \leq \sigma t$

f) $(m - (c_1 + c_2 + c_r + c_{app}), v_r, v_r') \in (\!|\sigma\tau_2|\!)_v$

Now, we can conclude as follows:

1. Using a), c) and e), we get $(c_1 + c_2 + c_r + c_{app}) - (c_1' + c_2' + c_r' + c_{app}) \leq \sigma t_1 + \sigma t_2 + \sigma t$
2. By f)

Case $$\frac{\Delta; \Phi; \Gamma \vdash e_1 \ominus e_1' \lesssim t_1 : \tau_1 \qquad \Delta; \Phi; \Gamma \vdash e_2 \ominus e_2' \lesssim t_2 : \tau_2}{\Delta; \Phi; \Gamma \vdash \langle e_1, e_2 \rangle \ominus \langle e_1', e_2' \rangle \lesssim t_1 + t_2 : \tau_1 \times \tau_2} \textbf{ r-prod}$$

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.
TS: $(m, \langle \delta e_1, \delta e_1 \rangle, \langle \delta e_1', \delta' e_2' \rangle) \in (\!|\sigma\tau_1 \times \sigma\tau_2|\!)_\varepsilon^{\sigma t_1 + \sigma t_2}$.
Following the definition of $(\!|\cdot|\!)_\varepsilon^{\cdot}$, assume that

$$\dfrac{\delta e_1 \Downarrow^{c_1} v_1 \quad (\star) \qquad \delta e_2 \Downarrow^{c_2} v_2 \quad ({\color{red}\diamond})}{\langle \delta e_1, \delta e_2 \rangle \Downarrow^{c_1+c_2} \langle v_1, v_2 \rangle}\ \textbf{prod} \text{ and } \dfrac{\delta' e_1 \Downarrow^{c'_1} v'_1 \quad (\star\star) \qquad \delta' e_2 \Downarrow^{c'_2} v'_2 \quad ({\color{red}\diamond\diamond})}{\langle \delta' e_1, \delta' e_2 \rangle \Downarrow^{c'_1+c'_2} \langle v'_1, v'_2 \rangle}\ \textbf{prod}$$

and $c_1 + c_2 < m$.

By IH 1 on the first premise, we get $(m, \delta e, \delta' e') \in (\!|\sigma\tau_1|\!)_{\varepsilon}^{\sigma t_1}$. Unrolling its definition with $(\star)$ and $(\star\star)$ and $c_1 < m$, we get

a) $c_1 - c'_1 \leq \sigma t_1$

b) $(m - c_1, v_1, v'_1) \in (\!|\sigma\tau_1|\!)_v$

By IH 1 on the first premise, we get $(m, \delta e, \delta' e') \in (\!|\sigma\tau_2|\!)_{\varepsilon}^{\sigma t_2}$. Unrolling its definition with $(\star)$ and $(\star\star)$ and $c_2 < m$, we get

c) $c_2 - c'_2 \leq \sigma t_2$

d) $(m - c_2, v_2, v'_2) \in (\!|\sigma\tau_2|\!)_v$

We can conclude as follows:

1. By a) and c), we get $(c_1 + c_2) - (c'_1 + c'_2) \leq \sigma t_1 + \sigma t_2$
2. By downward closure (Lemma 4) on b) using

$$m - (c_1 + c_2) \leq m - c_1$$

we get

$$(m - (c_1 + c_2), v_1, v_2) \in (\!|\sigma\tau_1|\!)_v \tag{1}$$

By downward closure (Lemma 4) on d) using

$$m - (c_1 + c_2) \leq m - c_2$$

we get

$$(m - (c_1 + c_2), v'_1, v'_2) \in (\!|\sigma\tau_2|\!)_v \tag{2}$$

By combining (1) and (2), we can show that $(m-(c_1+c_2), \langle v_1, v_2 \rangle, \langle v'_1, v'_2 \rangle) \in (\!|\sigma\tau_1 \times \sigma\tau_2|\!)_v$

Case $\dfrac{\Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t : \tau_1 \times \tau_2}{\Delta; \Phi; \Gamma \vdash \pi_1(e) \ominus \pi_1(e') \lesssim t : \tau_1}$ **r-proj1**

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS: $(m, \pi_1(\delta e), \pi_1(\delta' e')) \in (\!|\sigma\tau_1|\!)_{\varepsilon}^{\sigma t}$.

Following the definition of $(\!|\cdot|\!)_{\varepsilon}^{\cdot}$, assume that

$\dfrac{\delta e \Downarrow^{c} \langle v_1, v_2 \rangle \quad (\star)}{\pi_1(\delta e) \Downarrow^{c+c_{proj}} v_1}$ **proj1** and $\dfrac{\delta' e \Downarrow^{c} \langle v_1', v_2' \rangle \quad (\star\star)}{\pi_1(\delta' e) \Downarrow^{c'+c_{proj}} v_1'}$ **proj1** and $c + c_{proj} < m$.

By IH 1 on the first premise, we get $(m, \delta e, \delta' e') \in (\!|\sigma\tau_1|\!)_{\varepsilon}^{\sigma t}$. Unrolling its definition with $(\star)$ and $(\star\star)$ and $c < m$, we get

a) $c - c' \leq \sigma t$

b) $(m - c, \langle v_1, v_2 \rangle, \langle v_1', v_2' \rangle) \in (\!|\sigma\tau_1 \times \sigma\tau_2|\!)_v$

We can conclude as follows:

1. By a), $(c + c_{proj}) - (c' + c_{proj}) \leq \sigma t$
2. By unrolling the definition of b), we get $(m - c, v_1, v_1') \in (\!|\sigma\tau_1|\!)_v$.
   By downward closure (Lemma 4) on this using

$$m - (c + c_{proj}) \leq m - c$$

   we get $(m - (c + c_{proj})), v_1, v_1') \in (\!|\sigma\tau_1|\!)_v$.

Case $\dfrac{\Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t : \tau_1 \qquad \Delta; \Phi \vdash \tau_2 \ \texttt{wf}}{\Delta; \Phi; \Gamma \vdash \textbf{inl}\ e \ominus \textbf{inl}\ e' \lesssim t : \tau_1 + \tau_2}$ **r-inl**

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS: $(m, \mathrm{inl}\ (\delta e), \mathrm{inl}\ (\delta' e')) \in (\!|\sigma\tau_1 + \sigma\tau_2|\!)_{\varepsilon}^{\sigma t}$.

Following the definition of $(\!|\cdot|\!)_{\varepsilon}^{\cdot}$, assume that

$\dfrac{\delta e \Downarrow^{c} v \quad (\star)}{\mathrm{inl}\ \delta e \Downarrow^{c} \mathrm{inl}\ v}$ **inl** and $\dfrac{\delta' e' \Downarrow^{c'} v' \quad (\star\star)}{\mathrm{inl}\ \delta' e' \Downarrow^{c'} \mathrm{inl}\ v'}$ **inl** and $c < m$.

By IH 1 on the first premise, we get $(m, \delta e, \delta' e') \in (\!|\sigma\tau_1|\!)_{\varepsilon}^{\sigma t}$. Unrolling its definition with $(\star)$ and $(\star\star)$ and $c < m$, we get

a) $c - c' \leq \sigma t$

b) $(m - c, v, v') \in (\!|\sigma\tau_1|\!)_v$

We can conclude as follows:

1. By a), $c - c' \leq \sigma t$
2. Using b), we can show that $(m - c, \mathrm{inl}\ v, \mathrm{inl}\ v') \in (\!|\sigma\tau_1 + \sigma\tau_2|\!)_v$

$$\textbf{Case}\ \frac{\Delta;\Phi;\Gamma \vdash e \ominus e' \lesssim t : \tau_1 + \tau_2 \qquad \Delta;\Phi;x:\tau_1,\Gamma \vdash e_1 \ominus e_1' \lesssim t' : \tau \qquad \Delta;\Phi;y:\tau_2,\Gamma \vdash e_2 \ominus e_2' \lesssim t' : \tau}{\Delta;\Phi;\Gamma \vdash\ \textbf{case}\ (e, x.e_1, y.e_2) \ominus\ \textbf{case}\ (e', x.e_1', y.e_2') \lesssim t + t' : \tau}\ \textbf{r-case}$$

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS: $(m,\ \text{case}\ (\delta e, \delta e_1, \delta e_2),\ \text{case}\ (\delta' e', \delta' e_1', \delta' e_2')) \in (\!|\sigma\tau|\!)_\varepsilon^{\sigma t + \sigma t'}$.

Following the definition of $(\!|\cdot|\!)_\varepsilon^{\cdot}$, assume that

$\text{case}\ (\delta e, \delta e_1, \delta e_2) \Downarrow^{C} v_r$ and $\text{case}\ (\delta' e', \delta' e_1', \delta' e_2') \Downarrow^{C'} v_r'$ and $C < m$.

Depending on what δe and $\delta' e'$ evaluate to, there are two cases:

subcase 1: $\dfrac{\delta e \Downarrow^{c} \text{inl}\ v \quad (\star) \qquad \delta e_1[v/x] \Downarrow^{c_r} v_r \quad (\diamond)}{\text{case}\ (\delta e, x.\delta e_1, y.\delta e_2) \Downarrow^{c + c_r + c_{case}} v_r}$ **case-inl** and

$$\frac{\delta' e' \Downarrow^{c'} \text{inl}\ v' \quad (\star\star) \qquad \delta' e_1'[v'/x] \Downarrow^{c_r'} v_r' \quad (\diamond\diamond)}{\text{case}\ (\delta' e, x.\delta' e_1, y.\delta' e_2) \Downarrow^{c' + c_r' + c_{case}} v_r'}\ \textbf{case-inl}$$

Note that $C = c + c_r + c_{case} < m$.

By IH 1 on the first premise, we get $(m, \delta e, \delta' e') \in (\!|\sigma\tau_1 + \sigma\tau_2|\!)_\varepsilon^{\sigma t}$. Unrolling its definition with $(\star)$ and $(\star\star)$ and $c < m$, we get

a) $c - c' \leq \sigma t$

b) $(m - c, \text{inl}\ v, \text{inl}\ v') \in (\!|\sigma\tau_1 + \sigma\tau_2|\!)_v$

By IH 1 on the second premise using $(m - c, \delta[x \mapsto v], \delta'[x \mapsto v']) \in \mathcal{G}(\!|\sigma\Gamma, x : \sigma\tau_1|\!)$ obtained by

- $(m - c, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ by downward-closure (Lemma 4) on $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ using $m - c \leq m$
- $(m - c, v, v') \in (\!|\sigma\tau_1|\!)_v$ by unfolding b)

we get $(m - c, \delta e_1[v/x], \delta' e_1'[v'/x]) \in (\!|\sigma\tau|\!)_\varepsilon^{\sigma t'}$. Unrolling its definition with $(\diamond)$ and $(\diamond\diamond)$, and $c_r < m - c$, we get

c) $c_r - c_r' \leq \sigma t'$

d) $(m - (c + c_r), v_r, v_r') \in (\!|\sigma\tau|\!)_v$

Now, we can conclude this subcase by

1. By a) and c) $(c + c_r + c_{case}) - (c' + c_r' + c_{case}) \leq \sigma t + \sigma t'$
2. By downward closure (Lemma 4) on d) using

$$m - (c + c_r + c_{case}) \leq m - (c + c_r, c' + c_r')$$

we obtain $(m-(c+c_r+c_{case}), v_r, v'_r) \in (\!|\sigma\tau|\!)_v$.

subcase 2: $\dfrac{\delta e \Downarrow^{c} \text{inr } v \quad (\star) \qquad \delta e_2[v/y] \Downarrow^{c_r} v_r \quad (\diamond)}{\text{case } (\delta e, x.\delta e_1, y.\delta e_2) \Downarrow^{c+c_r+c_{case}} v_r}$ **case-inr** and

$\dfrac{\delta' e' \Downarrow^{c'} \text{inr } v' \quad (\star\star) \qquad \delta' e'_2[v'/y] \Downarrow^{c'_r} v'_r \quad (\diamond\diamond)}{\text{case } (\delta' e, x.\delta' e_1, y.\delta' e_2) \Downarrow^{c'+c'_r+c_{case}} v'_r}$ **case-inr**

This case is symmetric, hence we skip its proof.

Case $\dfrac{i :: S, \Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t : \tau \qquad i \notin \mathbf{FIV}(\Phi;\Gamma)}{\Delta; \Phi; \Gamma \vdash \Lambda e \ominus \Lambda e' \lesssim 0 : \forall i \overset{\mathbf{diff}(t)}{::} S.\,\tau}$ **r-iLam**

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS: $(m, \Lambda\delta e, \Lambda\delta' e') \in (\!|\forall i \overset{\text{diff}(\sigma t)}{::} S.\,\sigma\tau|\!)^{0}_{\varepsilon}$.

By Lemma 2, STS: $(m, \Lambda\delta e, \Lambda\delta' e') \in (\!|\forall i \overset{\text{diff}(\sigma t)}{::} S.\,\sigma\tau|\!)_v$.

By unrolling its definition, assume that $\vdash I :: S$.

There are two cases to show:

subcase 1: STS: $(m, \delta e, \delta' e') \in (\!|\sigma\tau\{I/i\}|\!)^{\sigma t[I/i]}_{\varepsilon}$.

This follows by IH 1 on the premise instantiated with the substitution $\sigma[i \mapsto I] \in \mathcal{D}[\![i :: S, \Delta]\!]$.

subcase 2: STS: $\forall j.(j, \delta e) \in [\![|\sigma\tau\{I/i\}|_1]\!]^{0,\infty}_{\varepsilon} \ \wedge\ (j, \delta' e') \in [\![|\sigma\tau\{I/i\}|_2]\!]^{0,\infty}_{\varepsilon}$.

Pick j.

subsubcase 1: STS1: $(j, \delta e) \in [\![|\sigma\tau\{I/i\}|_1]\!]^{0,\infty}_{\varepsilon}$

Follows by IH 3 on the premise using

- $\text{FV}(e) \subseteq \text{dom}(\Gamma)$ using Lemma 8 on the first premise
- $\sigma[i \mapsto I] \in \mathcal{D}[\![i :: S, \Delta]\!]$
- $(j, \delta) \in \mathcal{G}[\![|\sigma[i \mapsto I]\Gamma|_1]\!] \equiv \mathcal{G}[\![|\sigma\Gamma|_1]\!]$ by Lemma 3 on the main assumption (note that $i \notin \text{FV}(\Gamma;\Phi)$)

subsubcase 2: STS2: $(j, \delta' e') \in [\![|\sigma\tau\{I/i\}|_2]\!]^{0,\infty}_{\varepsilon}$

Follows by IH 3 on the premise using

- $\text{FV}(e') \subseteq \text{dom}(\Gamma)$ using Lemma 8 on the first premise
- $\sigma[i \mapsto I] \in \mathcal{D}[\![i :: S, \Delta]\!]$
- $(j, \delta') \in \mathcal{G}[\![|\sigma[i \mapsto I]\Gamma|_2]\!] \equiv \mathcal{G}[\![|\sigma\Gamma|_2]\!]$ by Lemma 3 on the main assumption (note that $i \notin \text{FV}(\Gamma;\Phi)$)

Case $\dfrac{\Delta;\Phi;\Gamma \vdash e \ominus e' \lesssim t : \forall i \overset{\textbf{diff}(t')}{::} S.\,\tau \qquad \Delta \vdash I : S}{\Delta;\Phi;\Gamma \vdash e[\,] \ominus e'[\,] \lesssim t + t'[I/i] : \tau\{I/i\}}$ **r-iApp**

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS: $(m, \delta e[\,], \delta' e'[\,]) \in (\!|\sigma\tau\{\sigma I/i\}|\!)_{\varepsilon}^{\sigma t + \sigma t'[\sigma I/i]}$.

Following the definition of $(\!|\cdot|\!)_{\varepsilon}^{\cdot}$, assume that

$\dfrac{\delta e \Downarrow^{c} \Lambda e_b \ \ (\star) \qquad e_b \Downarrow^{c_r} v_r \ \ (\diamond)}{\delta e[\,] \Downarrow^{c+c_r} v_r}$ **iApp** and $\dfrac{\delta' e' \Downarrow^{c'} \Lambda e'_b \ \ (\star\star) \qquad e'_b \Downarrow^{c'_r} v'_r \ \ (\diamond\diamond)}{\delta' e[\,] \Downarrow^{c'+c'_r} v'_r}$ **iApp** and

$(c + c_r) < m$.

By IH on the first premise, we get $(m, \delta e, \delta' e') \in (\!|\forall i \overset{\text{diff}(\sigma t')}{::} S.\,\sigma\tau|\!)_{\varepsilon}^{\sigma t}$.

By unrolling its definition with $(\star)$, $(\star\star)$ and $c < m$, we get

a) $c - c' \leq \sigma t$

b) $(m - c, \Lambda e_b, \Lambda e'_b) \in (\!|\forall i \overset{\text{diff}(\sigma t')}{::} S.\,\sigma\tau|\!)_{v}$

By Lemma 6 on the second premise using $\sigma \in \mathcal{D}[\![\Delta]\!]$, we get

$$\vdash \sigma I :: S \tag{1}$$

By unrolling the definition of b) with (1), we get

$$(m - c, e_b, e'_b) \in (\!|\sigma\tau\{\sigma I/i\}|\!)_{\varepsilon}^{\sigma t'[\sigma I/i]} \tag{2}$$

By unrolling the definition of (2) with $(\diamond)$ and $(\diamond\diamond)$ and $c_r < m - c$, we get

c) $c_r - c'_r \leq \sigma t'[\sigma I/i]$

d) $(m - (c + c_r), v_r, v'_r) \in (\!|\sigma\tau\{\sigma I/i\}|\!)_{v}$

We conclude as follows

1. By a) and c), we get $(c + c_r) - (c' + c'_r) \leq \sigma t + \sigma t'[\sigma I/i]$
2. By d)

Case $\dfrac{\Delta;\Phi;\Gamma \vdash e \ominus e' \lesssim t : \tau\{I/i\} \qquad \Delta \vdash I :: S}{\Delta;\Phi;\Gamma \vdash \textbf{pack}\ e \ominus \textbf{pack}\ e' \lesssim t : \exists i{::}S.\,\tau}$ **r-pack**

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS: $(m, \text{pack}\ \delta e, \text{pack}\ \delta' e') \in (\!|\exists i{::}S.\,\sigma\tau|\!)_{\varepsilon}^{\sigma t}$.

Following the definition of $(\!|\cdot|\!)_{\varepsilon}^{\cdot}$, assume that

$$\dfrac{\delta e \Downarrow^{c} v \quad (\star)}{\text{pack } \delta e \Downarrow^{c} \text{pack } v}\ \textbf{pack} \text{ and } \dfrac{\delta' e' \Downarrow^{c'} v' \quad (\star\star)}{\text{pack } \delta' e' \Downarrow^{c'} \text{pack } v'}\ \textbf{pack} \text{ and}$$

$c < m$.

By IH on the first premise, we get $(m, \delta e, \delta' e') \in (\!|\sigma\tau\{\sigma I/i\}|\!)_{\varepsilon}^{\sigma t}$.

By unrolling its definition with $(\star)$, $(\star\star)$ and $c < m$, we get

a) $c - c' \leq \sigma t$

b) $(m - c, v, v') \in (\!|\sigma\tau\{\sigma I/i\}|\!)_{v}$

By Lemma 6 on the second premise, we get

$$\vdash \sigma I :: S \tag{1}$$

We can conclude as follows

1. By a)
2. TS: $(m - c, \text{pack } e, \text{pack } v') \in (\!|\exists i{::}S.\, \sigma\tau|\!)_{v}$
   STS1: $\vdash \sigma I :: S$ follows directly by (1).
   STS2: $(m - c, v, v') \in (\!|\sigma\tau\{\sigma I/i\}|\!)_{v}$ follows by b)

Case $$\dfrac{\begin{array}{c}\Delta; \Phi; \Gamma \vdash e_1 \ominus e_1' \lesssim t_1 : \exists i{::}S.\, \tau_1 \\ i :: S, \Delta; \Phi; x : \tau_1, \Gamma \vdash e_2 \ominus e_2' \lesssim t_2 : \tau_2 \qquad i \notin FV(\Phi; \Gamma, \tau_2, t_2)\end{array}}{\Delta; \Phi; \Gamma \vdash \textbf{unpack } e_1 \textbf{ as } x \textbf{ in } e_2 \ominus \textbf{unpack } e_1' \textbf{ as } x \textbf{ in } e_2' \lesssim t_1 + t_2 : \tau_2}\ \textbf{r-unpack1}$$

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS: $(m, \text{unpack } \delta e_1 \text{ as } x \text{ in } \delta e_2, \text{unpack } \delta' e_1' \text{ as } x \text{ in } \delta' e_2') \in (\!|\sigma\tau_2|\!)_{\varepsilon}^{\sigma t_1 + \sigma t_2}$.

Following the definition of $(\!|\cdot|\!)_{\varepsilon}^{\cdot}$, assume that

$$\dfrac{\delta e_1 \Downarrow^{c_1} \text{pack } v \quad (\star) \qquad \delta e_2[v/x] \Downarrow^{c_2} v_r \quad (\diamond)}{\text{unpack } \delta e_1 \text{ as } x \text{ in } \delta e_2 \Downarrow^{c_1 + c_2} v_r}\ \textbf{unpack} \text{ and}$$

$$\dfrac{\delta' e_1' \Downarrow^{c_1'} \text{pack } v' \quad (\star\star) \qquad \delta' e_2'[v'/x] \Downarrow^{c_2'} v_r' \quad (\diamond\diamond)}{\text{unpack } \delta' e_1' \text{ as } x \text{ in } \delta' e_2' \Downarrow^{c_1' + c_2'} v_r'}\ \textbf{unpack} \text{ and}$$

$(c_1 + c_2) < m$.

By IH 1 on the first premise, we get $(m, \delta e_1, \delta' e_1') \in (\!|\exists i{::}S.\, \sigma\tau_1|\!)_{\varepsilon}^{\sigma t_1}$.

By unrolling its definition with $(\star)$, $(\star\star)$ and $c_1 < m$, we get

a) $c_1 - c_1' \leq \sigma t_1$

b) $(m - c_1, \text{pack } v, \text{pack } v') \in (\!|\exists i{::}S.\, \sigma\tau_1|\!)_{v}$

By unrolling the definition of b), we get

$$\vdash I :: S \tag{1}$$

$$(m - c_1, v, v') \in (\!|\sigma\tau_1\{I/i\}|\!)_v \tag{2}$$

By downward closure (Lemma 4) on $(m, \delta, \delta') \in \mathcal{G}(\!|\Gamma|\!)$, we have

$$(m - c_1, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!) \tag{3}$$

By IH 1 on the second premise using

- $\sigma[i \mapsto I] \in \mathcal{D}[\![i :: S, \Delta]\!]$ using (1)
- $(m - c_1, \delta[x \mapsto v], \delta'[x \mapsto v']) \in \mathcal{G}(\!|\sigma[i \mapsto I](\Gamma, x : \tau_1)|\!)$ using (2) and (3)

we get

$$(m - c_1, \delta e_2[v/x], \delta' e_2'[v'/x]) \in (\!|\sigma\tau_2|\!)_\varepsilon^{\sigma t_2} \tag{4}$$

By unrolling (4)'s definition using ($\diamond$), ($\diamond\diamond$) and $c_2 < m - c_1$, we get

c) $c_2 - c_2' \leq \sigma t_2$

d) $(m - (c_1 + c_2), v_r, v_r') \in (\!|\sigma\tau_2|\!)_v$

We can conclude as follows

1. By a) and c), we get $(c_1 + c_2) - (c_1' + c_2') \leq \sigma t_1 + \sigma t_2$
2. Follows by d)

Case $$\dfrac{\Delta; \Phi; \Gamma \vdash e_1 \ominus e_1' \lesssim t_1 : \tau_1 \qquad \Delta; \Phi; x : \tau_1, \Gamma \vdash e_2 \ominus e_2' \lesssim t_2 : \tau_2}{\Delta; \Phi; \Gamma \vdash \textbf{let } x = e_1 \textbf{ in } e_2 \ominus \textbf{let } x = e_1' \textbf{ in } e_2' \lesssim t_1 + t_2 : \tau_2}\ \textbf{r-let1}$$

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS: $(m, \text{let } x = \delta e_1 \text{ in } \delta e_2, \text{let } x = \delta' e_1' \text{ in } \delta' e_2') \in (\!|\sigma\tau_2|\!)_\varepsilon^{\sigma t_1 + \sigma t_2}$.

Following the definition of $(\!|\cdot|\!)_\varepsilon^{\cdot}$, assume that

$$\dfrac{\delta e_1 \Downarrow^{c_1} v_1 \quad (\diamond) \qquad \delta e_2[v_1/x] \Downarrow^{c_r} v_r \quad (\dagger)}{\text{let } x = \delta e_1 \text{ in } \delta e_2 \Downarrow^{c_1 + c_r + c_{let}} v_r}\ \textbf{let} \text{ and}$$

$$\dfrac{\delta' e_1' \Downarrow^{c_1'} v_1' \quad (\diamond\diamond) \qquad \delta' e_2'[v_1'/x] \Downarrow^{c_r'} v_r' \quad (\dagger\dagger)}{\text{let } x = \delta' e_1 \text{ in } \delta' e_2 \Downarrow^{c_1' + c_r' + c_{let}} v_r'}\ \textbf{let} \text{ and } (c_1 + c_r + c_{let}) < m.$$

By IH 1 on the first premise, we get $(m, \delta e_1, \delta' e_1') \in (\!|\sigma\tau_1|\!)_\varepsilon^{\sigma t_1}$. Unrolling its definition with ($\diamond$) and ($\diamond\diamond$) and $c_1 < m$, we get

a) $c_1 - c'_1 \leq \sigma t_1$

b) $(m - c_1, v_1, v'_1) \in (\!|\sigma\tau_1|\!)_v$

By IH 1 on the second premise using $(m - c_1, \delta[x \mapsto v_1], \delta'[x \mapsto v'_1]) \in \mathcal{G}(\!|\sigma\Gamma, x : \sigma\tau_1|\!)$ obtained by

- $(m - c_1, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ by downward closure (Lemma 4) on $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ using $m - c_1 \leq m$
- $(m - c_1, v, v') \in (\!|\sigma\tau_1|\!)_v$ by b)

we get $(m - c_1, \delta e_2[v_1/x], \delta' e'_2[v'_1/x]) \in (\!|\sigma\tau_2|\!)_\varepsilon^{\sigma t_2}$. Unrolling its definition with (†) and (††), and $c_r < m - c_1$, we get

c) $c_r - c'_r \leq \sigma t_2$

d) $(m - (c_1 + c_r), v_r, v'_r) \in (\!|\sigma\tau_2|\!)_v$

Now, we can conclude with

1. By a) and c) $(c_1 + c_r + c_{let}) - (c'_1 + c'_r + c_{let}) \leq \sigma t_1 + \sigma t_2$
2. By downward closure (Lemma 4) on d) using

$$m - (c_1 + c_r + c_{let}) \leq m - (c_1 + c_r)$$

   we obtain $(m - (c_1 + c_r + c_{let}), v_r, v'_r) \in (\!|\sigma\tau_2|\!)_v$.

Case $\dfrac{\Delta; \Phi; |\Gamma|_1 \vdash_{k_1}^{t_1} e_1 : A_1 \qquad \Delta; \Phi; |\Gamma|_2 \vdash_{k_2}^{t_2} e_2 : A_2}{\Delta; \Phi; \Gamma \vdash e_1 \ominus e_2 \lesssim t_1 - k_2 : U\,(A_1, A_2)}$ **switch**

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.
TS: $(m, \delta e_1, \delta' e_2) \in (\!|U\,(\sigma A_1, \sigma A_2)|\!)_\varepsilon^{\sigma t_1 - \sigma k_2}$.
Assume that

a) $\delta e_1 \Downarrow^{c_1} v_1$

b) $\delta' e_2 \Downarrow^{c_2} v_2$

c) $c_1 < m$

TS 1: $c_1 - c_2 \leq \sigma t_1 - \sigma k_2$
TS 2: $(m - c_1, v_1, v_2) \in (\!|U\,(\sigma A_1, \sigma A_2)|\!)_v$
We first show the second statement, the first one will be shown later.
By unrolling $(\!|U\,(\sigma A_1, \sigma A_2)|\!)_v$'s definition,

STS: $\forall m.(m, v_1) \in [\![\sigma A_1]\!]_v \;\wedge\; (m, v_2) \in [\![\sigma A_2]\!]_v$.
Pick m.

By IH 2 on the first premise using

- $\mathrm{FV}(e_1) \subseteq \mathrm{dom}(|\sigma\Gamma|_1)$ using Lemma 8 on the first premise
- $\models \sigma\Phi$
- $\forall j.(j, \delta) \in \mathcal{G}[\![|\sigma\Gamma|_1]\!]$ obtained by Lemma 3 on $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$

we get

$$\forall j.(j, \delta e_1) \in [\![\sigma A_1]\!]_\varepsilon^{\sigma k_1, \sigma t_1} \tag{1}$$

By IH 2 on the second premise using

- $\mathrm{FV}(e_2) \subseteq \mathrm{dom}(|\sigma\Gamma|_2)$ using Lemma 8 on the second premise
- $\models \sigma\Phi$
- $\forall j.(j, \delta') \in \mathcal{G}[\![|\sigma\Gamma|_2]\!]$ obtained by Lemma 3 on $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$

we get

$$\forall j.(j, \delta' e_2) \in [\![\sigma A_2]\!]_\varepsilon^{\sigma k_2, \sigma t_2} \tag{2}$$

We instantiate j with $m + \sigma t_1 + 1$ in (1) and we get

$$(m + \sigma t_1 + 1, \delta e_1) \in [\![\sigma A_1]\!]_\varepsilon^{\sigma k_1, \sigma t_1} \tag{3}$$

We instantiate j with $m + c_2 + 1$ in (2) and we get

$$(m + c_2 + 1, \delta' e_2) \in [\![\sigma A_2]\!]_\varepsilon^{\sigma k_2, \sigma t_2} \tag{4}$$

Next, unrolling first part of (3) using $\sigma t_1 < m + \sigma t_1 + 1$, we get

d) $\delta e_1 \Downarrow^{c_1} v_1$
e) $c_1 \leq \sigma t_1$
f) $(m + \sigma t_1 - c_1 + 1, v_1) \in [\![\sigma A_1]\!]_v$

Next, unrolling second part of (4) using b) and $c_2 < m + c_2 + 1$, we get

g) $\sigma k_2 \leq c_2$
h) $(m + 1, v_2) \in [\![\sigma A_2]\!]_v$

Now, we can conclude as follows:

1. By e) and g), we get $c_1 - c_2 \leq \sigma t_1 - \sigma k_2$
2. By downward closure (Lemma 4) on f) using

$$m \leq m + \sigma t_1 - c_1 + 1 \quad \text{(note that by e), } c_1 \leq \sigma t_1\text{)}$$

   we get $(m, v_1) \in [\![\sigma A_1]\!]_v$.
   By downward closure (Lemma 4) on h) using

$$m \leq m + 1$$

   we get $(m, v_2) \in [\![\sigma A_2]\!]_v$.

Case $\dfrac{\Delta; \Phi \models C \qquad \Delta; \Phi \wedge C; \Gamma \vdash e \ominus e' \lesssim t : \tau}{\Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t : C \,\&\, \tau}$ **c-andI**

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.
TS: $(m, \delta e, \delta' e') \in (\!|\sigma C \,\&\, \sigma\tau|\!)_{\varepsilon}^{\sigma t}$.
Following the definition of $(\!|\cdot|\!)_{\varepsilon}^{\cdot}$, assume that

a) $\delta e \Downarrow^{c} v$
b) $\delta' e' \Downarrow^{c'} v'$
c) $c < m$.

By IH 1 on the first premise using

- $\models \sigma(C \wedge \Phi)$ hold by the main assumption $\models \sigma\Phi$ and $\models \sigma C$ $(\star)$ obtained by Lemma 6 using the premise $\Delta; \Phi \models C$

we get $(m, \delta e, \delta' e') \in (\!|\sigma\tau|\!)_{\varepsilon}^{\sigma t}$. Unrolling its definition with (a-c), we get

d) $c - c' \leq \sigma t$
e) $(m - c, v, v') \in (\!|\sigma\tau|\!)_v$

We can conclude as follows:

1. By d), $c - c' \leq \sigma t$
2. Using e) and $(\star)$, we can show that $(m - c, v, v') \in (\!|\sigma C \,\&\, \sigma\tau|\!)_v$

Case $\dfrac{\Delta; \Phi; \Gamma \vdash e_1 \ominus e_1' \lesssim t_1 : C \,\&\, \tau_1 \qquad \Delta; \Phi \wedge C; x : \tau_1, \Gamma \vdash e_2 \ominus e_2' \lesssim t_2 : \tau_2}{\Delta; \Phi; \Gamma \vdash \mathbf{clet}\ e_1\ \mathbf{as}\ x\ \mathbf{in}\ e_2 \ominus \mathbf{clet}\ e_1'\ \mathbf{as}\ x\ \mathbf{in}\ e_2' \lesssim t_1 + t_2 : \tau_2}$ **r-c-andE**

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS: $(m, \text{clet } \delta e_1 \text{ as } x \text{ in } \delta e_2, \text{clet } \delta' e_1' \text{ as } x \text{ in } \delta' e_2') \in (\!|\sigma\tau_2|\!)_\varepsilon^{\sigma t_1 + \sigma t_2}$.

Following the definition of $(\!|\cdot|\!)_\varepsilon^{\cdot}$, assume that

$\dfrac{\delta e_1 \Downarrow^{c_1} v_1 \quad (\diamond) \qquad \delta e_2[v_1/x] \Downarrow^{c_r} v_r \quad (\dagger)}{\text{clet } \delta e_1 \text{ as } x \text{ in } \delta e_2 \Downarrow^{c_1 + c_r} v_r}$ **clet** and

$\dfrac{\delta' e_1' \Downarrow^{c_1'} v_1' \quad (\diamond\diamond) \qquad \delta' e_2'[v_1'/x] \Downarrow^{c_r'} v_r' \quad (\dagger\dagger)}{\text{clet } \delta' e_1 \text{ as } x \text{ in } \delta' e_2 \Downarrow^{c_1' + c_r'} v_r'}$ **clet** and $(c_1 + c_r) < m$.

By IH 1 on the first premise, we get $(m, \delta e_1, \delta' e_1') \in (\!|\sigma C \,\&\, \sigma\tau_1|\!)_\varepsilon^{\sigma t_1}$. Unrolling its definition with $(\diamond)$ and $(\diamond\diamond)$ and $c_1 < m$, we get

a) $c_1 - c_1' \leq \sigma t_1$

b) $(m - c_1, v_1, v_1') \in (\!|\sigma C \,\&\, \sigma\tau_1|\!)_v$

By IH 1 on the second premise using $(m - c_1, \delta[x \mapsto v_1], \delta'[x \mapsto v_1']) \in \mathcal{G}(\!|\sigma\Gamma, x : \sigma\tau_1|\!)$ obtained by

- $\models \sigma(C \wedge \Phi)$ hold by the main assumption $\models \sigma\Phi$ and $\models \sigma C$ obtained by unrolling the definition of b)
- $(m - c_1, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ by downward closure (Lemma 4) on $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ using $m - c_1 \leq m$
- $(m - c_1, v_1, v_1') \in (\!|\sigma\tau_1|\!)_v$ by unrolling the definition of b)

we get $(m - c_1, \delta e_2[v_1/x], \delta' e_2'[v_1'/x]) \in (\!|\sigma\tau_2|\!)_\varepsilon^{\sigma t_2}$. Unrolling its definition with $(\dagger)$ and $(\dagger\dagger)$, and $c_r < m - c_1$, we get

c) $c_r - c_r' \leq \sigma t_2$

d) $(m - (c_1 + c_r), v_r, v_r') \in (\!|\sigma\tau_2|\!)_v$

Now, we can conclude with

1. By a) and c) $(c_1 + c_r) - (c_1' + c_r') \leq \sigma t_1 + \sigma t_2$
2. By downward closure (Lemma 4) on d) using

$$m - (c_1 + c_r) \leq m - (c_1 + c_r)$$

   we obtain $(m - (c_1 + c_r), v_r, v_r') \in (\!|\sigma\tau_2|\!)_v$.

Case $\dfrac{\Delta; \Phi \wedge C; \Gamma \vdash e \ominus e' \lesssim t : \tau}{\Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t : C \supset \tau}$ **r-c-impI**

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS: $(m, \delta e, \delta' e') \in (\!|\sigma C \;\&\; \sigma\tau|\!)_{\varepsilon}^{\sigma t}$.

Following the definition of $(\!|\cdot|\!)_{\varepsilon}^{\cdot}$, assume that

a) $\delta e \Downarrow^{c} v$

b) $\delta' e' \Downarrow^{c'} v'$

c) $c < m$.

We first show the second statement.

TS2: $(m - c, v, v') \in (\!|\sigma C \supset \sigma\tau|\!)_{v}$

Assume that $\models \sigma C$ $(\star)$.

STS: $(m - c, v, v') \in (\!|\sigma\ \tau|\!)_{v}$

By IH 1 on the first premise using

- $\models \sigma(C \wedge \Phi)$ hold by the main assumption $\models \sigma\Phi$ and $\models \sigma C$ (by $\star$)

we get $(m, \delta e, \delta' e') \in (\!|\sigma\tau|\!)_{\varepsilon}^{\sigma t}$. Unrolling its definition with (a-c), we get

d) $c - c' \leq \sigma t$

e) $(m - c, v, v') \in (\!|\sigma\tau|\!)_{v}$

We can conclude as follows:

1. By d), $c - c' \leq \sigma t$
2. Using e) , we can show that $(m - c, v, v') \in (\!|\sigma C \supset \sigma\tau|\!)_{v}$

Case $\dfrac{\Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t : C \supset \tau \qquad \Delta; \Phi \models C}{\Delta; \Phi; \Gamma \vdash \textbf{celim}\ e \ominus \textbf{celim}\ e' \lesssim t : \tau}$ **r-c-implE**

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS: $(m, \text{celim}\ \delta e, \text{celim}\ \delta' e') \in (\!|\sigma\tau|\!)_{\varepsilon}^{\sigma t}$. Following the definition of $(\!|\cdot|\!)_{\varepsilon}^{\cdot}$, assume that

$\dfrac{\delta e \Downarrow^{c} v \quad (\diamond)}{\text{celim}\ \delta e \Downarrow^{c} v}$ **celim** and $\dfrac{\delta' e \Downarrow^{c} v \quad (\diamond\diamond)}{\text{celim}\ \delta' e \Downarrow^{c} v}$ **celim** and $c < m$ $(\star)$.

By IH 1 on the first premise, we get $(m, \delta e, \delta' e') \in (\!|\sigma C \supset \sigma\tau|\!)_{\varepsilon}^{\sigma t}$. Unrolling its definition using $(\diamond)$, $(\diamond\diamond)$ and $(\star)$, we get

a) $c - c' \leq \sigma t$

b) $(m - c, v, v') \in (\!|\sigma C \supset \sigma\tau|\!)_{v}$

We can conclude as follows:

1. By a), $c - c' \leq \sigma t$
2. Using b) and $\models \sigma C$ (obtained by Lemma 6 on the second premise) , we can show that $(m - c, v, v') \in (\!|\sigma\tau|\!)_v$

Case $\dfrac{\Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t : \tau_1 \qquad \Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t : \tau_2}{\Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t : \tau_1 \wedge \tau_2}$ **r-interI**

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

STS: $(m, \delta e, \delta' e') \in (\!|\sigma\tau_1 \wedge \sigma\tau_2|\!)_\varepsilon^{\sigma t}$.

Assume that

a) $\delta e \Downarrow^c v$
b) $\delta' e' \Downarrow^{c'} v'$
c) $c < m$

By IH 1 on the first premise using (a-c), we get $(m, \delta e, \delta' e') \in (\!|\sigma\tau_1|\!)_\varepsilon^{\sigma t}$.

By unrolling its definition, we get

d) $c - c' \leq \sigma t$
e) $(m - c, v, v') \in (\!|\sigma\tau_1|\!)_v$

By IH 1 on the second premise using (a-c), we get $(m, \delta e, \delta' e') \in (\!|\sigma\tau_2|\!)_\varepsilon^{\sigma t}$.

By unrolling its definition, we get

f) $c - c' \leq \sigma t$
g) $(m - c, v, v') \in (\!|\sigma\tau_2|\!)_v$

We can conclude as follows

1. By d) or f), we get $c - c' \leq \sigma t$
2. TS: $(m - c, v, v') \in (\!|\sigma\tau_1 \wedge \sigma\tau_2|\!)_v$.
   Directly follows by e) and g).

Case $\dfrac{\Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t : \tau_1 \wedge \tau_2}{\Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t : \tau_1}$ **r-interE$_1$**

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

STS: $(m, \delta e, \delta' e') \in (\!|\sigma\tau_1|\!)_\varepsilon^{\sigma t}$.

Assume that

a) $\delta e \Downarrow^{c} v$

b) $\delta' e' \Downarrow^{c'} v'$

c) $c < m$

By IH 1 on the first premise using (a-c), we get $(m, \delta e, \delta' e') \in (\!|\sigma\tau_1 \wedge \sigma\tau_2|\!)_{\varepsilon}^{\sigma t}$. By unrolling its definition, we get

d) $c - c' \leq \sigma t$

e) $(m - c, v, v') \in (\!|\sigma\tau_1 \wedge \sigma\tau_2|\!)_{v}$

We can conclude as follows

1. By d), we get $c - c' \leq \sigma t$
2. TS: $(m - c, v, v') \in (\!|\sigma\tau_1|\!)_{v}$.
   Directly follows by unrolling e).

Case $\dfrac{\Upsilon(\zeta) = \tau_1 \xrightarrow{\mathbf{diff}(t)} \tau_2 \qquad \Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t' : \tau_1}{\Delta; \Phi; \Gamma \vdash \zeta\ e \ominus \zeta\ e' \lesssim t + t' : \tau_2}$ **r-primapp**

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.
TS: $(m, \zeta\ \delta e, \zeta\ \delta' e') \in (\!|\sigma\tau_2|\!)_{\varepsilon}^{\sigma t + \sigma t'}$.
Following the definition of $(\!|\cdot|\!)_{\varepsilon}^{\cdot}$, assume that
$\dfrac{\delta e \Downarrow^{c} v \quad (\star) \qquad \hat{\zeta}(v) = (c_r, v_r) \quad (\diamond)}{\zeta\ \delta e \Downarrow^{c + c_r + c_{app}} v_r}$ **primapp** and
$\dfrac{\delta' e' \Downarrow^{c'} v' \quad (\star\star) \qquad \hat{\zeta}(v)' = (c'_r, v'_r) \quad (\diamond\diamond)}{\zeta\ \delta' e' \Downarrow^{c' + c'_r + c_{app}} v'_r}$ **primapp** and
$(c + c_r + c_{app}) < m$.
By IH 1 on the second premise, we get $(m, \delta e, \delta' e') \in (\!|\sigma\tau_1|\!)_{\varepsilon}^{\sigma t'}$. Unrolling its definition with $(\star)$ and $(\star\star)$, and $c < m$, we get

a) $c - c' \leq \sigma t'$

b) $(m - c, v, v') \in (\!|\sigma\tau_1|\!)_{v}$

Next, by Assumption (11) using $\zeta : \sigma\tau_1 \xrightarrow{\mathrm{diff}(\sigma t)} \sigma\tau_2$ (obtained by substitution on the first premise), b), $(\star)$ and $(\star\star)$, we get

c) $c_r - c'_r \leq \sigma t$

d) $(m - c, v_r, v'_r) \in (\!|\sigma\tau_2|\!)_{v}$

Now, we can conclude as follows:

1. Using a) and c), we get $(c + c_r + c_{app}) - (c' + c'_r + c_{app}) \leq \sigma t + \sigma t'$
2. By downward closure (Lemma 4) on d) using

$$m - (c + c_r + c_{app}) \leq m - c$$

we get $(m - (c + c_r + c_{app}), v_r, v'_r) \in (\!|\sigma\tau_2|\!)_v$

Case $\dfrac{\begin{array}{c}\Delta; \Phi; \Gamma \vdash e \ominus e \lesssim t : \tau \\ \forall x \in dom(\Gamma).\quad \Delta; \Phi \models \Gamma(x) \sqsubseteq \square\, \Gamma(x)\end{array}}{\Delta; \Phi; \Gamma, \Gamma'; \Omega \vdash e \ominus e \lesssim 0 : \square\, \tau}$ **nochange**

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma, \sigma\Gamma'|\!)$ and $\models \sigma\Phi$.
Then, $\delta = \delta_1 \cup \delta_2$ and $\delta' = \delta'_1 \cup \delta'_2$ such that $(m, \delta_1, \delta'_1) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $(m, \delta_2, \delta'_2) \in \mathcal{G}(\!|\sigma\Gamma'|\!)$.
TS: $(m, \delta e, \delta' e) \in (\!|\square\, \sigma\tau|\!)^0_\varepsilon$.
Since e doesn't have any free variables from Γ' by the first premise,
STS: $(m, \delta_1 e, \delta'_1 e) \in (\!|\square\, \sigma\tau|\!)^0_\varepsilon$.
Assume that

a) $\delta_1 e \Downarrow^c v$
b) $\delta'_1 e \Downarrow^{c'} v'$
c) $c < m$

TS 1: $c - c' \leq 0$
TS 2: $(m - c, v, v') \in (\!|\square\, \sigma\tau|\!)_v$
By IH 1 on the first premise using

- $(m, \delta_1, \delta'_1) \in \mathcal{G}(\!|\sigma\Gamma|\!)$
- $\models \sigma\Phi$

we get $(m, \delta_1 e, \delta'_1 e) \in (\!|\sigma\tau|\!)^{\sigma t}_\varepsilon$.
Unfolding its definition with a), b) and c), we get

d) $c - c \leq \sigma t$
e) $(m - c, v, v') \in (\!|\sigma\tau|\!)_v$

We can conclude as follows

1. By Lemma 5 using $(m, \delta_1, \delta'_1) \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and the second premise, we get $(m, \delta_1, \delta_1) \in \mathcal{G}(\!|\square\, \sigma\Gamma|\!)$. This means that $\delta_1 = \delta'_1$.
   Therefore, a) and b) are equal, that is $c = c'$ and $v = v'$. Hence, trivially we get $c - c \leq 0$.

2. Since $v = v'$ and $c = c'$, by using e), we get$(m - c, v, v) \in (\!|\square\, \sigma\tau|\!)_v$.

Case $\dfrac{\Delta; \Phi \wedge C; \Gamma \vdash e_1 \ominus e_2 \lesssim t : \tau \qquad \Delta; \Phi \wedge \neg C; \Gamma \vdash e_1 \ominus e_2 \lesssim t : \tau}{\Delta; \Phi; \Gamma \vdash e_1 \ominus e_2 \lesssim t : \tau}$ **r-split**

Assume that $\models \sigma\Phi$ and $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$.

TS: $(m, \delta e_1, \delta' e_2 \in (\!|\sigma\tau|\!)_\varepsilon^{\sigma k}$.

There are two cases:

subcase 1: $\models \sigma\Phi \wedge C$

Follows immediately by IH on the first premise.

subcase 2: $\models \sigma\Phi \wedge \neg C$

Follows immediately by IH on the second premise.

Case $\dfrac{\Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t : \tau \qquad \Delta; \Phi \models \tau \sqsubseteq \tau' \qquad \Delta; \Phi \models t \leq t'}{\Delta; \Phi; \Gamma \vdash e \ominus e' \lesssim t' : \tau'}$ **r-** $\sqsubseteq$

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS: $(m, \delta e, \delta' e') \in (\!|\sigma\tau'|\!)_\varepsilon^{\sigma t'}$.

Following the definition of $(\!|\cdot|\!)_\varepsilon^{\cdot}$, assume that

a) $\delta e \Downarrow^c v$

b) $\delta' e' \Downarrow^{c'} v'$

c) $c < m$.

By IH 1 on the first premise using (a-c), we get

d) $c - c' \leq \sigma t$

e) $(m - c, v, v') \in (\!|\sigma\tau|\!)_v$

We can conclude as

1. By Assumption (13) on the third premise, we get $\sigma t \leq \sigma t'$. Combining this with d), we get $c - c' \leq \sigma t'$.
2. By Lemma 5 on the second premise with e), we get $(m - c, v, v') \in (\!|\sigma\tau'|\!)_v$

Case $\dfrac{\Delta; \Phi; |\Gamma|_1 \vdash_{k_1}^{t_1} e_1 : A_1 \xrightarrow{\mathbf{exec}(k,t)} A_2 \qquad \Delta; \Phi; \Gamma \vdash e_2 \ominus e_2' \lesssim t_2 : U\,(A_1, A_2')}{\Delta; \Phi; \Gamma \vdash e_1\ e_2 \ominus e_2' \lesssim t_1 + t_2 + t + c_{app} : U\,(A_2, A_2')}$ **r-app-e**

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS: $(m, \delta e_1\ \delta e_2, \delta' e_2') \in (\!|U\,(\sigma A_2, \sigma A_2')|\!)_\varepsilon^{\sigma t_1 + \sigma t_2 + \sigma t + c_{app}}$.

Following the definition of $(\!|\cdot|\!)_{\varepsilon}^{\cdot}$, assume that

$$\frac{\delta e_1 \Downarrow^{c_1} \text{fix } f(x).e \quad (\star) \qquad \delta e_2 \Downarrow^{c_2} v_2 \quad (\diamond) \qquad e[v_2/x, (\text{fix } f(x).e)/f] \Downarrow^{c_r} v_r \quad (\dagger)}{\delta e_1\ \delta e_2 \Downarrow^{c_1+c_2+c_r+c_{app}} v_r}\ \textbf{app}$$ and

$\delta' e_2' \Downarrow^{c'} v'$ $(\diamond\diamond)$ and $c_1 + c_2 + c_r + c_{app} < m$.

TS1:$c_1 + c_2 + c_r + c_{app} - c' \leq \sigma t_1 + \sigma t_2 + \sigma t + c_{app}$

TS2: $(m - (c_1 + c_2 + c_r + c_{app}), v_r, v') \in (\!|U\ (\sigma A_2, \sigma A_2')|\!)_v$

We first show the second statement.

By unrolling the definition of $(\!|U\ (\sigma A_2, \sigma A_2')|\!)_v$,

STS: $\forall j.(j, v_r) \in [\![\sigma A_2]\!]_v \ \wedge \ (j, v') \in [\![\sigma A_2']\!]_v$.

Pick j.

By IH 2 on the first premise using

- $\text{FV}(e_1) \subseteq \text{dom}(|\sigma\Gamma|_2)$ using Lemma 8 on the first premise
- $\forall m.(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma|_1]\!]$ using Lemma 3 on $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$.

we get

$$\forall m.(m, \delta e_1) \in [\![\sigma A_1 \xrightarrow{\text{exec}(\sigma k, \sigma t)} \sigma A_2]\!]_{\varepsilon}^{\sigma k_1, \sigma t_1} \tag{1}$$

Instantiating (1) with $j + \sigma t + \sigma t_1 + 1 + c_{app}$, we get

$$(j + \sigma t + \sigma t_1 + 1 + c_{app}, \delta e_1) \in [\![\sigma A_1 \xrightarrow{\text{exec}(\sigma k, \sigma t)} \sigma A_2]\!]_{\varepsilon}^{\sigma k_1, \sigma t_1} \tag{2}$$

Unfolding the first part of the definition of (2) with $\sigma t_1 < j + \sigma t + \sigma t_1 + 1 + c_{app}$, we get

a) $\delta e_1 \Downarrow^{c_1} \text{fix } f(x).e$

b) $c_1 \leq \sigma t_1$

c) $(j + \sigma t + \sigma t_1 + 1 + c_{app} - c_1, \text{fix } f(x).e) \in [\![\sigma A_1 \xrightarrow{\text{exec}(\sigma k, \sigma t)} \sigma A_2]\!]_v$

By IH 1 on the second premise, we get $(m, \delta e_2, \delta' e_2') \in (\!|U\ (\sigma A_1, \sigma A_2')|\!)_{\varepsilon}^{\sigma t_2}$.

Unrolling its definition with $(\diamond)$ and $(\diamond\diamond)$, and $c_2 < m$, we get

d) $c_2 - c' \leq \sigma t_2$

e) $(m - c_2, v_2, v') \in (\!|U\ (\sigma A_1, \sigma A_2')|\!)_v$

By e), we get $\forall m.(m, v_2) \in [\![\sigma A_1]\!]_v \ \wedge \ (m, v') \in [\![\sigma A_2']\!]_v$.

Instantiating m with $j + \sigma t + c_{app}$, we get

$$(j + \sigma t + c_{app}, v_2) \in [\![\sigma A_1]\!]_v \tag{3}$$

$$(j + \sigma t + c_{app}, v') \in [\![\sigma A_2']\!]_v \tag{4}$$

Unrolling c) with (3) since $j + \sigma t + c_{app} < j + \sigma t + \sigma t_1 - c_1 + 1 + c_{app}$ and $c_1 \leq \sigma t_1$ by (b), we get

$$(j + \sigma t + c_{app}, e[v_2/x, (\text{fix } f(x).e)]) \in [\![\sigma A_2]\!]_{\varepsilon}^{\sigma k, \sigma t} \tag{5}$$

By unrolling first part of (5) with $\sigma t < j + \sigma t + c_{app}$, we get

f) $e[v_2/x, (\text{fix } f(x).e)] \Downarrow^{c_r} v_r$

g) $c_r \leq \sigma t$

h) $(j + c_{app} + \sigma t - c_r, v_r) \in [\![\sigma A_2]\!]_v$

Now, we can conclude as follows:

1. Using b), d) and g), we get $(c_1 + c_2 + c_r + c_{app}) - c' \leq \sigma t_1 + \sigma t_2 + \sigma t + c_{app}$
2. By downward closure (Lemma 4) on h) using

$$j \leq j + c_{app} + \sigma t - c_r \quad \text{since} \quad c_r \leq \sigma t \quad \text{by (g)}$$

we get $(j, v_r) \in [\![\sigma A_2]\!]_v$.
By downward closure (Lemma 4) on (4) using $j \leq j + \sigma t + c_{app}$, we get $(j, v') \in [\![\sigma A_2']\!]_v$.

Case $\dfrac{\Delta; \Phi; |\Gamma|_2 \vdash_{k_1}^{t_1} e_1' : A_1' \xrightarrow{\textbf{exec}(k,t)} A_2' \qquad \Delta; \Phi; \Gamma \vdash e_2 \ominus e_2' \lesssim t_2 : U\,(A_2, A_1')}{\Delta; \Phi; \Gamma \vdash e_2 \ominus e_1'\ e_2' \lesssim t_2 - k_1 - k - c_{app} : U\,(A_2, A_2')}$ **r-e-app**

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.
TS: $(m, \delta e_2, \delta' e_1'\ \delta e_2',) \in (\!|U\,(\sigma A_2, \sigma A_2')|\!)_{\varepsilon}^{\sigma t_2 - \sigma k_1 - \sigma k - c_{app}}$.
Following the definition of $(\!|\cdot|\!)_{\varepsilon}^{\cdot}$, assume that

$\delta e_2 \Downarrow^{c} v$ $(\diamond\diamond)$ and $\dfrac{\delta' e_1' \Downarrow^{c_1'} \text{fix } f(x).e' \ \ (\star) \qquad \delta' e_2' \Downarrow^{c_2'} v_2' \ \ (\diamond) \qquad e'[v_2'/x, (\text{fix } f(x).e')/f] \Downarrow^{c_r'} v_r' \ \ (\dagger)}{\delta' e_1'\ \delta' e_2' \Downarrow^{c_1' + c_2' + c_r' + c_{app}} v_r'}$ **app**

and $c < m$.
TS1: $c - (c_1' + c_2' + c_r' + c_{app}) \leq \sigma t_2 - \sigma k_1 - \sigma k - c_{app}$
TS2: $(m - c, v, v_r') \in (\!|U\,(\sigma A_2, \sigma A_2')|\!)_v$
We first show the second statement.
By unrolling the definition of $(\!|U\,(\sigma A_2, \sigma A_2')|\!)_v$,
STS: $\forall j. (j, v) \in [\![\sigma A_2]\!]_v \ \wedge \ (j, v_r') \in [\![\sigma A_2']\!]_v$.
Pick j.
By IH 2 on the first premise using

- $\mathrm{FV}(e_1') \subseteq \mathrm{dom}(|\sigma\Gamma|_2)$ using Lemma 8 on the first premise
- $\forall m.(m, \delta') \in \mathcal{G}[\![|\sigma\Gamma|_2]\!]$ using Lemma 3 on $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$.

we get

$$\forall m.(m, \delta' e_1') \in [\![\sigma A_1' \xrightarrow{\text{exec}(\sigma k, \sigma t)} \sigma A_2']\!]_\varepsilon^{\sigma k_1, \sigma t_1} \tag{1}$$

Instantiating (1) with $j + c_1 + c_r + 1 + c_{app}$, we get

$$(j + c_1 + c_r + 1 + c_{app}, \delta' e_1') \in [\![\sigma A_1' \xrightarrow{\text{exec}(\sigma k, \sigma t)} \sigma A_2']\!]_\varepsilon^{\sigma k_1, \sigma t_1} \tag{2}$$

Unfolding the second part of the definition of (2) with ($\star$) and $c_1' < j + c_1' + c_r' + c_{app} + 1$, we get

a) $\sigma k_1 \leq c_1'$
b) $(j + c_r' + c_{app} + 1, \text{fix } f(x).e') \in [\![\sigma A_1' \xrightarrow{\text{exec}(\sigma k, \sigma t)} \sigma A_2']\!]_v$

By IH 1 on the second premise, we get $(m, \delta e_2, \delta' e_2') \in (\!|U\,(\sigma A_2, \sigma A_1')|\!)_\varepsilon^{\sigma t_2}$. Unrolling its definition with ($\diamond\diamond$) and ($\diamond$), and $c < m$, we get

c) $c - c_2' \leq \sigma t_2$
d) $(m - c, v, v_2') \in (\!|U\,(\sigma A_2, \sigma A_1')|\!)_v$

By d), we get $\forall m.(m, v) \in [\![\sigma A_2]\!]_v \ \wedge \ (m, v_2') \in [\![\sigma A_1']\!]_v$.
Instantiating m with $j + c_r' + c_{app}$, we get

$$(j + c_r' + c_{app}, v) \in [\![\sigma A_2]\!]_v \tag{3}$$

$$(j + c_r' + c_{app}, v_2') \in [\![\sigma A_1']\!]_v \tag{4}$$

Unrolling b) with (4) since $j + c_r' + c_{app} < j + c_r' + c_{app} + 1$, we get

$$(j + c_r' + c_{app}, e'[v_2'/x, (\text{fix } f(x).e')]) \in [\![\sigma A_2']\!]_\varepsilon^{\sigma k, \sigma t} \tag{5}$$

By unrolling (5) with (†) and $c_r' < j + c_r' + c_{app}$, we get

e) $\sigma k \leq c_r'$
f) $(j + c_{app}, v_r') \in [\![\sigma A_2']\!]_v$

Now, we can conclude as follows:

1. Using a), c) and e), we get $c - (c_1' + c_2' + c_r' + c_{app}) \leq \sigma t_2 - \sigma k_1 - \sigma k - c_{app}$

2. By downward closure (Lemma 4) on (3) using

$$j \leq j + c'_r + c_{app}$$

we get $(j, v) \in [\![\sigma A_2]\!]_v$.

By downward closure (Lemma 4) on f) using

$$j \leq j + c_{app}$$

we get $(j, v'_r) \in [\![\sigma A'_2]\!]_v$.

Case $\dfrac{\Delta; \Phi; |\Gamma|_1 \vdash^{t_1}_{k_1} e_1 : A_1 \qquad \Delta; \Phi; x : U\,(A_1, A_1), \Gamma \vdash e_2 \ominus e \lesssim t_2 : \tau_2}{\Delta; \Phi; \Gamma \vdash \mathbf{let}\ x = e_1\ \mathbf{in}\ e_2 \ominus e \lesssim t_1 + t_2 + c_{let} : \tau_2}$ **r-let-e**

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.

TS: $(m, \text{let } x = \delta e_1 \text{ in } \delta e_2, \delta' e) \in (\!|\sigma\tau_2|\!)^{\sigma t_1 + \sigma t_2 + c_{let}}_{\varepsilon}$.

Following the definition of $(\!|\cdot|\!)^{\cdot}_{\varepsilon}$, assume that

$\dfrac{\delta e_1 \Downarrow^{c_1} v_1 \quad (\diamond) \qquad \delta e_2[v_1/x] \Downarrow^{c_r} v_r \quad (\dagger)}{\text{let } x = \delta e_1 \text{ in } \delta e_2 \Downarrow^{c_1 + c_r + c_{let}} v_r}$ **let** and $\delta' e \Downarrow^{c'} v'$ $(\star)$ and $(c_1 + c_r + c_{let}) < m$.

To be able to instantiate the IH 1 on the second premise, we first show

$$\forall m.(m, v_1) \in [\![\sigma A_1]\!]_v \tag{1}$$

Subproof. Pick m.

By IH 2 on the first premise using

- $\text{FV}(e_1) \subseteq \text{dom}(|\sigma\Gamma|_1)$ using Lemma 8 on the first premise
- $(m + \sigma t_1 + 1, \delta) \in \mathcal{G}[\![|\sigma\Gamma|_1]\!]$ obtained by Lemma 3 using $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$

we get

$$(m + \sigma t_1 + 1, \delta e_1) \in [\![\sigma A_1]\!]^{\sigma k_1, \sigma t_1}_{\varepsilon} \tag{2}$$

Unfolding the first part of the definition of (2) with $\sigma t_1 < m + \sigma t_1 + 1$, we get

a) $e_1 \Downarrow^{c_1} v_1$

b) $c_1 \leq \sigma t_1$

c) $(m + \sigma t_1 + 1 - c_1, v_1) \in [\![\sigma A_1]\!]_v$

RTS: $(m, v_1) \in [\![\sigma A_1]\!]_v$.
This follows by downward closure (Lemma 4) on c) using $m \leq m + \sigma t_1 - c_1 + 1$ and $0 \leq \sigma t_1 - c_1$ (by (b)). ∎

Next, we instantiate IH 1 on the second premise using

- $(m, \delta[x \mapsto v_1], \delta'[x \mapsto v_1]) \in \mathcal{G}(\!|\sigma\Gamma, x : U\,(\sigma A_1, \sigma A_1)|\!)$ using
  - $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$
  - $(m, v_1, v_1) \in (\!|U\,(\sigma A_1, \sigma A_1)|\!)_v$ using (1)

and we get $(m, \delta e_2[v_1/x], \delta' e[v_1/x]) \in (\!|\sigma\tau_2|\!)_\varepsilon^{\sigma t_2}$.
Since x doesn't occur free in e, we have
$(m, \delta e_2[v_1/x], \delta' e) \in (\!|\sigma\tau_2|\!)_\varepsilon^{\sigma t_2}$.
Unrolling its definition with (†) and ($\star$), and $c_r < m$, we get

g) $c_r - c' \leq \sigma t_2$

h) $(m - c_r, v_r, v') \in (\!|\sigma\tau_2|\!)_v$

Now, we can conclude by

1. By b) and g) $(c_1 + c_r + c_{let}) - c' \leq \sigma t_1 + \sigma t_2 + c_{let}$
2. By downward closure (Lemma 4) on h), using

$$m - (c_1 + c_r + c_{let}) \leq m - c_r$$

   we obtain $(m - (c_1 + c_r + c_{let}), v_r, v') \in (\!|\sigma\tau_2|\!)_v$.

$$\textbf{Case}\ \frac{\Delta; \Phi; |\Gamma|_1 \vdash_{-}^{t} e : A_1 + A_2 \qquad \Delta; \Phi; x : U\,(A_1, A_1), \Gamma \vdash e_1 \ominus e' \lesssim t' : \tau \qquad \Delta; \Phi; y : U\,(A_2, A_2), \Gamma \vdash e_2 \ominus e' \lesssim t' : \tau}{\Delta; \Phi; \Gamma \vdash \ \textbf{case}\ (e, x.e_1, y.e_2) \ominus e' \lesssim t' + t + c_{case} : \tau}\ \textbf{r-case-e}$$

Assume that $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$ and $\models \sigma\Phi$.
TS: $(m,\ \text{case}\ (\delta e, \delta e_1, \delta e_2), \delta' e') \in (\!|\sigma\tau|\!)_\varepsilon^{\sigma t + \sigma t'}$.
Following the definition of $(\!|\cdot|\!)_\varepsilon^{\cdot}$, assume that
$\text{case}\ (\delta e, \delta e_1, \delta e_2) \Downarrow^{C} v_r$ and $\delta' e' \Downarrow^{c'} v'$ (†) and $C < m$.
Depending on what δe evaluates to, there are two cases:

$$\textbf{subcase 1:}\ \frac{\delta e \Downarrow^{c} \text{inl}\ v \quad (\star) \qquad \delta e_1[v/x] \Downarrow^{c_r} v_r \quad (\diamond)}{\text{case}\ (\delta e, x.\delta e_1, y.\delta e_2) \Downarrow^{c + c_r + c_{case}} v_r}\ \textbf{case-inl}$$

Note that $C = c + c_r + c_{case} < m$.

To be able to instantiate the IH 1 on the second premise, we first show

$$\forall m.(m, \text{inl } v) \in [\![\sigma A_1 + \sigma A_2]\!]_v \tag{1}$$

Subproof. Pick m.
By IH 2 on the first premise using

- $\text{FV}(e) \subseteq \text{dom}(|\sigma\Gamma|_1)$ using Lemma 8 on the first premise
- $(m + \sigma t + 1, \delta) \in \mathcal{G}[\![|\sigma\Gamma|_1]\!]$ obtained by Lemma 3 using $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$

we get

$$(m + \sigma t + 1, \delta e_1) \in [\![\sigma A_1 + \sigma A_2]\!]_\varepsilon^{\sigma k, \sigma t} \tag{2}$$

Unfolding the first part of the definition of (2) with $\sigma t < m + \sigma t + 1$, we get

a) $e \Downarrow^c \text{inl } v$
b) $c \leq \sigma t$
c) $(m + \sigma t + 1 - c, \text{inl } v) \in [\![\sigma A_1 + \sigma A_2]\!]_v$

RTS: $(m, \text{inl } v) \in [\![\sigma A_1 + \sigma A_2]\!]_v$.
This follows by downward closure (Lemma 4) on c) using $m \leq m + \sigma t - c + 1$ and $0 \leq \sigma t - c$ (by (b)). ■

Next, we instantiate IH 1 on the second premise using

- $(m, \delta[x \mapsto v], \delta'[x \mapsto v]) \in \mathcal{G}(\!|\sigma\Gamma, x : U\,(\sigma A_1, \sigma A_1)|\!)$ using
  - $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$
  - $(m, v, v) \in (\!|U\,(\sigma A_1, \sigma A_1)|\!)_v$ by unrolling the definition of (1)

and we get $(m, \delta e_1[v/x], \delta' e'[v/x]) \in (\!|\sigma\tau|\!)_\varepsilon^{\sigma t'}$.
Since x doesn't occur free in e', we have
$(m, \delta e_1[v/x], \delta' e') \in (\!|\sigma\tau|\!)_\varepsilon^{\sigma t'}$.
Unrolling its definition with ($\diamond$) and ($\dagger$), and $c_r < m$, we get

i) $c_r - c' \leq \sigma t'$
j) $(m - c_r, v_r, v') \in (\!|\sigma\tau|\!)_v$

Now, we can conclude by

1. By b) and i) $(c + c_r + c_{case}) - c' \leq \sigma t + \sigma t' + c_{case}$

2. By downward closure (Lemma 4) on j), using

$$m - (c + c_r + c_{case}) \leq m - c_r$$

we obtain $(m - (c + c_r + c_{case}), v_r, v') \in (\!|\sigma\tau|\!)_v$.

subcase 2: $\dfrac{\delta e \Downarrow^{c} \text{inr } v \quad (\star) \qquad \delta e_2[v/y] \Downarrow^{c_r} v_r \quad (\diamond)}{\text{case } (\delta e, x.\delta e_1, y.\delta e_2) \Downarrow^{c+c_r+c_{case}} v_r}$ **case-inr**

Note that $C = c + c_r + c_{case} < m$.

Like in the previous case, we have

$$\forall m.(m, \text{inr } v) \in [\![\sigma A_1 + \sigma A_2]\!]_v \tag{3}$$

Next, we instantiate IH 1 on the third premise using

- $(m, \delta[y \mapsto v], \delta'[y \mapsto v]) \in \mathcal{G}(\!|\sigma\Gamma, y : U\ (\sigma A_2, \sigma A_2)|\!)$ using
  - $(m, \delta, \delta') \in \mathcal{G}(\!|\sigma\Gamma|\!)$
  - $(m, v, v) \in (\!|U\ (\sigma A_2, \sigma A_2)|\!)_v$ by unrolling the definition of (3)

and we get $(m, \delta e_2[v/y], \delta' e'[v/y]) \in (\!|\sigma\tau|\!)_\varepsilon^{\sigma t'}$.

Since y doesn't occur free in e', we have

$(m, \delta e_2[v/y], \delta' e') \in (\!|\sigma\tau|\!)_\varepsilon^{\sigma t'}$.

Unrolling its definition with $(\diamond)$ and $(\dagger)$, and $c_r < m$, we get

k) $c_r - c' \leq \sigma t'$

l) $(m - c_r, v_r, v') \in (\!|\sigma\tau|\!)_v$

Now, we can conclude by

1. By b) and k) $(c + c_r + c_{case}) - c' \leq \sigma t + \sigma t' + c_{case}$
2. By downward closure (Lemma 4) on l), using

$$m - (c + c_r + c_{case}) \leq m - c_r$$

we obtain $(m - (c + c_r + c_{case}), v_r, v') \in (\!|\sigma\tau|\!)_v$.

□

Proof of Statement (2). Remember the statement (2) of Theorem 14:
Assume that $\Delta; \Phi; \Omega \vdash_k^t e : A$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$ and there exists Ω' s.t. $\text{FV}(e) \subseteq \text{dom}(\Omega')$ and $\Omega' \subseteq \Omega$ and $(m, \gamma) \in \mathcal{G}[\![\sigma\Omega']\!]$. Then, $(m, \gamma e) \in [\![\sigma A]\!]_\varepsilon^{\sigma k, \sigma t}$.

Proof is by induction on the typing of e. We show a few selected cases.

Case $\dfrac{\Omega(x) = A}{\Delta; \Phi; \Omega \vdash_0^0 x : A}$ **var**

Assume that $\models \sigma\Phi$ and there exists Ω' s.t. $\mathrm{FV}(x) \subseteq \mathrm{dom}(\Omega')$ and $\Omega' \subseteq \Omega$ and $(m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$

TS: $(m, \gamma(x)) \in [\![\sigma A]\!]_\varepsilon^{0,0}$.

By Value Lemma (Lemma 2),

STS: $(m, \gamma(x)) \in [\![\sigma A]\!]_v$.

Note that $x \in \mathrm{dom}(\Omega')$ and $\Omega' \subseteq \Omega$, therefore $\Omega'(x) = A$,

RTS: $(m, \gamma(x)) \in [\![\sigma A]\!]_v$.

This follows by $\Omega(x) = A$ and $(m, \gamma) \in \mathcal{G}(\!|\sigma\Omega|\!)$

Case $\dfrac{\Delta; \Phi; \Omega \vdash_{k_1}^{t_1} e_1 : A \qquad \Delta; \Phi; \Omega \vdash_{k_2}^{t_2} e_2 : \textbf{list}[n]\, A}{\Delta; \Phi; \Omega \vdash_{k_1+k_2}^{t_1+t_2} \textbf{cons}(e_1, e_2) : \textbf{list}[n+1]\, A}$ **cons**

Assume that $\models \sigma\Phi$ and there exists Ω' s.t. $\mathrm{FV}(\mathrm{cons}(e_1, e_2)) \subseteq \mathrm{dom}(\Omega')$ and $\Omega' \subseteq \Omega$ and $(m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$

TS: $(m, \mathrm{cons}(\gamma e_1, \gamma e_2)) \in [\![\mathrm{list}[\sigma n + 1]\, \sigma A]\!]_\varepsilon^{\sigma k_1 + \sigma k_2, \sigma t_1 + \sigma t_2}$.

Following the definition of $[\![\cdot]\!]_\varepsilon^{\cdot,\cdot}$, there are two parts to show

- Assume that $\sigma t_1 + \sigma t_2 < m$.
  By IH 2 on the first premise using

  $$\mathrm{FV}(e_1) \subseteq \mathrm{dom}(\Omega') \text{ and } \Omega' \subseteq \Omega \text{ and } (m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$$

  we get $(m, \gamma e_1) \in [\![\sigma A]\!]_\varepsilon^{\sigma k_1, \sigma t_1}$. Unrolling its definition with $\sigma t_1 < m$, we get

  a) $\gamma e_1 \Downarrow^{c_1} v_1$
  b) $c_1 \leq \sigma t_1$
  c) $(m - c_1, v_1) \in [\![\sigma A]\!]_v$

  By IH 2 on the second premise using

  $$\mathrm{FV}(e_2) \subseteq \mathrm{dom}(\Omega') \text{ and } \Omega' \subseteq \Omega \text{ and } (m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$$

  we get $(m, \gamma e_2) \in [\![\mathrm{list}[\sigma n]\, \sigma A]\!]_\varepsilon^{\sigma k_2, \sigma t_2}$.
  Unrolling its definition with $\sigma t_2 < m$, we get

  d) $\gamma e_2 \Downarrow^{c_2} v_2$
  e) $c_2 \leq \sigma t_2$
  f) $(m - c_2, v_2) \in [\![\mathrm{list}[\sigma n]\, \sigma A]\!]_v$

  Now, we can conclude as follows:

  1. Using a) and d), we get
     $\dfrac{e_1 \Downarrow^{c_1} v_1 \qquad e_2 \Downarrow^{c_2} v_2}{\mathrm{cons}(e_1, e_2) \Downarrow^{c_1 + c_2} \mathrm{cons}(v_1, v_2)}$ **cons**
  2. Using b) and e), we get $(c_1 + c_2) \leq \sigma t_1 + \sigma t_2$

3. By downward closure (Lemma 4) on c) using

$$m - (c_1 + c_2) \leq m - c_1$$

we get $(m - (c_1 + c_2), v_1) \in [\![\sigma A]\!]_v$.
By downward closure (Lemma 4) on f) using

$$m - (c_1 + c_2) \leq m - c_2$$

we get $(m - (c_1 + c_2), v_2) \in [\![\mathrm{list}[\sigma n]\, \sigma A]\!]_v$.
By combining these two statements, we can conclude as $(m-(c_1+c_2), \mathrm{cons}(v_1, v_2)) \in [\![\mathrm{list}[\sigma n + 1]\, \sigma A]\!]_v$

- Assume that $\dfrac{\gamma e_1 \Downarrow^{c_1} v_1 \quad (\star) \qquad \gamma e_2 \Downarrow^{c_2} v_2 \quad (\diamond)}{\mathrm{cons}(\gamma e_1, \gamma e_2) \Downarrow^{c_1+c_2} \mathrm{cons}(v_1, v_2)}$ **cons** and $c_1 + c_2 < m$.
By IH 2 on the first premise using

$$\mathrm{FV}(e_1) \subseteq \mathrm{dom}(\Omega') \text{ and } \Omega' \subseteq \Omega \text{ and } (m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$$

we get $(m, \gamma e_1) \in [\![\sigma A]\!]_\varepsilon^{\sigma k_1, \sigma t_1}$. Unrolling its definition with $(\star)$ and $c_1 < m$, we get

a) $\sigma k_1 \leq c_1$
b) $(m - c_1, v_1) \in [\![\sigma A]\!]_v$

By IH 2 on the second premise using

$$\mathrm{FV}(e_2) \subseteq \mathrm{dom}(\Omega') \text{ and } \Omega' \subseteq \Omega \text{ and } (m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$$

we get $(m, \gamma e_2) \in [\![\mathrm{list}[\sigma n]\, \sigma A]\!]_\varepsilon^{\sigma k_2, \sigma t_2}$.
Unrolling its definition with $(\diamond)$ and $c_2 < m$, we get

c) $\sigma k_2 \leq c_2$
d) $(m - c_2, v_2) \in [\![\mathrm{list}[\sigma n]\, \sigma A]\!]_v$

Now, we can conclude as follows:

1. Using a) and c), we get $\sigma k_1 + \sigma k_2 \leq (c_1 + c_2)$
2. By downward closure (Lemma 4) on b) using

$$m - (c_1 + c_2) \leq m - c_1$$

we get $(m - (c_1 + c_2), v_1) \in [\![\sigma A]\!]_v$.
By downward closure (Lemma 4) on d) using

$$m - (c_1 + c_2) \leq m - c_2$$

we get $(m - (c_1 + c_2), v_2) \in [\![\mathrm{list}[\sigma n]\, \sigma A]\!]_v$.
By combining these two statements, we can conclude as $(m-(c_1+c_2), \mathrm{cons}(v_1, v_2)) \in [\![\mathrm{list}[\sigma n + 1]\, \sigma A]\!]_v$

Case $\dfrac{\Delta; \Phi \vdash^{\mathsf{A}} A_1 \xrightarrow{\mathbf{exec}(k,t)} A_2 \ \mathtt{wf} \qquad \Delta; \Phi; x : A_1, f : A_1 \xrightarrow{\mathbf{exec}(k,t)} A_2, \Omega \vdash_k^t e : A_2}{\Delta; \Phi; \Omega \vdash_0^0 \mathbf{fix}\ f(x).e : A_1 \xrightarrow{\mathbf{exec}(k,t)} A_2}$ **fix**

Assume that $\models \sigma\Phi$ and there exists Ω' s.t. $\mathrm{FV}(\text{fix } f(x)) \subseteq \mathrm{dom}(\Omega')$ and $\Omega' \subseteq \Omega$ and $(m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$

TS: $(m, \text{fix } f(x).\gamma e) \in [\![\sigma A_1 \xrightarrow{\text{exec}(\sigma k, \sigma t)} \sigma A_2]\!]_{\varepsilon}^{0,0}$.

By Lemma 2, STS: $(m, \text{fix } f(x).\gamma e) \in [\![\sigma A_1 \xrightarrow{\text{exec}(\sigma k, \sigma t)} \sigma A_2]\!]_{v}$.

We prove the more general statement

$$\forall m' \leq m.\ (m', \text{fix } f(x).\gamma e) \in [\![\sigma A_1 \xrightarrow{\text{exec}(\sigma k, \sigma t)} \sigma A_2]\!]_{v}$$

by subinduction on m'.

There are two cases:

subcase 1: $m' = 0$

Since there is no non-negative j such that $j < 0$, the goal is vacuously true.

subcase 2: $m' = m'' + 1 \leq m$

By sub-IH

$$(m'', \text{fix } f(x).\gamma e) \in [\![\sigma A_1 \xrightarrow{\text{exec}(\sigma k, \sigma t)} \sigma A_2]\!]_{v} \tag{1}$$

STS: $(m'' + 1, \text{fix } f(x).\gamma e) \in [\![\sigma A_1 \xrightarrow{\text{exec}(\sigma k, t)} \sigma A_2]\!]_{v}$.

Pick $j < m'' + 1$ and assume that $(j, v) \in [\![\sigma A_1]\!]_v$.

STS: $(j, \gamma e[v/x, (\text{fix } f(x).\gamma e)/f]) \in [\![\sigma A_2]\!]_{\varepsilon}^{\sigma k, \sigma t}$.

This follows by IH on the premise instantiated with

- $\mathrm{FV}(e) \subseteq \mathrm{dom}(x : A_1, f : A_1 \xrightarrow{\text{exec}(k,t)} A_2, \Omega')$ and $x : A_1, f : A_1 \xrightarrow{\text{exec}(k,t)} A_2, \Omega' \subseteq x : A_1, f : A_1 \xrightarrow{\text{exec}(k,t)} A_2, \Omega$
- $(j, \gamma[x \mapsto v, f \mapsto (\text{fix } f(x).\gamma e)]) \in \mathcal{G}[\![\sigma\Omega', x : \sigma A_1, f : \sigma A_1 \xrightarrow{\text{exec}(\sigma k, \sigma t)} \sigma A_2]\!]$ which holds because
  - $(j, \gamma) \in \mathcal{G}[\![\sigma\Omega']\!]$ obtained by downward closure (Lemma 4) on $(m, \gamma) \in \mathcal{G}[\![\sigma\Omega']\!]$ using $j < m'' + 1 \leq m$.
  - $(j, v) \in [\![\sigma A_1]\!]_v$, from the assumption above
  - $(j, \text{fix } f(x).\gamma e) \in [\![\sigma A_1 \xrightarrow{\text{exec}(\sigma k, \sigma t)} \sigma A_2]\!]_v$, obtained by downward closure (Lemma 4) on (1) using $j \leq m''$

Case $\dfrac{\Delta; \Phi; \Omega \vdash_{k_1}^{t_1} e_1 : A_1 \xrightarrow{\mathbf{exec}(k,t)} A_2 \qquad \Delta; \Phi; \Omega \vdash_{k_2}^{t_2} e_2 : A_1}{\Delta; \Phi; \Omega \vdash_{k_1+k_2+k+c_{app}}^{t_1+t_2+t+c_{app}} e_1\ e_2 : A_2}$ **app**

Assume that $\models \sigma\Phi$ and there exists Ω' s.t. $\mathrm{FV}(e_1\ e_2) \subseteq \mathrm{dom}(\Omega')$ and $\Omega' \subseteq \Omega$ and $(m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$

TS: $(m, \gamma e_1\ \gamma e_2) \in [\![\sigma A_2]\!]_{\varepsilon}^{\sigma k_1 + \sigma k_2 + \sigma k + c_{app}, \sigma t_1 + \sigma t_2 + \sigma t + c_{app}}$.

Following the definition of $[\![\cdot]\!]_{\varepsilon}^{\cdot,\cdot}$, there are two cases:

- Assume that $\sigma t_1 + \sigma t_2 + \sigma t + c_{app} < m$.
  By IH 2 on the first premise using

$$\mathrm{FV}(e_1) \subseteq \mathrm{dom}(\Omega') \text{ and } \Omega' \subseteq \Omega \text{ and } (m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$$

we get $(m, \gamma e_1) \in [\![\sigma A_1 \xrightarrow{\text{exec}(\sigma k, \sigma t)} \sigma A_2]\!]_\varepsilon^{\sigma k_1, \sigma t_1}$.
Unrolling its definition with $\sigma t_1 < m$, we get

a) $\gamma e_1 \Downarrow^{c_1} \text{fix } f(x).e$
b) $c_1 \leq \sigma t_1$
c) $(m - c_1, \text{fix } f(x).e) \in [\![\sigma A_1 \xrightarrow{\text{exec}(\sigma k, \sigma t)} \sigma A_2]\!]_v$

By IH 2 on the second premise using

$$\text{FV}(e_2) \subseteq \text{dom}(\Omega') \text{ and } \Omega' \subseteq \Omega \text{ and } (m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$$

we get $(m, \gamma e_2) \in [\![\sigma A_1]\!]_\varepsilon^{\sigma k_2, \sigma t_2}$. Unrolling its definition with $\sigma t_2 < m$, we get

d) $\gamma e_2 \Downarrow^{c_2} v_2$
e) $c_2 \leq \sigma t_2$
f) $(m - c_2, v_2) \in [\![\sigma A_1]\!]_v$

By downward closure (Lemma 4) on f) using $m - c_1 - c_2 - c_{app} \leq m - c_2$, we get

$$(m - (c_1 + c_2 + c_{app}), v_2) \in [\![\sigma A_1]\!]_v \tag{1}$$

Next, we unroll c) with (1) using $m - (c_1 + c_2 + c_{app}) < m - c_1$ to obtain

$$(m - (c_1 + c_2 + c_{app}), e[v_2/x, (\text{fix } f(x).e)]) \in [\![\sigma A_2]\!]_\varepsilon^{\sigma k, \sigma t} \tag{2}$$

To unroll first part of (2)'s definition, we need to show that $\sigma t < m - (c_1 + c_2 + c_{app})$ or $\sigma t + (c_1 + c_2 + c_{app}) < m$.
By b) and e), we know that

$$c_1 + c_2 \leq \sigma t_1 + \sigma t_2 \tag{3}$$

By adding $\sigma t + c_{app}$ to the both sides of (3), we have

$$c_1 + c_2 + \sigma t + c_{app} \leq \sigma t_1 + \sigma t_2 + \sigma t + c_{app} \tag{4}$$

By the main assumption, we know that $\sigma t_1 + \sigma t_2 + \sigma t + c_{app} < m$.
Therefore, we know that $c_1 + c_2 + \sigma t + c_{app} < m$. Now, by unfolding we get

g) $e[v_2/x, (\text{fix } f(x).e)] \Downarrow^{c_r} v_r$
h) $c_r \leq \sigma t$
i) $(m - (c_1 + c_2 + c_r + c_{app}), v_r) \in [\![\sigma A_2]\!]_v$

Now, we can conclude as follows:

1. Using a), d) and g), we get

$$\frac{e_1 \Downarrow^{c_1} \text{fix } f(x).e \qquad e_2 \Downarrow^{c_2} v_2 \qquad e[v_2/x, (\text{fix } f(x).e)/f] \Downarrow^{c_r} v_r}{e_1\ e_2 \Downarrow^{c_1 + c_2 + c_r + c_{app}} v_r} \textbf{ app}$$

2. Using b), e) and h), we get $(c_1 + c_2 + c_r + c_{app}) \leq \sigma t_1 + \sigma t_2 + \sigma t + c_{app}$
3. By i)

- Assume that $\dfrac{\gamma e_1 \Downarrow^{c_1} \text{fix } f(x).e \quad (\star) \qquad \gamma e_2 \Downarrow^{c_2} v_2 \quad (\diamond) \qquad e[v_2/x, (\text{fix } f(x).e)/f] \Downarrow^{c_r} v_r \quad (\dagger)}{\gamma e_1\ \gamma e_2 \Downarrow^{c_1+c_2+c_r+c_{app}} v_r}$ **app** and $c_1 + c_2 + c_r + c_{app} < m$.
  By IH 2 on the first premise using
  $$\text{FV}(e_1) \subseteq \text{dom}(\Omega') \text{ and } \Omega' \subseteq \Omega \text{ and } (m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$$
  we get $(m, \gamma e_1) \in [\![\sigma A_1 \xrightarrow{\text{exec}(\sigma k, \sigma t)} \sigma A_2]\!]_\varepsilon^{\sigma k_1, \sigma t_1}$.
  Unrolling its definition with $(\star)$ and $c_1 < m$, we get
  a) $\sigma k_1 \leq c_1$
  b) $(m - c_1, \text{fix } f(x).e) \in [\![\sigma A_1 \xrightarrow{\text{exec}(\sigma k, \sigma t)} \sigma A_2]\!]_v$

  By IH 2 on the second premise using
  $$\text{FV}(e_2) \subseteq \text{dom}(\Omega') \text{ and } \Omega' \subseteq \Omega \text{ and } (m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$$
  we get $(m, \gamma e_2) \in [\![\sigma A_1]\!]_\varepsilon^{\sigma k_2, \sigma t_2}$. Unrolling its definition with $(\diamond)$ and $c_2 < m$, we get
  c) $\sigma k_2 \leq c_2$
  d) $(m - c_2, v_2) \in [\![\sigma A_1]\!]_v$

  By downward closure (Lemma 4) on d) using $m - c_1 - c_2 - c_{app} \leq m - c_2$, we get
  $$(m - (c_1 + c_2 + c_{app}), v_2) \in [\![\sigma A_1]\!]_v \tag{5}$$
  Next, we unroll b) with (5) and $m - (c_1 + c_2 + c_{app}) < m - c_1$ to obtain
  $$(m - (c_1 + c_2 + c_{app}), e[v_2/x, (\text{fix } f(x).e)]) \in [\![\sigma A_2]\!]_\varepsilon^{\sigma k, \sigma t} \tag{6}$$
  By unrolling second part of (6)'s definition using $(\dagger)$ and $c_r < m - (c_1 + c_2 + c_{app})$, we get
  e) $\sigma k \leq c_r$
  f) $(m - (c_1 + c_2 + c_r + c_{app}), v_r) \in [\![\sigma A_2]\!]_v$

  Now, we can conclude as follows:
  1. Using a), c) and e), we get $\sigma k_1 + \sigma k_2 + \sigma k + c_{app} \leq (c_1 + c_2 + c_r + c_{app})$
  2. By f)

Case $\dfrac{\Delta; \Phi; \Omega \vdash_k^t e : A_1 \qquad \Delta; \Phi \vdash^{\mathsf{A}} A_2 \ \texttt{wf}}{\Delta; \Phi; \Omega \vdash_k^t \textbf{inl } e : A_1 + A_2}$ **inl**

Assume that $\models \sigma\Phi$ and there exists Ω' s.t. $\text{FV}(\text{inl } e) \subseteq \text{dom}(\Omega')$ and $\Omega' \subseteq \Omega$ and $(m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$
TS: $(m, \text{inl } (\gamma e)) \in [\![\sigma A_1 + \sigma A_2]\!]_\varepsilon^{\sigma k, \sigma t}$.
Following the definition of $[\![\cdot]\!]_\varepsilon^{\cdot,\cdot}$, there are two cases:

- Assume that $\sigma t < m$.
  By IH 2 on the first premise using
  $$\text{FV}(e) \subseteq \text{dom}(\Omega') \text{ and } \Omega' \subseteq \Omega \text{ and } (m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$$

we get $(m, \gamma e) \in [\![\sigma A]\!]_{\varepsilon}^{\sigma k, \sigma t}$. Unrolling the first part of its definition with $\sigma t < m$, we get

a) $\gamma e \Downarrow^c v$
b) $c \leq \sigma t$
c) $(m - c, v) \in [\![\sigma A]\!]_v$

We can conclude as follows:

1. By a),
$$\frac{\gamma e \Downarrow^c v}{\text{inl } \gamma e \Downarrow^c \text{inl } v} \ \textbf{inl}$$
2. By b), $c \leq \sigma t$
3. By c), we can show that $(m - c, \text{inl } v) \in [\![\sigma A_1 + \sigma A_2]\!]_v$

- Assume that $\dfrac{\gamma e \Downarrow^c v \quad (\star)}{\text{inl } \gamma e \Downarrow^c \text{inl } v}$ **inl** and $c < m$.
By IH 2 on the first premise using

$$\text{FV}(e) \subseteq \text{dom}(\Omega') \text{ and } \Omega' \subseteq \Omega \text{ and } (m, \delta) \in \mathcal{G}[\![\sigma \Omega']\!]$$

we get $(m, \gamma e) \in [\![\sigma A]\!]_{\varepsilon}^{\sigma k, \sigma t}$. Unrolling the second part of its definition with $(\star)$ and $c < m$, we get

a) $\sigma k \leq c$
b) $(m - c, v) \in [\![\sigma A]\!]_v$

We can conclude as follows:

1. By a), $\sigma k \leq c$
2. By b), we can show that $(m - c, \text{inl } v) \in [\![\sigma A_1 + \sigma A_2]\!]_v$

Case $\dfrac{\Delta; \Phi; \Omega \vdash_k^t e : A_1 + A_2 \qquad \Delta; \Phi; x : A_1, \Omega \vdash_{k'}^{t'} e_1 : A \qquad \Delta; \Phi; y : A_2, \Omega \vdash_{k'}^{t'} e_2 : A}{\Delta; \Phi; \Omega \vdash_{k+k'+c_{case}}^{t+t'+c_{case}} \textbf{case } (e, x.e_1, y.e_2) : A}$ **case**

Assume that $\models \sigma\Phi$ and there exists Ω' s.t. $\text{FV}(\text{ case } (e, e_1, e_2)) \subseteq \text{dom}(\Omega')$ and $\Omega' \subseteq \Omega$ and $(m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$
TS: $(m, \text{ case } (\gamma e, \gamma e_1, \gamma e_2)) \in [\![\sigma A]\!]_{\varepsilon}^{\sigma k, \sigma t + \sigma t' + c_{case}}$.
Following the definition of $[\![\cdot]\!]_{\varepsilon}^{\cdot,\cdot}$, there are two cases:

- Assume that $\sigma t + \sigma t' + c_{case} < m$.
By IH 2 on the first premise using

$$\text{FV}(e) \subseteq \text{dom}(\Omega') \text{ and } \Omega' \subseteq \Omega \text{ and } (m, \delta) \in \mathcal{G}[\![\sigma \Omega']\!]$$

we get $(m, \gamma e) \in [\![\sigma A_1 + \sigma A_2]\!]_{\varepsilon}^{\sigma k, \sigma t}$.
Unrolling the first part of its definition with $\sigma t < m$, there are two cases. We only show one, as the other is very similar. We have

a) $\gamma e \Downarrow^c \text{inl } v$
b) $c \leq \sigma t$

c) $(m - c, \text{inl } v) \in [\![\sigma A_1 + \sigma A_2]\!]_v$

By IH 2 on the second premise using $(m - c - c_{case}, \gamma[x \mapsto v]) \in \mathcal{G}[\![\sigma\Omega', x : \sigma A_1]\!]$ obtained by

- $\text{FV}(e_1) \subseteq \text{dom}(x : A_1, \Omega')$ and $x : A_1, \Omega' \subseteq x : A_1, \Omega$
- $(m - c - c_{case}, \gamma) \in \mathcal{G}[\![\sigma\Omega']\!]$ by downward closure (Lemma 4) on $(m, \gamma) \in \mathcal{G}[\![\sigma\Omega']\!]$ using $m - c - c_{case} \leq m$
- $(m - c - c_{case}, v) \in [\![\sigma A_1]\!]_v$ by downward closure (Lemma 4) on c), and unfolding its definition

we get

$$(m - c - c_{case}, \gamma e_1[v/x]) \in [\![\sigma A]\!]_\varepsilon^{\sigma k', \sigma t'} \tag{1}$$

To unroll the first part of (1)'s definition, we need to show that $\sigma t' < m - (c + c_{case})$ or $\sigma t' + (c + c_{case}) < m$.
By b), we know that

$$c \leq \sigma t \tag{2}$$

By adding $\sigma t' + c_{case}$ to both sides of (2), we have

$$c + \sigma t' + c_{case} \leq \sigma t + \sigma t' + c_{case} \tag{3}$$

By the main assumption, we know that $\sigma t + \sigma t' + c_{case} < m$.
Therefore, we know that $c + \sigma t' + c_{case} < m$. Now, we can unroll to obtain

d) $\gamma e_1[v/x] \Downarrow^{c_r} v_r$
e) $c_r \leq \sigma t'$
f) $(m - (c + c_r + c_{case}), v_r) \in [\![\sigma A]\!]_v$

Now, we can conclude as follows

1. By a) and d), we get
$$\frac{\gamma e \Downarrow^{c} \text{inl } v \qquad \gamma e_1[v/x] \Downarrow^{c_r} v_r}{\text{case } (\gamma e, x.\gamma e_1, y.\gamma e_2) \Downarrow^{c + c_r + c_{case}} v_r} \textbf{ case-inl}$$
2. By b) and e) $(c + c_r + c_{case}) \leq \sigma t + \sigma t' + c_{case}$
3. By f)

- Assume that $\dfrac{\gamma e \Downarrow^{c} \text{inl } v \quad (\star) \qquad \gamma e_1[v/x] \Downarrow^{c_r} v_r \quad (\diamond)}{\text{case } (\gamma e, x.\gamma e_1, y.\gamma e_2) \Downarrow^{c + c_r + c_{case}} v_r}$ **case-inl** and $c + c_r + c_{case} < m$.
By IH 2 on the first premise using

$$\text{FV}(e) \subseteq \text{dom}(\Omega') \text{ and } \Omega' \subseteq \Omega \text{ and } (m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$$

we get $(m, \gamma e) \in [\![\sigma A_1 + \sigma A_2]\!]_\varepsilon^{\sigma k, \sigma t}$.
Unrolling second part of its definition with $(\star)$ and $c < m$, we get

a) $\sigma t \leq c$
b) $(m - c, \text{inl } v) \in [\![\sigma A_1 + \sigma A_2]\!]_v$

By IH 2 on the second premise using $(m - c - c_{case}, \gamma[x \mapsto v]) \in \mathcal{G}[\![\sigma\Omega', x : \sigma A_1]\!]$ obtained by

- $\mathrm{FV}(e_1) \subseteq \mathrm{dom}(x : A_1, \Omega')$ and $x : A_1, \Omega' \subseteq x : A_1, \Omega$
- $(m - c - c_{case}), \gamma) \in \mathcal{G}[\![\sigma\Omega']\!]$ by downward-closure (Lemma 4) on $(m, \gamma) \in \mathcal{G}[\![\sigma\Omega']\!]$ using $m - c - c_{case} \leq m$
- $(m - c - c_{case}, v) \in [\![\sigma A_1]\!]_v$ by downward closure (Lemma 4) on c), and unfolding its definition

we get

$$(m - c - c_{case}, \gamma e_1[v/x]) \in [\![\sigma A]\!]_\varepsilon^{\sigma k', \sigma t'} \tag{4}$$

By unrolling second part of (4)'s definition using ($\diamond$) and $c_r < m - c - c_{case}$, we get

c) $\sigma t' \leq c_r$

d) $(m - (c + c_r + c_{case}), v_r) \in [\![\sigma A]\!]_v$

Now, we can conclude as follows

1. By a) and c) $\sigma k' + \sigma t' + c_{case} \leq (c + c_r + c_{case})$
2. By d)

Case $$\frac{\Delta; \Phi; \Omega \vdash_k^t e : \mathbf{list}[n]\, A \qquad \Delta; \Phi \wedge n = 0; \Omega \vdash_{k'}^{t'} e_1 : A' \qquad i, \Delta; \Phi \wedge n = i + 1; h : A, tl : \mathbf{list}[i]\, A, \Omega \vdash_{k'}^{t'} e_2 : A'}{\Delta; \Phi; \Omega \vdash_{k+k'+c_{caseL}}^{t+t'+c_{caseL}} \mathbf{case}\ e\ \mathbf{of\ nil}\ \rightarrow e_1 \mid h :: tl \rightarrow e_2 : A'}\ \mathbf{caseL}$$

Assume that $\models \sigma\Phi$ and there exists Ω' s.t. $\mathrm{FV}(\text{ case } e \text{ of nil } \rightarrow e_1 \mid h :: tl \rightarrow e_2) \subseteq \mathrm{dom}(\Omega')$ and $\Omega' \subseteq \Omega$ and
TS: $(m, \text{ case } \gamma e \text{ of nil } \rightarrow \gamma e_1 \mid h :: tl \rightarrow \gamma e_2) \in [\![\sigma A']\!]_\varepsilon^{\sigma k + \sigma k' + c_{caseL}, \sigma t + \sigma t' + c_{caseL}}$.
Following the definition of $[\![\cdot]\!]_\varepsilon^{\cdot,\cdot}$, there are two parts to show:

- Assume that $\sigma t + \sigma t' + c_{caseL} < m$.
  By IH 2 on the first premise using

$$\mathrm{FV}(e) \subseteq \mathrm{dom}(\Omega') \text{ and } \Omega' \subseteq \Omega \text{ and } (m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$$

we get $(m, \gamma e) \in [\![\mathrm{list}[\sigma n]\, \sigma A]\!]_\varepsilon^{\sigma k, \sigma t}$. Unrolling its definition with $\sigma t < m$, we get

a) $\gamma e \Downarrow^c v$

b) $c \leq \sigma t$

c) $(m - c, v) \in [\![\mathrm{list}[\sigma n]\, \sigma A]\!]_v$

Depending on what γe evaluates to, there are two cases.

subcase 1: $\gamma e \Downarrow^c \mathrm{nil}$
By c), $\sigma n = 0$ since $v = \mathrm{nil}$.
Then, we can instantiate IH 2 on the second premise using

- $\mathrm{FV}(e) \subseteq \mathrm{dom}(\Omega')$ and $\Omega' \subseteq \Omega$ and $(m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$
- $\models \sigma\Phi \wedge \sigma n \doteq 0$ obtained by combining $\models \sigma\Phi$ with $\models \sigma n \doteq 0$

we get $(m, \gamma e_1) \in [\![\sigma A']\!]_\varepsilon^{\sigma k', \sigma t'}$.
Unrolling its definition using $\sigma t' < m$, we get

d) $\gamma e_1 \Downarrow^{c_r} v_r$
e) $c_r \leq \sigma t'$
f) $(m - c_r, v_r) \in [\![\sigma A']\!]_v$

We conclude with

1. By a) and d), we get $\dfrac{\gamma e \Downarrow^c \text{nil} \qquad \gamma e_1 \Downarrow^{c_r} v_r}{\text{case } \gamma e \text{ of nil } \to \gamma e_1 \mid h :: tl \to \gamma e_2 \Downarrow^{c+c_r+c_{caseL}} v_r}$ **caseL-nil**
2. By b) and e), we get $c + c_r + c_{caseL} \leq \sigma t + \sigma t' + c_{caseL}$
3. By downward closure (Lemma 4) on f) using

$$m - (c + c_r + c_{caseL}) \leq m - c_r$$

we get $(m - (c + c_r + c_{caseL}), v_r) \in [\![\sigma A']\!]_v$.

subcase 2: $\gamma e \Downarrow^c \text{cons}(v_1, v_2)$

By c), $\sigma n = I + 1$ and we have

$$(m - c, v_1) \in [\![\sigma A]\!]_v \tag{1}$$

$$(m - c, v_2) \in [\![\text{list}[I]\, \sigma A]\!]_v \tag{2}$$

Then, we can instantiate IH 2 on the third premise using

- $\text{FV}(e_2) \subseteq \text{dom}(h : A, tl : \text{list}[i]\, A, \Omega')$ and $h : A, tl : \text{list}[i]\, A, \Omega' \subseteq h : A, tl : \text{list}[i]\, A, \Omega$
- $\sigma[i \mapsto I] \in \mathcal{D}[\![i :: \mathbb{N}, \Delta]\!]$
- $\models \sigma[i \mapsto I](\Phi \wedge n \doteq i + 1)$ obtained by combining $\models \sigma\Phi$ with $\models \sigma n \doteq I + 1$,
- $(m - c, \gamma[h \mapsto v_1, tl \mapsto v_2]) \in \mathcal{G}[\![\sigma[i \mapsto I](\Omega', x : A, tl : \text{list}[i]\, A)]\!]$ using (1) and (2) and $(m - c, \gamma) \in \mathcal{G}[\![\sigma\Omega']\!]$ (obtained by downward closure (Lemma 4)).

we get $(m - c, \gamma e_2[v_1/h, v_2/tl]) \in [\![\sigma[i \mapsto I]A]\!]_\varepsilon^{\sigma[i \mapsto I]k', \sigma[i \mapsto I]t'}$.
Since, $i \notin FV(k', t', A, A')$, we have $(m - c, \gamma e_2[v_1/h, v_2/tl]) \in [\![\sigma A']\!]_\varepsilon^{\sigma k', \sigma t'}$.
To unroll its definition, we need to show that $\sigma t' < m - c$ or $c + \sigma t' < m$.
By b), we know that

$$c \leq \sigma t \tag{3}$$

By adding $\sigma t'$ to both sides of (3), we have

$$c + \sigma t' \leq \sigma t + \sigma t' \tag{4}$$

By the main assumption, we know that $\sigma t + \sigma t' + c_{case} < m$.
Therefore, we know that $c + \sigma t' \leq \sigma t + \sigma t' < \sigma t + \sigma t' + c_{caseL} < m$. Now, we can unroll to obtain

g) $\gamma e_1[v_1/h, v_2/tl] \Downarrow^{c_r} v_r$
h) $c_r \leq \sigma t'$
i) $(m - c - c_r, v_r) \in [\![\sigma A']\!]_v$

We conclude with

1. By a) and g), we get $\dfrac{\gamma e \Downarrow^c \text{cons}(v_1, v_2) \qquad \gamma e_2[v_1/h, v_2/tl] \Downarrow^{c_r} v_r}{\text{case } \gamma e \text{ of nil } \to \gamma e_1 \mid h :: tl \to \gamma e_2 \Downarrow^{c+c_r+c_{caseL}} v_r}$ **caseL-cons**

2. By b) and h), we get $c + c_r + c_{caseL} \leq \sigma t + \sigma t' + c_{caseL}$
3. By downward closure (Lemma 4) on i) using

$$m - c - c_r - c_{caseL} \leq m - c - c_r,$$

we get $(m - (c + c_r + c_{caseL}), v_r) \in [\![\sigma A']\!]_v$.

- Assume that case γe of nil $\to \gamma e_1 \mid h :: tl \to \gamma e_2 \Downarrow^C v_r$ and $C < m$. Depending on what γe evaluates to, there are two cases.

subcase 1: $$\frac{\gamma e \Downarrow^c \text{nil} \quad (\star) \qquad \gamma e_1 \Downarrow^{c_r} v_r \quad (\diamond)}{\text{case } \gamma e \text{ of nil } \to \gamma e_1 \mid h :: tl \to \gamma e_2 \Downarrow^{c + c_r + c_{caseL}} v_r} \textbf{caseL-nil}$$

By IH 2 on the first premise using

$$\text{FV}(e) \subseteq \text{dom}(\Omega') \text{ and } \Omega' \subseteq \Omega \text{ and } (m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$$

we get $(m, \gamma e) \in [\![\text{list}[\sigma n]\, \sigma A]\!]_\varepsilon^{\sigma k, \sigma t}$. Unrolling its definition with $(\star)$ and $c < m$, we get

a) $\sigma k \leq c$
b) $(m - c, \text{nil}) \in [\![\text{list}[\sigma n]\, \sigma A]\!]_v$

By b), $\sigma n = 0$ since $v = \text{nil}$.
Then, we can instantiate IH 2 on the second premise using

- $\text{FV}(e_1) \subseteq \text{dom}(\Omega')$ and $\Omega' \subseteq \Omega$ and $(m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$
- $\models \sigma\Phi \wedge \sigma n \doteq 0$ obtained by combining $\models \sigma\Phi$ with $\models \sigma n \doteq 0$

we get $(m, \gamma e_1) \in [\![\sigma A']\!]_\varepsilon^{\sigma k', \sigma t'}$.
Unrolling its definition using $(\diamond)$ and $c_r < m$, we get

c) $\sigma k' \leq c_r$
d) $(m - c_r, v_r) \in [\![\sigma A']\!]_v$

We conclude with

1. By a) and c), we get $\sigma k + \sigma k' + c_{caseL} \leq c + c_r + c_{caseL}$
2. By downward closure (Lemma 4) on d) using

$$m - c - c_r - c_{caseL} \leq m - c - c_r$$

we get $(m - (c + c_r + c_{caseL}), v_r) \in [\![\sigma A']\!]_v$.

subcase 2: $$\frac{\gamma e \Downarrow^c \text{cons}(v_1, v_2) \quad (\star) \qquad \gamma e_2[v_1/h, v_2/tl] \Downarrow^{c_r} v_r \quad (\diamond\diamond)}{\text{case } \gamma e \text{ of nil } \to \gamma e_1 \mid h :: tl \to \gamma e_2 \Downarrow^{c + c_r + c_{caseL}} v_r} \textbf{caseL-cons}$$

By IH 2 on the first premise, we get $(m, \gamma e) \in [\![\text{list}[\sigma n]\, \sigma A]\!]_\varepsilon^{\sigma k, \sigma t}$. Unrolling its definition with $(\star)$ and $c < m$, we get

a) $\sigma t \leq c$
b) $(m - c, \text{cons}(v_1, v_2)) \in [\![\text{list}[\sigma n]\, \sigma A]\!]_v$

By b), $\sigma n = I + 1$ for some I and we have

$$(m - c, v_1) \in [\![\sigma A]\!]_v \tag{5}$$

$$(m - c, v_2) \in \llbracket \text{list}[I]\, \sigma A \rrbracket_v \tag{6}$$

Then, we can instantiate IH 2 on the third premise using

- $\text{FV}(e_2) \subseteq \text{dom}(h : A, tl : \text{list}[i]\, A, \Omega')$ and $h : A, tl : \text{list}[i]\, A, \Omega' \subseteq h : A, tl : \text{list}[i]\, A, \Omega$
- $\sigma[i \mapsto I] \in \mathcal{D}\llbracket i :: \mathbb{N}, \Delta \rrbracket$
- $\models \sigma[i \mapsto I](\Phi \wedge n \doteq i + 1)$ obtained by combining $\models \sigma\Phi$ with $\models \sigma n \doteq I + 1$,
- $(m - c, \gamma[h \mapsto v_1, tl \mapsto v_2]) \in \mathcal{G}\llbracket \sigma[i \mapsto I](\Omega', x : A, tl : \text{list}[i]\, A) \rrbracket$ using (5) and (6) and $(m - c, \gamma) \in \mathcal{G}\llbracket \sigma\Omega' \rrbracket$ (obtained by downward closure (Lemma 4)).

we get $(m, \gamma e_2[v_1/h, v_2/tl]) \in \llbracket \sigma[i \mapsto I] A \rrbracket_\varepsilon^{\sigma[i \mapsto I]k', \sigma[i \mapsto I]t'}$.
Since, $i \notin FV(k', t', A, A')$, we have $(m, \gamma e_2[v_1/h, v_2/tl]) \in \llbracket \sigma A' \rrbracket_\varepsilon^{\sigma k', \sigma t'}$.
Unrolling its definition using ($\diamond\diamond$) and $c_r < m - c$, we get

c) $\sigma t' \leq c_r$
d) $(m - c - c_r, v_r) \in \llbracket \sigma A' \rrbracket_v$

We conclude with

1. By a) and c), we get $\sigma k + \sigma k' + c_{caseL} \leq c + c_r + c_{caseL}$
2. By downward closure (Lemma 4) on d) using

$$m - c - c_r - c_{caseL} \leq m - c - c_r$$

we get $(m - (c + c_r + c_{caseL}), v_r) \in \llbracket \sigma A' \rrbracket_v$.

Case $\dfrac{\Delta; \Phi; \Omega \vdash_k^t e : A\{I/i\} \qquad \Delta \vdash I :: S}{\Delta; \Phi; \Omega \vdash_k^t \textbf{pack } e : \exists i{::}S.\, A}$ **pack**

Assume that $\models \sigma\Phi$ and there exists Ω' s.t. $\text{FV}(\text{pack } e) \subseteq \text{dom}(\Omega')$ and $\Omega' \subseteq \Omega$ and $(m, \delta) \in \mathcal{G}\llbracket \sigma\Omega' \rrbracket$
TS: $(m, \text{pack } \gamma e) \in \llbracket \exists i{::}S.\, A \rrbracket_\varepsilon^{\sigma k, \sigma t}$.
Following the definition of $\llbracket \cdot \rrbracket_\varepsilon^{\cdot, \cdot}$, there are two parts to show:

- Assume that $\sigma t < m$.
  By IH 2 on the first premise using

$$\text{FV}(e) \subseteq \text{dom}(\Omega') \text{ and } \Omega' \subseteq \Omega \text{ and } (m, \delta) \in \mathcal{G}\llbracket \sigma\Omega' \rrbracket$$

  we get $(m, \gamma e) \in \llbracket \sigma A\{\sigma I/i\} \rrbracket_\varepsilon^{\sigma k, \sigma t}$.
  Unrolling its definition with $\sigma t < m$, we get

  a) $\gamma e \Downarrow^c v$
  b) $c \leq \sigma t$
  c) $(m - c, v) \in \llbracket \sigma A\{\sigma I/i\} \rrbracket_v$

  Then we can conclude as follows:

  1. By a), we get $\dfrac{\gamma e \Downarrow^c v}{\text{pack } \gamma e \Downarrow^c \text{pack } v}$ **pack**
  2. By b), $c \leq \sigma t$

3. TS: $(m - c, \text{pack } v) \in [\![\exists i{::}S.\, A]\!]_v$.
   By Lemma 6 on the second premise we know that $\vdash \sigma I :: S$.
   STS: $(m - c, v) \in [\![\sigma A\{\sigma I/i\}]\!]_v$.
   This follows by c).

- Assume that
  $$\frac{\gamma e \Downarrow^c v \quad (\star)}{\text{pack } \gamma e \Downarrow^c \text{pack } v}\ \textbf{pack}$$ and $c < m$.
  By IH 2 on the first premise using

  $$\text{FV}(e) \subseteq \text{dom}(\Omega') \text{ and } \Omega' \subseteq \Omega \text{ and } (m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$$

  we get $(m, \gamma e) \in [\![\sigma A\{\sigma I/i\}]\!]_\varepsilon^{\sigma k, \sigma t}$.
  Unrolling its definition with $(\star)$ and $c < m$, we get

  a) $\sigma k \leq c$
  b) $(m - c, v) \in [\![\sigma A\{\sigma I/i\}]\!]_v$

  Then we can conclude as follows:

  1. By a), $\sigma k \leq c$
  2. TS: $(m - c, \text{pack } v) \in [\![\exists i{::}S.\, A]\!]_v$.
     By Lemma 6 on the second premise we know that $\vdash \sigma I :: S$.
     STS: $(m - c, v) \in [\![\sigma A\{\sigma I/i\}]\!]_v$.
     This follows by b).

Case $$\frac{\Delta; \Phi; \Omega \vdash^{t_1}_{k_1} e_1 : A_1 \qquad \Delta; \Phi; x : A_1, \Omega \vdash^{t_2}_{k_2} e_2 : A_2}{\Delta; \Phi; \Omega \vdash^{t_1+t_2+c_{let}}_{k_1+k_2+c_{let}} \textbf{let } x = e_1 \textbf{ in } e_2 : A_2}\ \textbf{let}$$
Assume that $\models \sigma\Phi$ and there exists Ω' s.t. $\text{FV}(\text{let } x = e_1 \text{ in } e_2) \subseteq \text{dom}(\Omega')$ and $\Omega' \subseteq \Omega$ and $(m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$
TS: $(m, \text{let } x = \gamma e_1 \text{ in } \gamma e_2) \in [\![\sigma A_2]\!]_\varepsilon^{\sigma k_1 + \sigma k_2 + c_{let}, \sigma t_1 + \sigma t_2 + c_{let}}$.
Following the definition of $[\![\cdot]\!]_\varepsilon^{\cdot,\cdot}$, there are two parts to show

- Assume that $\sigma t_1 + \sigma t_2 + c_{let} < m$.
  By IH 2 on the first premise using

  $$\text{FV}(e_1) \subseteq \text{dom}(\Omega') \text{ and } \Omega' \subseteq \Omega \text{ and } (m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$$

  we get $(m, \gamma e_1) \in [\![\sigma A_1]\!]_\varepsilon^{\sigma k_1, \sigma t_1}$. Unrolling its definition with $\sigma t_1 < m$, we get

  a) $\gamma e_1 \Downarrow^{c_1} v_1$
  b) $c_1 \leq \sigma t_1$
  c) $(m - c_1, v_1) \in [\![\sigma A_1]\!]_v$

  By IH 2 on the second premise using $(m - c_1 - c_{let}, \gamma[x \mapsto v]) \in \mathcal{G}[\![\sigma\Omega', x : \sigma A_1]\!]$ obtained by

  - $\text{FV}(e_2) \subseteq \text{dom}(x : A_1, \Omega')$ and $x : A_1, \Omega' \subseteq x : A_1, \Omega$
  - $(m - c_1 - c_{let}, \gamma) \in \mathcal{G}[\![\sigma\Omega']\!]$ by downward closure (Lemma 4) on $(m, \gamma) \in \mathcal{G}[\![\sigma\Omega']\!]$ using $m - c_1 - c_{let} \leq m$

– $(m - c_1 - c_{let}, v) \in [\![\sigma A_1]\!]_v$ by downward closure (Lemma 4) on c)

we get

$$(m - c_1 - c_{let}, \gamma e_1[v/x]) \in [\![\sigma A_2]\!]_\varepsilon^{\sigma k_2, \sigma t_2} \tag{1}$$

To unroll the first part of (1)'s definition, we need to show that $\sigma t_2 < m - (c + c_{let})$ or $\sigma t_2 + (c + c_{let}) < m$.
By b), we know that

$$c \leq \sigma t_1 \tag{2}$$

By adding $\sigma t_2 + c_{let}$ to both sides of (2), we have

$$c + \sigma t_2 + c_{let} \leq \sigma t_1 + \sigma t_2 + c_{let} \tag{3}$$

By the main assumption, we know that $\sigma t_1 + \sigma t_2 + c_{let} < m$.
Therefore, we know that $c + \sigma t_2 + c_{let} < m$. Now, we can unroll to obtain

d) $\gamma e_1[v/x] \Downarrow^{c_r} v_r$
e) $c_r \leq \sigma t_2$
f) $(m - (c_1 + c_2 + c_{let}), v_r) \in [\![\sigma A]\!]_v$

Now, we can conclude as follows

1. By a) and d), we get
$$\dfrac{\gamma e_1 \Downarrow^{c_1} v_1 \qquad \gamma e_2[v_1/x] \Downarrow^{c_r} v_r}{\text{let } x = \gamma e_1 \text{ in } \gamma e_2 \Downarrow^{c_1 + c_r + c_{let}} v_r}\ \textbf{let}$$
2. By b) and e) $(c_1 + c_r + c_{let}) \leq \sigma t_1 + \sigma t_2 + c_{let}$
3. By f)

- Assume that $\dfrac{\gamma e_1 \Downarrow^{c_1} v_1 \quad (\star) \qquad \gamma e_2[v_1/x] \Downarrow^{c_r} v_r \quad (\diamond)}{\text{let } x = \gamma e_1 \text{ in } \gamma e_2 \Downarrow^{c_1 + c_r + c_{let}} v_r}\ \textbf{let}$ and $c_1 + c_r + c_{let} < m$.
By IH 2 on the first premise using

$$\text{FV}(e_1) \subseteq \text{dom}(\Omega') \text{ and } \Omega' \subseteq \Omega \text{ and } (m, \delta) \in \mathcal{G}[\![\sigma \Omega']\!]$$

we get $(m, \gamma e_1) \in [\![\sigma A_1]\!]_\varepsilon^{\sigma k_1, \sigma t_1}$. Unrolling its definition with $(\star)$ and $c_1 < m$, we get

a) $\sigma k_1 \leq c_1$
b) $(m - c_1, v_1) \in [\![\sigma A_1]\!]_v$

By IH 2 on the second premise using $(m - c_1 - c_{let}, \gamma[x \mapsto v]) \in \mathcal{G}[\![\sigma \Omega', x : \sigma A_1]\!]$ obtained by

– $\text{FV}(e_2) \subseteq \text{dom}(x : A_1, \Omega')$ and $x : A_1, \Omega' \subseteq x : A_1, \Omega$
– $(m - c_1 - c_{let}, \gamma) \in \mathcal{G}[\![\sigma \Omega']\!]$ by downward closure (Lemma 4) on $(m, \gamma) \in \mathcal{G}[\![\sigma \Omega']\!]$ using $m - c_1 - c_{let} \leq m$
– $(m - c_1 - c_{let}, v) \in [\![\sigma A_1]\!]_v$ by downward closure (Lemma 4) on c)

we get

$$(m - c_1 - c_{let}, \gamma e_1[v/x]) \in [\![\sigma A_2]\!]_\varepsilon^{\sigma k_2, \sigma t_2} \tag{4}$$

Unrolling (4)'s definition using $(\diamond)$ and $c_r < m - c_1 - c_{let}$, we get

c) $\sigma k_2 \leq c_r$

d) $(m - (c_1 + c_2 + c_{let}), v_r) \in [\![\sigma A]\!]_v$

Now, we can conclude as follows

1. By a) and c) $\sigma k_1 + \sigma k_2 + c_{let} \leq (c_1 + c_r + c_{let})$
2. By d)

Case $\dfrac{\Upsilon(\zeta) = A_1 \xrightarrow{\mathbf{exec}(k,t)} A_2 \qquad \Delta; \Phi; \Omega \vdash^{t'}_{k'} e : A_1}{\Delta; \Phi; \Omega \vdash^{t+t'+c_{app}}_{k+k'+c_{app}} \zeta\ e : A_2}$ **primapp**

Assume that $\models \sigma\Phi$ and there exists Ω' s.t. $\mathrm{FV}(\zeta\ e) \subseteq \mathrm{dom}(\Omega')$ and $\Omega' \subseteq \Omega$ and $(m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$

TS: $(m, \zeta\ \gamma e) \in [\![\sigma A_2]\!]_{\varepsilon}^{\sigma k + \sigma k' + c_{app}, \sigma t + \sigma t' + c_{app}}$.

Following the definition of $[\![\cdot]\!]_{\varepsilon}^{\cdot,\cdot}$, there are two parts to show

- Assume that $\sigma t + \sigma t' + c_{app} < m$.
  By IH 2 on the second premise using

  $$\mathrm{FV}(e) \subseteq \mathrm{dom}(\Omega') \text{ and } \Omega' \subseteq \Omega \text{ and } (m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$$

  we get $(m, \gamma e) \in [\![\sigma A_1]\!]_{\varepsilon}^{\sigma k', \sigma t'}$. Unrolling its definition with $\sigma t' < m$, we get

  a) $\gamma e \Downarrow^c v$

  b) $\sigma k' \leq c \leq \sigma t'$

  c) $(m - c, v) \in [\![\sigma A_1]\!]_v$

  Next, by Assumption (12) using $\zeta : \sigma A_1 \xrightarrow{\mathrm{exec}(\sigma k, \sigma t)} \sigma A_2$ (obtained by substitution on the first premise) and c), we get

  d) $\hat{\zeta}(v) = (c_r, v_r)$

  e) $\sigma k \leq c_r \leq \sigma t$

  f) $(m - c, v_r) \in [\![\sigma A_2]\!]_v$

  Now, we can conclude as follows:

  1. Using a) and d), we get $\dfrac{\gamma e \Downarrow^c v \qquad \hat{\zeta}(v) = (c_r, v_r)}{\zeta\ \gamma e \Downarrow^{c + c_r + c_{app}} v_r}$ **primapp**
  2. Using b) and e), we get $\sigma k + \sigma k' + c_{app} \leq (c + c_r + c_{app}) \leq \sigma t + \sigma t' + c_{app}$
  3. By downward closure (Lemma 4) on f) using

     $$m - (c + c_r + c_{app}) \leq m - c$$

     we get $(m - (c + c_r + c_{app}), v_r) \in [\![\sigma A_2]\!]_v$

Case $\dfrac{\Delta; \Phi; \Omega \vdash^{t}_{k} e : A \qquad \Delta; \Phi \models A \sqsubseteq A' \qquad \Delta; \Phi \models k' \leq k \qquad \Delta; \Phi \models t \leq t'}{\Delta; \Phi; \Omega \vdash^{t'}_{k'} e : A'}$ $\sqsubseteq$ **exec**

Assume that $\models \sigma\Phi$ and there exists Ω' s.t. $\mathrm{FV}(e) \subseteq \mathrm{dom}(\Omega')$ and $\Omega' \subseteq \Omega$ and $(m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$

TS: $(m, \gamma e) \in [\![\sigma A']\!]_{\varepsilon}^{\sigma k', \sigma t'}$.

Following the definition of $[\![\cdot]\!]_{\varepsilon}^{\cdot,\cdot}$, there are two parts to show

subcase 1: Assume that $\sigma t' < m$.

By Assumption (13) on the fourth premise, we get

$$\sigma t \leq \sigma t' \tag{1}$$

By IH 2 on the first premise using

$$\mathrm{FV}(e) \subseteq \mathrm{dom}(\Omega') \text{ and } \Omega' \subseteq \Omega \text{ and } (m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$$

we get $(m, \gamma e) \in [\![\sigma A]\!]_{\varepsilon}^{\sigma k', \sigma t'}$. Unrolling its definition with $\sigma t \leq \sigma t' < m$, we get

a) $\gamma e \Downarrow^{c} v$
b) $c \leq \sigma t$
c) $(m - c, v) \in [\![\sigma A]\!]_{v}$

We can conclude this subcase

1. By a)
2. By b) and (1), we get $c - c' \leq \sigma t'$
3. By Lemma 5 on the second premise using c), we get $(m - c, v) \in [\![\sigma A']\!]_{v}$

subcase 2: Assume that

a) $\gamma e \Downarrow^{c} v$
b) $c < m$.

By IH 2 on the first premise using

$$\mathrm{FV}(e) \subseteq \mathrm{dom}(\Omega') \text{ and } \Omega' \subseteq \Omega \text{ and } (m, \delta) \in \mathcal{G}[\![\sigma\Omega']\!]$$

we get $(m, \gamma e) \in [\![\sigma A]\!]_{\varepsilon}^{\sigma k', \sigma t'}$. Unrolling its definition with a) and $c < m$, we get

c) $\sigma k \leq c$
d) $(m - c, v) \in [\![\sigma A]\!]_{v}$

We can conclude this subcase

1. By Assumption (13) on the third premise, we get $\sigma k' \leq \sigma k$. By c) we know $\sigma k \leq c$, therefore we get $\sigma k' \leq c$
2. By Lemma 5 on the second premise using c), we get $(m - c, v) \in [\![\sigma A']\!]_{v}$

□

Proof of Statement (3). Remember the statement (3) of Theorem 14:
Assume that $\Delta; \Phi; \Gamma \vdash e_1 \ominus e_2 \lesssim t : \tau$ and $\sigma \in \mathcal{D}[\![\Delta]\!]$ and $\models \sigma\Phi$. Then for $i \in \{1, 2\}$, if there exists Γ'_i s.t. $\mathrm{FV}(e_i) \subseteq \mathrm{dom}(\Gamma'_i)$ and $\Gamma'_i \subseteq \Gamma$ and $(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'_i|_i]\!]$, then $(m, \delta e_i) \in [\![|\sigma\tau|_i]\!]_{\varepsilon}^{0,\infty}$.

For the structural rules, we will only show the left side since the right side is similar. For asynchronous rules, we first show the left side and then the right side in the same case.

Case $\dfrac{\Gamma(x) = \tau}{\Delta; \Phi; \Gamma \vdash x \ominus x \lesssim 0 : \tau}$ **r-var**

Assume that $\models \sigma\Phi$ and there exists Γ' s.t. $\mathrm{FV}(x) \subseteq \mathrm{dom}(\Gamma')$ and $\Gamma' \subseteq \Gamma$ and $(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_1]\!]$.
TS: $(m, \delta(x)) \in [\![|\sigma\tau|_1]\!]_\varepsilon^{0,\infty}$.
By Lemma 2, STS: $(m, \delta(x)) \in [\![|\sigma\tau|_1]\!]_v$.
By $(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_1]\!]$ and $x \in \mathrm{dom}(\Gamma')$, we can conclude that $(m, \delta(x)) \in [\![|\sigma\tau|_1]\!]_v$.

Case $\dfrac{\Delta; \Phi \vdash \tau_1 \xrightarrow{\mathbf{diff}(t)} \tau_2 \ \mathtt{wf} \qquad \Delta; \Phi; x : \tau_1, f : \tau_1 \xrightarrow{\mathbf{diff}(t)} \tau_2, \Gamma \vdash e_1 \ominus e_2 \lesssim t : \tau_2}{\Delta; \Phi; \Gamma \vdash \mathbf{fix}\ f(x).e_1 \ominus \mathbf{fix}\ f(x).e_2 \lesssim 0 : \tau_1 \xrightarrow{\mathbf{diff}(t)} \tau_2}$ **r-fix**

Assume that $\models \sigma\Phi$ and there exists Γ' s.t. $\mathrm{FV}(\mathrm{fix}\ f(x).e) \subseteq \mathrm{dom}(\Gamma')$ and $\Gamma' \subseteq \Gamma$ and $(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_1]\!]$.
TS: $(m, \mathrm{fix}\ f(x).\delta e_1) \in [\![|\sigma\tau_1|_1 \xrightarrow{\mathrm{exec}(0,\infty)} |\sigma\tau_2|_1]\!]_\varepsilon^{0,\infty}$.
By Lemma 2, STS: $(m, \mathrm{fix}\ f(x).\delta e_1) \in [\![|\sigma\tau_1|_1 \xrightarrow{\mathrm{exec}(0,\infty)} |\sigma\tau_2|_1]\!]_v$.
We prove the more general statement

$$\forall m' \leq m.\ (m', \mathrm{fix}\ f(x).\delta e_1) \in [\![|\sigma\tau_1|_1 \xrightarrow{\mathrm{exec}(0,\infty)} |\sigma\tau_2|_1]\!]_v$$

by subinduction on m'.
There are two cases:

subcase 1: $m' = 0$
Since there is no non-negative j such that $j < 0$, the goal is vacuously true.

subcase 2: $m' = m'' + 1 \leq m$
By sub-IH

$$(m'', \mathrm{fix}\ f(x).\delta e_1) \in [\![|\sigma\tau_1|_1 \xrightarrow{\mathrm{exec}(0,\infty)} |\sigma\tau_2|_1]\!]_v \tag{1}$$

STS: $(m'' + 1, \mathrm{fix}\ f(x).\delta e_1) \in [\![|\sigma\tau_1|_1 \xrightarrow{\mathrm{exec}(0,\infty)} |\sigma\tau_2|_1]\!]_v$.
Pick $j < m'' + 1$ and assume that $(j, v) \in [\![|\sigma\tau_1|_1]\!]_v$.
STS: $(j, \delta e_1[v/x, (\mathrm{fix}\ f(x).\delta e_1)/f]) \in [\![|\sigma\tau_2|_1]\!]_\varepsilon^{0,\infty}$.
This follows by IH 3 on the premise instantiated with

- $(j, \delta[x \mapsto v, f \mapsto (\mathrm{fix}\ f(x).\delta e_1)]) \in \mathcal{G}[\![x : |\sigma\tau_1|_1, f : |\sigma\tau_1|_1 \xrightarrow{\mathrm{exec}(0,\infty)} |\sigma\tau_2|_1, |\sigma\Gamma|_1]\!]$ which holds because
  - $\mathrm{FV}(e_1) \subseteq \mathrm{dom}(x : \tau_1, f : \tau_1 \xrightarrow{\mathrm{diff}(t)} \tau_2), \Gamma'$ and $x : \tau_1, f : \tau_1 \xrightarrow{\mathrm{diff}(t)} \tau_2, \Gamma' \subseteq x : \tau_1, f : \tau_1 \xrightarrow{\mathrm{diff}(t)} \tau_2, \Gamma$
  - $(j, \delta) \in \mathcal{G}[\![|\sigma\Gamma|_1]\!]$ using downward closure (Lemma 4) on $(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma|_1]\!]$ using $j < m'' + 1 \leq m$.
  - $(j, v) \in [\![|\sigma\tau_1|_1]\!]_v$, from the assumption above
  - $(j, \mathrm{fix}\ f(x).\delta e_1) \in [\![|\sigma\tau_1|_1 \xrightarrow{\mathrm{exec}(0,\infty)} |\sigma\tau_2|_1]\!]_v$, obtained by downward closure (Lemma 4) on (1) using $j \leq m''$

Case $\dfrac{\begin{array}{c}\Delta; \Phi; \Gamma \vdash e_1 \ominus e_1' \lesssim t_1 : \tau_1 \xrightarrow{\mathbf{diff}(t)} \tau_2 \\ \Delta; \Phi; \Gamma \vdash e_2 \ominus e_2' \lesssim t_2 : \tau_1\end{array}}{\Delta; \Phi; \Gamma \vdash e_1\ e_2 \ominus e_1'\ e_2' \lesssim t_1 + t_2 + t : \tau_2}$ **r-app**

Assume that $\models \sigma\Phi$ and there exists Γ' s.t. $\mathrm{FV}(e_1\ e_2) \subseteq \mathrm{dom}(\Gamma')$ and $\Gamma' \subseteq \Gamma$ and $(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_1]\!]$.

TS: $(m, \delta e_1\ \delta e_2) \in [\![|\sigma\tau_2|_1]\!]_\varepsilon^{0,\infty}$.
Following the definition of $[\![\cdot]\!]_\varepsilon^{\cdot,\cdot}$, there are two cases:

- Assume that $\infty < m$.
  Since m is finite, this case is vacuously true.
- Assume that
  $\dfrac{\delta e_1 \Downarrow^{c_1} \text{fix } f(x).e \quad (\star) \qquad \delta e_2 \Downarrow^{c_2} v_2 \quad (\diamond) \qquad e[v_2/x, (\text{fix } f(x).e)/f] \Downarrow^{c_r} v_r \quad (\dagger)}{\delta e_1\ \delta e_2 \Downarrow^{c_1+c_2+c_r+c_{app}} v_r}$ **app** and
  $c_1 + c_2 + c_r + c_{app} < m$.
  By IH 3 on the first premise using

  $$\text{FV}(e_1) \subseteq \text{dom}(\Gamma') \text{ and } \Gamma' \subseteq \Gamma \text{ and } (m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_1]\!]$$

  we get $(m, \delta e_1) \in [\![|\sigma\tau_1|_1 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_1]\!]_\varepsilon^{0,\infty}$.
  Unrolling its definition with $(\star)$ and $c_1 < m$, we get

  a) $0 \leq c_1$
  b) $(m - c_1, \text{fix } f(x).e) \in [\![|\sigma\tau_1|_1 \xrightarrow{\text{exec}(0,\infty)} |\sigma\tau_2|_1]\!]_v$

  By IH 3 on the second premise using

  $$\text{FV}(e_2) \subseteq \text{dom}(\Gamma') \text{ and } \Gamma' \subseteq \Gamma \text{ and } (m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_1]\!]$$

  we get $(m, \delta e_2) \in [\![|\sigma\tau_1|_1]\!]_\varepsilon^{0,\infty}$. Unrolling its definition with $(\diamond)$ and $c_2 < m$, we get

  c) $0 \leq c_2$
  d) $(m - c_2, v_2) \in [\![|\sigma\tau_1|_1]\!]_v$

  By downward closure (Lemma 4) on d) using $m - c_1 - c_2 - c_{app} \leq m - c_2$, we get

  $$(m - (c_1 + c_2 + c_{app}), v_2) \in [\![|\sigma\tau_1|_1]\!]_v \tag{1}$$

  Next, we unroll b) with (1) and $m - (c_1 + c_2 + c_{app}) < m - c_1$ to obtain

  $$(m - (c_1 + c_2 + c_{app}), e[v_2/x, (\text{fix } f(x).e)]) \in [\![|\sigma\tau_2|_1]\!]_\varepsilon^{0,\infty} \tag{2}$$

  By unrolling second part of (2)'s definition using $(\dagger)$ and $c_r < m - (c_1 + c_2 + c_{app})$, we get

  e) $0 \leq c_r$
  f) $(m - (c_1 + c_2 + c_r + c_{app}), v_r) \in [\![|\sigma\tau_2|_1]\!]_v$

  Now, we can conclude as follows:

  1. We can trivially show $0 \leq (c_1 + c_2 + c_r + c_{app})$
  2. By f)

Case $\dfrac{\Delta; \Phi; \Gamma \vdash e_1 \ominus e_1' \lesssim t_1 : \tau \qquad \Delta; \Phi; \Gamma \vdash e_2 \ominus e_2' \lesssim t_2 : \textbf{list}[n]^\alpha\, \tau}{\Delta; \Phi; \Gamma \vdash \textbf{cons}(e_1, e_2) \ominus \textbf{cons}(e_1', e_2') \lesssim t_1 + t_2 : \textbf{list}[n+1]^{\alpha+1}\, \tau}$ **r-cons1**
Assume that $\models \sigma\Phi$ and there exists Γ' s.t. $\text{FV}(\text{cons}(e_1, e_2)) \subseteq \text{dom}(\Gamma')$ and $\Gamma' \subseteq \Gamma$ and $(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_1]\!]$.
TS: $(m, \text{cons}(\delta e_1, \delta e_2)) \in [\![|\text{list}[\sigma n + 1]^{\sigma\alpha+1}\, \sigma\tau|_1]\!]_\varepsilon^{0,\infty} \equiv [\![\text{list}[\sigma n + 1]\, |\sigma\tau|_1]\!]_\varepsilon^{0,\infty}$.
Following the definition of $[\![\cdot]\!]_\varepsilon^{\cdot,\cdot}$, there are two parts to show

- Assume that $\infty < m$.
This is vacuously true.

- Assume that $\dfrac{\delta e_1 \Downarrow^{c_1} v_1 \quad (\star) \qquad \delta e_2 \Downarrow^{c_2} v_2 \quad (\diamond)}{\mathrm{cons}(\delta e_1, \delta e_2) \Downarrow^{c_1+c_2} \mathrm{cons}(v_1, v_2)}$ **cons** and $c_1 + c_2 < m$.
By IH 3 on the first premise using

$$\mathrm{FV}(e_1) \subseteq \mathrm{dom}(\Gamma') \text{ and } \Gamma' \subseteq \Gamma \text{ and } (m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_1]\!]$$

we get $(m, \delta e_1) \in [\![|\sigma\tau|_1]\!]_\varepsilon^{0,\infty}$. Unrolling its definition with $(\star)$ and $c_1 < m$, we get

a) $0 \leq c_1$
b) $(m - c_1, v_1) \in [\![|\sigma\tau|_1]\!]_v$

By IH 3 on the second premise using

$$\mathrm{FV}(e_2) \subseteq \mathrm{dom}(\Gamma') \text{ and } \Gamma' \subseteq \Gamma \text{ and } (m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_1]\!]$$

we get $(m, \delta e_2) \in [\![|\mathrm{list}[\sigma n]^{\sigma\alpha}\, \sigma\tau|_1]\!]_\varepsilon^{0,\infty}$.
Unrolling its definition with $(\diamond)$ and $c_2 < m$, we get

c) $0 \leq c_2$
d) $(m - c_2, v_2) \in [\![\mathrm{list}[\sigma n]\, |\sigma\tau|_1]\!]_v$

Now, we can conclude as follows:

1. We can trivially show that $0 \leq (c_1 + c_2)$
2. By downward closure (Lemma 4) on b) and d), we get $(m - (c_1 + c_2), v_1) \in [\![|\sigma\tau|_1]\!]_v$ and $(m - (c_1 + c_2), v_2) \in [\![\mathrm{list}[\sigma n]\, |\sigma\tau|_1]\!]_v$, when combined, gives us $(m - (c_1 + c_2), \mathrm{cons}(v_1, v_2)) \in [\![\mathrm{list}[\sigma n + 1]\, |\sigma\tau|_1]\!]_v \equiv [\![|\mathrm{list}[\sigma n]^{\sigma\alpha+1}\, \sigma\tau|_1]\!]_v$

Case $\dfrac{\Delta; \Phi; \Gamma \vdash e_1 \ominus e_1' \lesssim t_1 : \square\, \tau \qquad \Delta; \Phi; \Gamma \vdash e_2 \ominus e_2' \lesssim t_2 : \mathbf{list}[n]^\alpha\, \tau}{\Delta; \Phi; \Gamma \vdash \mathbf{cons}(e_1, e_2) \ominus \mathbf{cons}(e_1', e_2') \lesssim t_1 + t_2 : \mathbf{list}[n+1]^\alpha\, \tau}$ **r-cons2**
Assume that $\models \sigma\Phi$ and there exists Γ' s.t. $\mathrm{FV}(\mathrm{cons}(e_1, e_2)) \subseteq \mathrm{dom}(\Gamma')$ and $\Gamma' \subseteq \Gamma$ and $(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_1]\!]$.
TS: $(m, \mathrm{cons}(\delta e_1, \delta e_2)) \in [\![|\mathrm{list}[\sigma n + 1]^{\sigma\alpha}\, \sigma\tau|_1]\!]_\varepsilon^{0,\infty} \equiv [\![\mathrm{list}[\sigma n + 1]\, |\sigma\tau|_1]\!]_\varepsilon^{0,\infty}$.
Following the definition of $[\![\cdot]\!]_\varepsilon^{\cdot,\cdot}$, there are two parts to show

- Assume that $\infty < m$.
This is vacuously true.

- Assume that $\dfrac{\delta e_1 \Downarrow^{c_1} v_1 \quad (\star) \qquad \delta e_2 \Downarrow^{c_2} v_2 \quad (\diamond)}{\mathrm{cons}(\delta e_1, \delta e_2) \Downarrow^{c_1+c_2} \mathrm{cons}(v_1, v_2)}$ **cons** and $c_1 + c_2 < m$.
By IH 3 on the first premise using

$$\mathrm{FV}(e_1) \subseteq \mathrm{dom}(\Gamma') \text{ and } \Gamma' \subseteq \Gamma \text{ and } (m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_1]\!]$$

we get $(m, \delta e_1) \in [\![|\square\, \sigma\tau|_1]\!]_\varepsilon^{0,\infty}$. Unrolling its definition with $(\star)$ and $c_1 < m$, we get

a) $0 \leq c_1$
b) $(m - c_1, v_1) \in [\![|\square\, \sigma\tau|_1]\!]_v \equiv [\![|\sigma\tau|_1]\!]_v$

By IH 3 on the second premise using

$$\mathrm{FV}(e_2) \subseteq \mathrm{dom}(\Gamma') \text{ and } \Gamma' \subseteq \Gamma \text{ and } (m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_1]\!]$$

we get $(m, \delta e_2) \in [\![|\mathrm{list}[\sigma n]^{\sigma\alpha}\, \sigma\tau|_1]\!]_{\varepsilon}^{0,\infty}$.
Unrolling its definition with ($\diamond$) and $c_2 < m$, we get

c) $0 \leq c_2$
d) $(m - c_2, v_2) \in [\![\mathrm{list}[\sigma n]\, |\sigma\tau|_1]\!]_v$

Now, we can conclude as follows:

1. We can trivially show that $0 \leq (c_1 + c_2)$
2. By downward closure (Lemma 4) on b) and d), we get $(m - (c_1 + c_2), v_1) \in [\![|\sigma\tau|_1]\!]_v$ and $(m - (c_1 + c_2), v_2) \in [\![\mathrm{list}[\sigma n]\, |\sigma\tau|_1]\!]_v$, when combined, gives us $(m - (c_1 + c_2), \mathrm{cons}(v_1, v_2)) \in [\![\mathrm{list}[\sigma n + 1]\, |\sigma\tau|_1]\!]_v \equiv [\![|\mathrm{list}[\sigma n]^{\sigma\alpha}\, \sigma\tau|_1]\!]_v$

Case $\dfrac{\Delta; \Phi; |\Gamma|_1 \vdash_{k_1}^{t_1} e_1 : A_1 \qquad \Delta; \Phi; |\Gamma|_2 \vdash_{k_2}^{t_2} e_2 : A_2}{\Delta; \Phi; \Gamma \vdash e_1 \ominus e_2 \lesssim t_1 - k_2 : U\,(A_1, A_2)}$ **switch**

The are two parts to show.

subcase 1: Assume that $\models \sigma\Phi$ and there exists Γ' s.t. $\mathrm{FV}(e_1) \subseteq \mathrm{dom}(\Gamma')$ and $\Gamma' \subseteq \Gamma$ and $(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_1]\!]$.
TS: $(m, \delta e_1) \in [\![|U\,(\sigma A_1, \sigma A_2)|_1]\!]_{\varepsilon}^{0,\infty} \equiv [\![\sigma A_1]\!]_{\varepsilon}^{0,\infty}$.
There are two parts to show.

- Assume that $\infty < m$.
  This is vacuously true.
- Assume that

  a) $\delta e_1 \Downarrow^{c_r} v_r$
  b) $c_r < m$.

  By IH 2 on the first premise using

  $$\mathrm{FV}(e_1) \subseteq \mathrm{dom}(|\Gamma'|_1) \text{ and } |\Gamma'|_1 \subseteq |\Gamma|_1 \text{ and } (m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_1]\!]$$

  we get $(m, \delta e_1) \in [\![\sigma A_1]\!]_{\varepsilon}^{\sigma k_1, \sigma t_1}$
  By unrolling second part of its definition with a) and b), we get

  a) $\sigma k_1 \leq c_r$
  b) $(m - c_r, v_r) \in [\![\sigma A_1]\!]_v$

  We can conclude as follows

  1. Trivially, $0 \leq c_r$
  2. By d)

subcase 2: Assume that $\models \sigma\Phi$ and there exists Γ' s.t. $\mathrm{FV}(e_2) \subseteq \mathrm{dom}(\Gamma')$ and $\Gamma' \subseteq \Gamma$ and $(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_2]\!]$.
TS: $(m, \delta e_2) \in [\![|U\,(\sigma A_1, \sigma A_2)|_2]\!]_{\varepsilon}^{0,\infty} \equiv [\![\sigma A_2]\!]_{\varepsilon}^{0,\infty}$.
There are two parts to show.

- Assume that $\infty < m$.
  This is vacuously true.
- Assume that
  a) $\delta e_2 \Downarrow^{c_r} v_r$
  b) $c_r < m$.

  By IH 2 on the second premise using

  $$\mathrm{FV}(e_2) \subseteq \mathrm{dom}(|\Gamma'|_2) \text{ and } |\Gamma'|_2 \subseteq |\Gamma|_2 \text{ and } (m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_2]\!]$$

  we get $(m, \delta e_2) \in [\![\sigma A_2]\!]_{\varepsilon}^{\sigma k_2, \sigma t_2}$
  By unrolling second part of its definition with a) and b), we get
  a) $\sigma k_2 \leq c_r$
  b) $(m - c_r, v_r) \in [\![\sigma A_2]\!]_v$

  We can conclude as follows
  1. Trivially, $0 \leq c_r$
  2. By d)

Case $$\frac{\Delta; \Phi; |\Gamma|_1 \vdash_{k_1}^{t_1} e_1 : A_1 \xrightarrow{\mathbf{exec}(k,t)} A_2 \qquad \Delta; \Phi; \Gamma \vdash e_2 \ominus e_2' \lesssim t_2 : U\,(A_1, A_2')}{\Delta; \Phi; \Gamma \vdash e_1\ e_2 \ominus e_2' \lesssim t_1 + t_2 + t + c_{app} : U\,(A_2, A_2')}\ \textbf{r-app-e}$$

There are two parts to show: left and right sides. We first show the left side.

subcase 1: Assume that $\models \sigma\Phi$ and there exists Γ' s.t. $\mathrm{FV}(e_1\ e_2) \subseteq \mathrm{dom}(\Gamma')$ and $\Gamma' \subseteq \Gamma$ and $(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_1]\!]$.
TS: $(m, \delta e_1\ \delta e_2) \in [\![|U\,(\sigma A_2, \sigma A_2')|_1]\!]_{\varepsilon}^{0,\infty} \equiv [\![\sigma A_2]\!]_{\varepsilon}^{0,\infty}$.
Following the definition of $[\![\cdot]\!]_{\varepsilon}^{\cdot,\cdot}$, there are two cases:

- Assume that $\infty < m$.
  Since m is finite, this case is vacuously true.
- Assume that

  $$\frac{\delta e_1 \Downarrow^{c_1} \text{fix } f(x).e \quad (\star) \qquad \delta e_2 \Downarrow^{c_2} v_2 \quad ({\diamond}) \qquad e[v_2/x, (\text{fix } f(x).e)/f] \Downarrow^{c_r} v_r \quad (\dagger)}{\delta e_1\ \delta e_2 \Downarrow^{c_1 + c_2 + c_r + c_{app}} v_r}\ \textbf{app}$$

  and $c_1 + c_2 + c_r + c_{app} < m$.
  By IH 2 on the first premise using

  $$\mathrm{FV}(e_1) \subseteq \mathrm{dom}(|\Gamma'|_1) \text{ and } |\Gamma'|_1 \subseteq |\Gamma|_1 \text{ and } (m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_1]\!]$$

  we get $(m, \delta e_1) \in [\![\sigma A_1 \xrightarrow{\mathrm{exec}(\sigma k, \sigma t)} \sigma A_2]\!]_{\varepsilon}^{\sigma k_1, \sigma t_1}$.
  Unrolling its definition with $(\star)$ and $c_1 < m$, we get
  a) $\sigma k_1 \leq c_1$
  b) $(m - c_1, \text{fix } f(x).e) \in [\![\sigma A_1 \xrightarrow{\mathrm{exec}(\sigma k, \sigma t)} \sigma A_2]\!]_v$

  By IH 3 on the second premise using $(m - c, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_1]\!]$ which hold since
  - $\mathrm{FV}(e_2) \subseteq \mathrm{dom}(\Gamma')$ and $\Gamma' \subseteq \Gamma$
  - $(m - c, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_1]\!]$ by downward closure (Lemma 4) on $(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma|_2]\!]$ using $m - c \leq m$

we get $(m, \delta e_2) \in [\![\sigma A_1]\!]_\varepsilon^{0,\infty}$. Unrolling its definition with ($\diamond$) and $c_2 < m$, we get

c) $0 \leq c_2$
d) $(m - c_2, v_2) \in [\![\sigma A_2]\!]_v$

By downward closure (Lemma 4) on d) using $m - c_1 - c_2 - c_{app} \leq m - c_2$, we get

$$(m - (c_1 + c_2 + c_{app}), v_2) \in [\![\sigma A_1]\!]_v \tag{1}$$

Next, we unroll b) with (1) and $m - (c_1 + c_2 + c_{app}) < m - c_1$ to obtain

$$(m - (c_1 + c_2 + c_{app}), e[v_2/x, (\text{fix } f(x).e)]) \in [\![\sigma A]\!]_\varepsilon^{\sigma k, \sigma t} \tag{2}$$

By unrolling second part of (2)'s definition using ($\dagger$) and
$c_r < m - (c_1 + c_2 + c_{app})$, we get

e) $\sigma k \leq c_r$
f) $(m - (c_1 + c_2 + c_r + c_{app}), v_r) \in [\![\sigma A_2]\!]_v$

Now, we can conclude as follows:

1. We can trivially show $0 \leq (c_1 + c_2 + c_r + c_{app})$
2. By f)

subcase 2: Assume that $\models \sigma\Phi$ and there exists Γ' s.t. $\text{FV}(e_2') \subseteq \text{dom}(\Gamma')$ and $\Gamma' \subseteq \Gamma$ and $(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_2]\!]$.
TS: $(m, \delta e_2') \in [\![|U\,(\sigma A_2, \sigma A_2')|_2]\!]_\varepsilon^{0,\infty} \equiv [\![A_2']\!]_\varepsilon^{0,\infty}$
Following the definition of $[\![\cdot]\!]_\varepsilon^{\cdot,\cdot}$, there are two cases:

- Assume that $\infty < m$.
  Since m is finite, this case is vacuously true.
- The conclusion follows by IH 3 on the second premise using $(m - c, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_2]\!]$ which hold since
  - $\text{FV}(e_2') \subseteq \text{dom}(\Gamma')$ and $\Gamma' \subseteq \Gamma$
  - $(m - c, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_2]\!]$ by downward closure (Lemma 4) on $(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma|_2]\!]$ using $m - c \leq m$

  i.e. we get $(m, \delta e_2) \in [\![\sigma A_2']\!]_\varepsilon^{0,\infty}$.

Case $$\frac{\Delta; \Phi; |\Gamma|_2 \vdash_{k_1}^{t_1} e_1 : A_1 \qquad \Delta; \Phi; x : U\,(A_1, A_1), \Gamma \vdash e \ominus e_2 \lesssim t_2 : \tau_2}{\Delta; \Phi; \Gamma \vdash e \ominus \textbf{let } x = e_1 \textbf{ in } e_2 \lesssim t_2 - k_1 - c_{let} : \tau_2} \textbf{ r-e-let}$$
There are two parts to show.

subcase 1: Assume that $\models \sigma\Phi$ and there exists Γ' s.t. $\text{FV}(e) \subseteq \text{dom}(\Gamma')$ and $\Gamma' \subseteq \Gamma$ and $(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_1]\!]$.
TS: $(m, \delta e) \in [\![|\sigma\tau_2|_1]\!]_\varepsilon^{0,\infty}$
There are two parts to show

- Assume that $\infty < m$.
  This is vacuously true.
- By IH 3 on the second premise using $(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_1]\!]$ since we know that $\text{FV}(e) \subseteq \text{dom}(\Gamma')$ and $\Gamma' \subseteq \Gamma, x : A_1$ since x doesn't occur free in e, we get immediately $(m, \delta e) \in [\![|\sigma\tau_2|_1]\!]_\varepsilon^{0,\infty}$.

subcase 2: Assume that $\models \sigma\Phi$ and there exists Γ' s.t. $\mathrm{FV}(\text{let } x = e_1 \text{ in } e_2) \subseteq \mathrm{dom}(\Gamma')$ and $\Gamma' \subseteq \Gamma$ and $(m, \delta) \in \mathcal{G}[\![|\sigma$
TS: $(m, \text{let } x = \delta e_1 \text{ in } \delta e_2) \in [\![|\sigma\tau_2|_2]\!]_{\varepsilon}^{0,\infty}$
There are two parts to show

- Assume that $\infty < m$.
  This is vacuously true.
- Assume that

  a) $\dfrac{\delta e_1 \Downarrow^{c_1} v_1 \quad (\star) \qquad \delta e_2[v_1/x] \Downarrow^{c_r} v_r \quad (\diamond)}{\text{let } x = \delta e_1 \text{ in } \delta e_2 \Downarrow^{c_1+c_r+c_{let}} v_r}$ **let**

  b) $c_1 + c_r + c_{let} < m$

  By IH 2 on the first premise using

  $$\mathrm{FV}(e_1) \subseteq \mathrm{dom}(|\Gamma'|_2) \text{ and } |\Gamma'|_2 \subseteq |\Gamma|_2 \text{ and } (m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_2]\!]$$

  we get $(m, \delta e_1) \in [\![\sigma A_1]\!]_{\varepsilon}^{\sigma k_1, \sigma t_1}$.
  Unrolling second part of its definition using $(\star)$ and $c_1 < m$, we get

  c) $\sigma k_1 \leq c_1$

  d) $(m - c_1, v_1) \in [\![\sigma A_1]\!]_v$

  By IH 3 on the second premise using $(m - c, \delta[x \mapsto v_1]) \in \mathcal{G}[\![x : \sigma A_1, |\sigma\Gamma'|_2]\!]$ which hold since

  - $\mathrm{FV}(e_2) \subseteq \mathrm{dom}(x : U\,(A_1, A_1), \Gamma')$ and $x : U\,(A_1, A_1), \Gamma' \subseteq x : U\,(A_1, A_1), \Gamma$
  - $(m - c, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_2]\!]$ by downward closure (Lemma 4) on $(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma|_2]\!]$ using $m - c \leq m$
  - $(m - c, v_1) \in [\![\sigma A_1]\!]_v$

  we get $(m - c, \delta e_2[v_1/x]) \in [\![|\sigma\tau_2|_2]\!]_{\varepsilon}^{0,\infty}$.
  Unfolding its definition using $(\diamond)$ and $c_r < m - c_1$, we get

  e) $0 \leq c_r$

  f) $(m - (c_1 + c_r), v_r) \in [\![|\sigma\tau_2|_2]\!]_v$

  Then we can conclude as follows

  1. Trivially, $0 \leq c_1 + c_r + c_{let}$
  2. By downward closure (Lemma 4) on f) using

  $$m - (c_1 + c_r + c_{let}) \leq m - (c_1 + c_r)$$

  we get $(m - (c_1 + c_r + c_{let}), v_r) \in [\![|\sigma\tau_2|_2]\!]_v$

Case $\dfrac{\Delta; \Phi; |\Gamma|_2 \vdash_{\bar{k}'} e' : A_1 + A_2 \qquad \Delta; \Phi; x : U\,(A_1, A_1), \Gamma \vdash e \ominus e_1' \lesssim t : \tau \qquad \Delta; \Phi; y : U\,(A_2, A_2), \Gamma \vdash e \ominus e_2' \lesssim t : \tau}{\Delta; \Phi; \Gamma \vdash e \ominus \textbf{ case } (e', x.e_1', y.e_2') \lesssim t - k' - c_{case} : \tau}$ **r-e-case**

There are two parts to show.

subcase 1: Assume that $\models \sigma\Phi$ and there exists Γ' s.t. $\mathrm{FV}(e) \subseteq \mathrm{dom}(\Gamma')$ and $\Gamma' \subseteq \Gamma$ and $(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_1]\!]$.

TS: $(m, \delta e) \in [\![|\sigma\tau|_1]\!]_{\varepsilon}^{0,\infty}$
There are two parts to show

- Assume that $\infty < m$.
  This is vacuously true.
- By IH 3 on the second premise using $(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_1]\!]$ since we know that $\mathrm{FV}(e) \subseteq \mathrm{dom}(\Gamma')$ and $\Gamma' \subseteq \Gamma, x : A_1$ since x doesn't occur free in e, we get immediately $(m, \delta e) \in [\![|\sigma\tau|_1]\!]_\varepsilon^{0,\infty}$.

subcase 2: Assume that $\models \sigma\Phi$ and there exists Γ' s.t. $\mathrm{FV}(\text{ case } (e', e_1', e_2')) \subseteq \mathrm{dom}(\Gamma')$ and $\Gamma' \subseteq \Gamma$ and $(m, \delta) \in \mathcal{G}[\![|\sigma$
TS: $(m, \text{ case } (\delta e', \delta e_1', \delta e_2')) \in [\![|\sigma\tau|_2]\!]_\varepsilon^{0,\infty}$
There are two parts to show

- Assume that $\infty < m$.
  This is vacuously true.
- There are also two parts to show here depending on what δe evaluates to. We only show one for brevity, the other one is similar.
  Assume that

  a) $\dfrac{\delta e' \Downarrow^{c'} \text{inl } v' \quad (\star) \qquad \delta e_1'[v'/x] \Downarrow^{c_r'} v_r' \quad (\diamond)}{\text{case } (\delta e, x.\delta e_1, y.\delta e_2) \Downarrow^{c'+c_r'+c_{case}} v_r'}$ **case-inl**

  b) $c' + c_r' + c_{case} < m$

  By IH 2 on the first premise using
  $\mathrm{FV}(e') \subseteq \mathrm{dom}(|\Gamma'|_2)$ and $|\Gamma'|_2 \subseteq |\Gamma|_2$ and $(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_2]\!]$
  we get $(m, \delta e') \in [\![\sigma A_1 + \sigma A_2]\!]_\varepsilon^{\sigma k', -'}$.
  Unrolling second part of its definition using $(\star)$ and $c < m$, we get

  c) $\sigma k' \leq c'$

  d) $(m - c', \text{inl } v') \in [\![\sigma A_1 + \sigma A_2]\!]_v$

  By IH 3 on the second premise using $(m - c', \delta[x \mapsto v']) \in \mathcal{G}[\![x : \sigma A_1, |\sigma\Gamma'|_2]\!]$ which hold since

  - $\mathrm{FV}(e_1') \subseteq \mathrm{dom}(x : U\,(A_1, A_1), \Gamma')$ and $x : U\,(A_1, A_1), \Gamma' \subseteq x : U\,(A_1, A_1), \Gamma$
  - $(m - c', \delta) \in \mathcal{G}[\![|\sigma\Gamma'|_2]\!]$ by downward closure (Lemma 4) on $(m, \delta) \in \mathcal{G}[\![|\sigma\Gamma|_2]\!]$ using $m - c \leq m$
  - $(m - c', v') \in [\![\sigma A_1]\!]_v$

  we get $(m - c', \delta e_2'[v'/x]) \in [\![|\sigma\tau|_2]\!]_\varepsilon^{0,\infty}$.
  Unfolding its definition using $(\diamond)$ and $c_r' < m - c'$, we get

  e) $0 \leq c_r'$

  f) $(m - (c' + c_r'), v_r') \in [\![|\sigma\tau|_2]\!]_v$

  Then we can conclude as follows

  1. Trivially, $0 \leq c' + c_r' + c_{let}$
  2. By downward closure (Lemma 4) on f) using
  $$m - (c' + c_r' + c_{let}) \leq m - (c' + c_r')$$
  we get $(m - (c' + c_r' + c_{let}), v_r') \in [\![|\sigma\tau|_2]\!]_v$

$\square$

□

2 Example Programs

2.1 Two-dimensional count (in-depth)

The following example demonstrates that by using relational analysis we can show that one program is faster than the other in a case where non-relational reasoning does not suffice to do so. Let us consider `2Dcount` an algorithm that counts how many rows of a two-dimensional matrix contain a key x and satisfy a predicate p. Such an algorithm could be used in many scenarios, e.g. in the context of web analytics, a data analyst might be interested in how many rows of a matrix storing the number of top m frequently visited websites contains "google.com" and satisfy a predicate. We can define `2Dcount` as follows:

$$
\begin{array}{l}
\text{fix } \texttt{2Dcount}(\texttt{find}).\lambda x.\lambda M.\texttt{case } M \texttt{ of} \\
\quad \texttt{nil} \ \to 0 \\
\mid l :: M' \ \to \ \text{let } r = \texttt{2Dcount find } x\ M' \text{ in} \\
\qquad\qquad\quad \text{if } p\ l \text{ then } r + (\texttt{find } x\ l) \\
\qquad\qquad\qquad\quad \text{else } \ r
\end{array}
$$

Suppose that we have the following two different implementations for `find`.

$$
\begin{array}{l}
\text{fix } \texttt{find1}(x).\lambda l.\texttt{case } l \texttt{ of} \\
\quad \texttt{nil} \ \to 0 \\
\mid h :: tl \ \to \ \text{if } h = x \text{ then } 1 \text{ else } \texttt{find1 } x\ tl
\end{array}
$$

$$
\begin{array}{l}
\text{fix } \texttt{find2}(x).\lambda l.\texttt{case } l \texttt{ of} \\
\quad \texttt{nil} \ \to 0 \\
\mid h :: tl \ \to \ \text{if } (\texttt{find2 } x\ tl) = 1 \text{ then } 1 \text{ else if } (h = x) \text{ then } 1 \text{ else } 0
\end{array}
$$

The first function `find1` scans the row from left to right and returns 1 for the first element that matches the key whereas the second function `find2` scans the list from right to left and returns 1 for the last element that matches the key. Assume that the matrix M has type $\text{list}[m]^0\,(\text{list}[n]^0\,\text{int})$ and the predicate p has the same worst- and best-case execution cost. For simplicity, let us also assume that we only account for application steps; the analysis generalizes to non-zero costs for other elimination steps as well. What can we say about the relative cost of `2Dcount` with respect to these two `find` implementations?

A naive non-relational reasoning reveals that the minimum and the maximum execution costs for `find1` are 1 and $3 \cdot n$, respectively whereas the minimum and maximum execution costs for `find2` are $3 \cdot n$ and $4 \cdot n$, respectively. Unlike the `sam` example where we used a representative linear cost function, for this example, we show the concrete costs to emphasize the importance of precise cost counting. [1]

$$\texttt{find1} : \text{int} \to \forall n{::}\mathbb{N}.\, \text{list}[n]\,\text{int} \xrightarrow{\text{exec}(1,3\cdot n)} \text{int}.$$

[1] If t is omitted from $\tau_1 \xrightarrow{\text{diff}(t)} \tau_2$, it is assumed to be zero (similarly for unary functions).

$$\texttt{find2} : \text{int} \to \forall n{::}\mathbb{N}.\, \text{list}[n]\, \text{int} \xrightarrow{\text{exec}(3\cdot n, 4\cdot n)} \text{int}.$$

So, we can conclude that `find1` is faster than `find2` because the difference in `find1`'s maximum and `find2`'s minimum execution cost is less than or equal to 0. However, a similar naive analysis for the whole top-level program cannot establish that `2Dcount` with `find1` is faster than `2Dcount` with `find2`. When the predicate is always false, `2Dcount find1` runs slowest with cost $3 \cdot n \cdot m + 7 \cdot m$ and when the predicate is always true, `2Dcount find2` runs fastest with cost $4 \cdot m$, i.e. their difference is not upper bounded by 0.

Instead, we can establish this by performing a relational analysis of `2Dcount` and using the fact that we are interested in the relative cost of the same matrix M. We can type `2Dcount` as follows:

$$\begin{aligned}&\vdash \texttt{2Dcount} \ominus \texttt{2Dcount} \lesssim 0 :\\ &(U\,(\text{int}, \text{int}) \to \forall n{::}\mathbb{N}.\, U\,(\text{list}[n]\,\text{int}, \text{list}[n]\,\text{int}) \xrightarrow{\text{diff}(0)} U\,(\text{bool}, \text{bool})) \to \text{int}_r \to \forall m, n{::}\mathbb{N}.\\ &\text{list}[m]^0\,(\text{list}[n]^0\,\text{int}_r) \xrightarrow{\text{diff}(0)} U\,(\text{int}, \text{int}).\end{aligned}$$

Note that `2Dcount` takes as input the find function with 0 relative cost (`find`'s type is given in parentheses above). We can instantiate `find` with `find1` or `find2`. To show that their relative cost is upper bounded by 0, based on the non-relational types of `find1` and `find2` obtained above, we use the following subtyping rule

$$\frac{}{\Delta; \Phi \models U\,(A_1 \xrightarrow{\text{exec}(k,t)} A_2, A_1' \xrightarrow{\text{exec}(k',t')} A_2') \sqsubseteq U\,(A_1, A_1') \xrightarrow{\text{diff}(t-k')} U\,(A_2, A_2')} \to \textbf{execdiff}$$

Next, let us see how we establish 0 relative cost for `2Dcount` in RelCost. In particular, let us focus on the more interesting case in which the two matrices contain at least one row. Since the rows do not differ between the two executions, the result of the predicate p will be the same, hence both programs will take the same branch in the two runs. It suffices to show that the two "then" branches and the two "else" branches are related. For the "then" branches, the analysis is trivial since we recursively call `2Dcount` which is assumed to have 0 relative cost. For the "else" branches, since we know that the relative cost of `find` and `2Dcount` are 0, we can immediately establish 0 cost.

Comment on how this example is typed in the type system in the paper Note that in the above example, the subtyping rule $\to$ **execdiff** is more general than the following subtyping rule that is shown in the paper:

$$\frac{}{\Delta; \Phi \models U\,(A_1 \xrightarrow{\text{exec}(k,t)} A_2) \sqsubseteq U\,A_1 \xrightarrow{\text{diff}(t-k)} U\,A_2} \to \textbf{execdiff}$$

The above subtyping rule is more restricting: it forces the two functions to have the same lower and upper bounds. So, if we don't allow unrelated types to talk about two different unary types, we would be forced to subtype the functions `find1` and `find2` to have the same costs as follows:

$$\texttt{find1} : (\text{int} \to \forall n{::}\mathbb{N}.\, \text{list}[n]\,\text{int} \xrightarrow{\text{exec}(1, 3\cdot n)} \text{int}) \sqsubseteq (\text{int} \to \forall n{::}\mathbb{N}.\, \text{list}[n]\,\text{int} \xrightarrow{\text{exec}(1, 4\cdot n)} \text{int})$$

$\texttt{find2} : (\mathrm{int} \to \forall n{::}\mathbb{N}.\, \mathrm{list}[n]\, \mathrm{int} \xrightarrow{\mathrm{exec}(3\cdot n, 4\cdot n)} \mathrm{int}) \sqsubseteq (\mathrm{int} \to \forall n{::}\mathbb{N}.\, \mathrm{list}[n]\, \mathrm{int} \xrightarrow{\mathrm{exec}(1, 4\cdot n)} \mathrm{int}).$

Then, we cannot show that `find1` and `find2` have 0 execution cost difference using the rule $\to_{\textbf{execdiff}}$ since $4 \cdot n - 1 \neq 0$. For the system in the paper, the proof is completed using the following rule which can be shown sound.

$$\frac{|\Gamma| \vdash^{t_1}_{k_1} e_1 : A_1 \xrightarrow{\mathrm{exec}(k_1', t_1')} A_2 \qquad |\Gamma| \vdash^{t_2}_{k_2} e_2 : A_1 \xrightarrow{\mathrm{exec}(k_2', t_2')} A_2}{\Gamma \vdash e_1 \ominus e_2 \lesssim t_1 - k_2 : U\, A_1 \xrightarrow{\mathrm{diff}(t_1' - k_2')} U\, A_2}\ \textbf{fun-switch}.$$

In the generalized system with $U\,(A_1, A_2)$, we do not need **fun-switch** since we can derive an analogous rule using the generic subtyping rule $\to_{\textbf{execdiff}}$.

2.2 Loop unswitching

Next, we consider a compiler optimization technique known as *loop unswitching* that moves a conditional inside a loop to the outside. For simplicity, we consider a variant in which the else branch just returns a unit. Consider the function `loop` that iterates over a list l.

fix $\texttt{loop}(l).\texttt{case}\ l\ \texttt{of}$
$\ \ \texttt{nil}\ \to ()$
$\mid h :: tl\ \to\ \text{if } b \text{ then } \boxed{\text{let } \_ = f\ h \text{ in } \texttt{loop}\ tl}\ \text{ else } ().$

This program can be transformed to a version that pulls out the conditional from the loop body as follows:

$\texttt{loopOp} = \text{if } b \text{ then}$
$\qquad \text{fix } \texttt{loop}'(l).\texttt{case}\ l\ \texttt{of}$
$\qquad\ \ \texttt{nil}\ \to ()$
$\qquad \mid h :: tl\ \to\ \boxed{\text{let } \_ = f\ h \text{ in } \texttt{loop}'\ tl}$
$\text{else } \lambda l.()$

Suppose that the list l has type $\mathrm{list}[n]^0\, \mathrm{int}_r$, i.e. it is substituted by the same list for two programs, and the function f has type $\mathrm{int}_r \xrightarrow{\mathrm{diff}(0)} \mathrm{int}_r$, i.e. given related integers, it returns related integers with 0 execution cost difference. Assuming that the boolean input b is equal between two runs, what can we say about the relative cost of these two programs? Intuitively, `loopOp` is an optimization: rather than checking b at each iteration, it only checks it once outside of the function definition. Here, we do a more fine-grained cost counting and assume all elimination forms to have a unit cost. Then, intuitively one would expect that the execution cost difference between these two programs is n.

If we resort to the non-relational execution cost analysis, using the switch rule we have introduced in Example 1 from the paper, we can establish the following type

$$\vdash \lambda b.\texttt{loop} \ominus \lambda b.\texttt{loopOp} \lesssim 0 : \quad U\,((\mathrm{bool} \to \forall n{::}\mathbb{N}.\ \mathrm{list}[n]\, \mathrm{int} \xrightarrow{\mathrm{exec}(5\cdot n+1, 1)} \mathrm{unit}), .)$$

by typing the two programs independently. Then, via subtyping $U\,((A_1 \xrightarrow{\mathrm{exec}(k,t)} A_2), (A_1 \xrightarrow{\mathrm{exec}(k,t)} A_2)) \sqsubseteq U\,(A_1, A_1) \xrightarrow{\mathrm{diff}(t-k)} U\,(A_2, A_2)$, we can establish a relative execution cost difference of $5 \cdot n$ for these two functions. However, this bound is not precise enough: it is 5 times more than what we expected, because it completely ignores the fact that b doesn't change between the two programs.

Instead, we can obtain a more precise bound using relational analysis. To achieve this, we make use of asynchronous rules that allows us to compare programs with different structure. For instance, we can compare an arbitrary expression e to an if statement as follows:

$$\frac{|\Gamma|_2 \vdash_k^t e' : \text{bool} \qquad \Gamma \vdash e \ominus e'_1 \lesssim t' : \tau \qquad \Gamma \vdash e \ominus e'_2 \lesssim t' : \tau}{\Gamma \vdash e \ominus (\text{if } e' \text{ then } e'_1 \text{ else } e'_2) \lesssim t' - k - 1 : \tau}\ \textbf{e-if}$$

In this rule we relate e to the branches of the conditional and separately establish lower and upper bounds on the execution cost of the guard of the conditional. This rule allows us to compare `loop` to the inner recursive function $\texttt{loop}'$ in `loopOp`. Similarly, using a symmetric variant of **e-if** rule, we can compare the inner conditional branch of `loop` to the body of $\texttt{loop}'$ (shown in shaded boxes above). Note that, in the latter, we want to avoid comparing the "`else`" branch () to let $\_ = f\ h$ in $\texttt{loop}'\ tl$. This can be taken care of by refining the boolean type with its value as follows: $\text{bool}_r[B]$. [2] Then, we can type these two programs with a more precise relative cost n

$$\vdash \lambda b.\texttt{loop} \ominus \lambda b.\texttt{loopOp} \lesssim 0 :\ \forall B :: \{\text{true}, \text{false}\}.$$
$$\text{bool}[B] \xrightarrow{\text{diff}(-1)} \forall n{::}\mathbb{N}.\, \text{list}[n]^0\, \text{int}_r \xrightarrow{\text{diff}(n)} \text{unit}_r.$$

The negative cost 1 comes from the fact that the optimized version incurs a unit cost for the outer "`if`" statement and the expected cost n comes from the fact that the conditional elimination incurs a unit cost for each recursive call.

2.3 Selection Sort

Consider the standard selection sort algorithm that finds the smallest element in a list and then sorts the remaining list recursively. In RelCost, we can show that `ssort` is a constant time algorithm, i.e. its relative cost is 0.

We briefly explain its typing. The first ingredient is the function `minR` that takes a non-empty list of size n and returns the minimum element and the remaining list of size n.

$$\begin{array}{l}
\text{fix } \texttt{minR}(l).\texttt{case } l \texttt{ of} \\
\quad \texttt{nil} \ \rightarrow contra \\
\mid h :: tl \ \rightarrow \ \texttt{case } tl \texttt{ of} \\
\qquad\qquad\qquad \texttt{nil} \ \rightarrow (h, \text{nil}\,) \\
\qquad\qquad\quad \mid h' :: tl' \ \rightarrow \ \text{let } (m, r) = \texttt{minR}\ tl \text{ in} \\
\qquad\qquad\qquad\qquad\qquad \text{if } h < m \text{ then } \text{cons}(h, \text{cons}(m, r)) \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{else } \ \text{cons}(m, \text{cons}(h, r))
\end{array}$$

It can be given the following type with the same minimum and maximum case execution cost $h(n)$, where h is a linear funtion in n.

$$\vdash_0^0 \texttt{minR} :\ \forall n{::}\mathbb{N}.\, n > 0 \;\&\; \text{list}[n]\, \text{int} \xrightarrow{\text{exec}(h(n),h(n))} (\text{int} \times \text{list}[n-1]\, \text{int}).$$

[2]Although we do not consider indexed booleans in this paper, they can be easily simulated by lists.

Note that its type constraints the input sizes to be non-zero by using the constrained type $C \,\&\, \tau$. Then, by using the following subtyping rule

$$\frac{}{\Delta; \Phi \models U\,(A_1 \xrightarrow{\text{exec}(k,t)} A_2, A_1' \xrightarrow{\text{exec}(k',t')} A_2') \sqsubseteq U\,(A_1, A_1') \xrightarrow{\text{diff}(t-k')} U\,(A_2, A_2')} \rightarrow \textbf{execdiff}$$

we can give minR zero relative cost as follows:

$$\vdash \texttt{minR} \ominus \texttt{minR} \lesssim 0 : \ \forall n{::}\mathbb{N}.\, n > 0 \,\&\, U\,((\text{list}[n+1]\,\text{int}), (\text{list}[n+1]\,\text{int})) \xrightarrow{\text{diff}(0)} U\,((\text{int} \times \text{list}[n]\,\text{int}), (\text{int} \times \text{list}$$

The selection sort function ssort first finds the minimum element and the rest of the list members and then returns the minimum element appended to the rest of the sorted list.

fix $\texttt{ssort}(l).\texttt{case}\ l\ \texttt{of}$
  $\texttt{nil}\ \rightarrow \text{nil}$
$\mid h :: tl\ \rightarrow\ \text{let}\ (m, r) = \texttt{minR}\ l\ \text{in}\ \text{cons}(m, \texttt{ssort}\ r)$

Then, we can relationally show that ssort has zero relative cost with respect to two lists that differ by α elements.

$$\vdash \texttt{ssort} \ominus \texttt{ssort} \lesssim 0 : \ \forall n, \alpha{::}\mathbb{N}.\, \text{list}[n]^{\alpha}\, U\,(\text{int}, \text{int}) \xrightarrow{\text{diff}(0)} U\,((\text{list}[n]\,\text{int}), (\text{list}[n]\,\text{int})).$$

We briefly explain how we derived this type. We focus on the part where the list has at least one element. From the type above, we know that minR's relative cost is 0 and "cons"'ing is constant time. In addition, we assumed that recursively, ssort incurs 0 cost. Hence we can conclude that relative cost of ssort is 0.

2.4 Approximate sum

The next example is from the approximate computing domain in which one often runs an approximate version of the program to save resources. For instance, consider two implementations of a calculation that computes the mean of a list of real numbers. The first function computes the sum of a list of reals and divides the sum by the length of the list whereas the second function (its approximate version) only computes the sum of the half of the elements, divides this sum by the total length of the list and then doubles the result afterwards. The first version could be operating over precise real numbers whereas the second–approximate–version could be operating over approximate numbers. How can we type these two implementations in RelCost?

We first show the two helper functions sum and sumAppr that correspond to precise and approximate summation over a list of numbers.

fix $\texttt{sum}(l).\lambda acc.\texttt{case}\ l\ \texttt{of}$
  $\texttt{nil}\ \rightarrow acc$
$\mid h :: tl\ \rightarrow\ \texttt{case}\ tl\ \texttt{of}$
      $\texttt{nil}\ \rightarrow h + acc$
    $\mid h' :: tl'\ \rightarrow\ h +$ h ' + (sum tl' acc)

$$
\begin{array}{l}
\text{fix } \texttt{sumAppr}(l).\lambda acc.\texttt{case } l \texttt{ of} \\
\quad \texttt{nil} \ \rightarrow acc \\
\mid h :: tl \ \rightarrow \ \texttt{case } tl \texttt{ of} \\
\qquad\qquad \texttt{nil} \ \rightarrow h + acc \\
\qquad\qquad \mid h' :: tl' \ \rightarrow \ \colorbox{lightgray}{h + (\texttt{sumAppr} tl' acc)}
\end{array}
$$

Assume that addition and division operations are constant time and the helper function `length` can be given the following type

$$\vdash \texttt{length} \ominus \texttt{length} \lesssim 0 : \forall n{::}\mathbb{N}.\, \text{list}[n]^{\alpha}\, U\,(\text{int}, \text{int}) \xrightarrow{\text{diff}(0)} \text{int}_r$$

Then, we can show that the two helper functions `sum` and `sumAppr` can be given the following relational type with relative cost n.

$$\vdash \texttt{sum} \ominus \texttt{sumAppr} \lesssim 0 : \forall n{::}\mathbb{N}.\, \text{list}[n]^{\alpha}\, U\,(\text{int}, \text{int}) \xrightarrow{\text{diff}(2n)} U\,(\text{int}, \text{int})\ .$$

Intuitively, these two functions only differ by an addition operation for each recursive call, therefore we obtain $2n$ difference cost in their execution time: for each recursive call, a unit cost for the addition and a unit cost for the primitive application. To type `sum` and `sumAppr` in RelCost we make use of the following asynchronous typing rule that allows us to compare the argument of a function application to an arbitrary expression

$$\frac{\Delta; \Phi; |\Gamma|_1 \vdash_{k_1}^{t_1} e_1 : A_1 \xrightarrow{\text{exec}(k,t)} A_2 \qquad \Delta; \Phi; \Gamma \vdash e_2 \ominus e_2' \lesssim t_2 : U\,(A_1, A_2')}{\Delta; \Phi; \Gamma \vdash e_1\ e_2 \ominus e_2' \lesssim t_1 + t_2 + t + c_{app} : U\,(A_2, A_2')}\ \textbf{r-app-e}$$

In particular, in the bodies of `sum` and `sumAppr`, we can relationally reason about the shaded parts and independently establish the maximum execution cost of the addition operation in `sum`. Then we can type these two functions as follows:

$$\vdash \left(\lambda l.\ \frac{\texttt{sum}\ l\ 0}{\texttt{length}\ l}\right) \ominus \left(\lambda l.\ 2 \cdot \frac{\texttt{sumAppr}\ l\ 0}{\texttt{length}\ l}\right) \lesssim 0 : \forall n{::}\mathbb{N}.\, \text{list}[n]^{\alpha}\, U\,(\text{int}, \text{int}) \xrightarrow{\text{diff}(2n-2)} U\,(\text{int}, \text{int})\ .$$

Since the approximate version performs an additional multiplication operation, we use the symmetric version of the rule **r-app-e** and subtract two unit costs: one for the cost of the multiplication and one for the cost of the application of the primitive application.