The FMLP$^+$: An Asymptotically Optimal Real-Time Locking Protocol for Suspension-Aware Analysis

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***Abstract*—Multiprocessor real-time locking protocols that are asymptotically optimal under suspension-oblivious schedulability analysis (where suspensions are pessimistically modeled as processor demand) are known for partitioned, global, and clustered job-level fixed priority (JLFP) scheduling. However, for the case of more accurate suspension-aware schedulability analysis (where suspensions are accounted for explicitly), asymptotically optimal protocols are known only for partitioned JLFP scheduling. In this paper, the gap is closed with the introduction of the first semaphore protocol for suspension-aware analysis that is asymptotically optimal under global and clustered JLFP scheduling. To this end, a new progress mechanism that avoids repeated priority inversions is developed and analyzed, based on the key observation that if lock-holding, low-priority jobs are priority-boosted, then certain other non-lock-holding, higher-priority jobs must be *co-boosted*.**

I. Introduction

The purpose of suspension-based real-time locking protocols is to limit *priority inversions* [22], which, intuitively, occur when a high-priority task that should be scheduled is instead delayed by a remote or lower-priority task. Such locking-related delay, also called *priority inversion blocking* (*pi-blocking*), is problematic because it can result in deadline misses. However, some pi-blocking is unavoidable when using locks and thus must be bounded and accounted for during schedulability analysis.

Clearly, an "optimal" locking protocol should minimize pi-blocking to the extent possible. Formally, a locking protocol is asymptotically optimal if it ensures that, for *any* task set, maximum pi-blocking is bounded within a constant factor of the pi-blocking unavoidable in *some* task set [11]. Interestingly, there exist two classes of schedulability analysis that yield *different* lower bounds: under *suspension-oblivious* (*s-oblivious*) analysis, $\Omega(m)$ pi-blocking is fundamental, whereas under *suspension-aware* (*s-aware*) analysis, $\Omega(n)$ pi-blocking is unavoidable in the general case [7, 11], where m and n denote the number of processors and tasks, respectively. As the names imply, the key difference is that suspensions are accounted for explicitly under s-aware analysis, whereas they are (pessimistically) modeled as processor demand in the s-oblivious case. In principle, s-aware schedulability analysis is preferable, but s-oblivious analysis is easier to derive and permits simpler pi-blocking bounds.

And indeed, for the simpler s-oblivious case, asymptotically optimal locking protocols are known for partitioned, global, and clustered *job-level fixed-priority*[1] (JLFP) scheduling [10–12]. In contrast, the s-aware case is analytically much more challenging and less understood: asymptotically optimal protocols are known so far only for partitioned JLFP scheduling [7, 11]. The general problem of optimal s-aware locking under global and clustered JLFP scheduling, however, has remained unsolved.

A. Contributions

We answer this fundamental question by introducing the generalized *FIFO Multiprocessor Locking Protocol* (FMLP$^+$), the first semaphore protocol for clustered scheduling that ensures $O(n)$ maximum s-aware pi-blocking under any JLFP policy.

While it was initially assumed [11] that a variant of Block *et al.*'s *Flexible Multiprocessor Locking Protocol* (FMLP) [6]—which uses $O(n)$ FIFO queues together with *priority inheritance* [22]—is asymptotically optimal under global scheduling, we show in Sec. III that this holds only under some, but not all global JLFP schedulers. In fact, we show that both priority inheritance and (unrestricted) *priority boosting* [22], which are the two mechanisms used in all prior locking protocols for s-aware analysis to avoid unbounded pi-blocking, can give rise to non-optimal $\Omega(\Phi)$ pi-blocking, where Φ is the ratio of the longest and the shortest period (and not bounded by m or n).

To overcome this lower bound, we introduce in Sec. IV-A a new progress mechanism called "restricted segment boosting," which boosts at most one carefully chosen lock-holding job in each cluster while simultaneously "co-boosting" certain other, non-lock-holding jobs to interfere with the underlying JLFP schedule as little as possible. Together with simple FIFO queues, this ensures $O(n)$ maximum s-aware pi-blocking (within about a factor of two of the lower bound, see Sec. IV-C). Notably, our analysis permits non-uniform cluster sizes, allows each cluster to use a different JLFP policy, supports self-suspensions within critical sections (Sec. IV-F), and can be easily combined with prior work [25] to support nested critical sections (Sec. IV-G).

Finally, while answering the s-aware blocking optimality question in the general case is the main contribution of this paper, Sec. V presents a schedulability study that shows the FMLP$^+$ to outperform s-oblivious approaches if the underlying s-aware schedulability analysis is sufficiently accurate.

B. Related Work

On uniprocessors, the blocking optimality problem has long been solved: both the classic *Stack Resource Policy* [3] and the *Priority Ceiling Protocol* [22, 24] limit pi-blocking to at most one (outermost) critical section, which is obviously optimal.

On multiprocessors, there are two major lock types: *spin locks*, wherein blocked jobs busy-wait, and suspension-based *semaphores*. Spin locks are well understood and it is not difficult to see that non-preemptable FIFO spin locks, which ensure $O(m)$

[1] See Sec. II for definitions and a review of essential background.

blocking [6, 13, 15], are asymptotically optimal. Intuitively, spin locks are appropriate for short critical sections, whereas busy-waiting becomes problematic with longer critical sections (and especially if critical sections contain self-suspensions).

Numerous semaphore protocols have been proposed in recent years (*e.g.*, [6, 14, 16, 17, 19, 20]; see [7, 8, 13] for recent overviews); due to space constraints, we focus here on the most relevant related work pertaining to blocking optimality.

Blocking optimality in multiprocessor real-time systems was first considered in [11], which established lower bounds on s-aware and s-oblivious pi-blocking and introduced protocols with (asymptotically) matching upper bounds, namely the $O(m)$ *Locking Protocol* (OMLP) for partitioned and global scheduling under s-oblivious analysis, and a simple, but impractical proof-of-existence protocol for partitioned scheduling under s-aware analysis. It was also suggested in [11] that the global FMLP [6] is asymptotically optimal under global scheduling w.r.t. s-aware analysis, which, however, is only the case under certain JLFP schedulers, as we discuss in Sec. III. The OMLP was later extended to clustered scheduling [12], and the $O(m)$ *Independence-Preserving Locking Protocol* (OMIP) [10] was introduced as an (also asymptotically optimal) alternative to the OMLP for systems with stringent latency requirements.

A precursor to this paper is the *partitioned FIFO Multiprocessor Locking Protocol* (P-FMLP$^+$), which was introduced in [7] as a refinement of Block *et al.*'s earlier partitioned FMLP [6]. The original partitioned FMLP [6] is not asymptotically optimal under s-aware analysis due to a limiting tie-breaking rule, which the P-FMLP$^+$ corrects to ensure asymptotically optimal s-aware pi-blocking [7]. However, the P-FMLP$^+$ is based on priority boosting and hence achieves asymptotic optimality only under partitioned scheduling. In this paper, we generalize the P-FMLP$^+$ to clustered scheduling, starting from first principles. To disambiguate the earlier variant from the generalized version developed in this paper, we refer to the earlier partitioned version [7] exclusively as the P-FMLP$^+$ and reserve the name FMLP$^+$ for the new, generalized protocol introduced in Sec. IV.

Most recently, Ward and Anderson [25, 26] presented the RNLP, which is the first multiprocessor real-time locking protocol to support fine-grained locking. The generalized FMLP$^+$ presented in this paper can be integrated with the RNLP to support nested critical sections, which we discuss in Sec. IV-G.

Finally, to the best of our knowledge, no asymptotically optimal s-aware locking protocol for the general case of clustered JLFP scheduling has been proposed in prior work. We present our solution in Sec. IV after first introducing needed background in Sec. II and demonstrating the sub-optimality of existing progress mechanisms in Sec. III.

II. Definitions and Assumptions

We consider a real-time workload consisting of n sporadic tasks $\tau = \{T_1, \ldots, T_n\}$ scheduled on m identical processors. We denote a task T_i's *worst-case execution cost* as e_i, its *minimum inter-arrival time* (or *period*) as p_i, and its *relative deadline* as d_i. Task T_i has an *implicit* deadline if $d_i = p_i$, a *constrained* deadline if $d_i \leq p_i$, and an *arbitrary* deadline otherwise. We let $J_{i,j}$ denote the j^{th} job of T_i, and let J_i denote an arbitrary job of T_i. A task's *utilization* is defined as $u_i = e_i/p_i$. A job J_i is *pending* from its release until it completes. T_i's *worst-case response time* r_i denotes the maximum duration that any J_i remains pending. If a job $J_{i,j+1}$ is released before its predecessor $J_{i,j}$ has completed (*i.e.*, due to a deadline miss or if $d_i > p_i$), then $J_{i,j+1}$ is not eligible to execute until $J_{i,j}$ has completed (*i.e.*, tasks are sequential).

While pending, a job is either *ready* (and can be scheduled) or *suspended* (and not available for scheduling). Jobs suspend either when waiting for a contended lock (Sec. II-A), or may also *self-suspend* for other, locking-unrelated reasons. We model locking-unrelated self-suspensions explicitly because they affect the locking protocol presented in Sec. IV. To this end, we let w_i denote the maximum number of self-suspensions of any J_i.

For simplicity, we assume integral time: all points in time and all task parameters are integer multiples of a smallest quantum (*e.g.*, a processor cycle), and a point in time t represents the interval $[t, t+1)$.

A. Shared Resources

Besides the m processors, the tasks share n_r serially-reusable shared resources $\ell_1, \ldots, \ell_{n_r}$ (*e.g.*, I/O ports, network links, data structures, *etc.*). We let $N_{i,q}$ denote the maximum number of times that any J_i accesses ℓ_q, and let $L_{i,q}$ denote T_i's *maximum critical section length*, that is, the maximum time that any J_i uses ℓ_q as part of a single access ($L_{i,q} = 0$ if $N_{i,q} = 0$). As a shorthand, we define $N_i \triangleq \sum_{q=1}^{n_r} N_{i,q}$ and $L^{max} \triangleq \max\{L_{i,q} \mid T_i \in \tau \land 1 \leq q \leq n_r\}$. The worst-case execution cost e_i includes all critical sections.

A request for a shared resource is *nested* if the requesting job already holds a resource, and *outermost* otherwise. We assume that jobs release all resources before completion, and that they must be scheduled to issue requests for shared resources.

To ensure mutual exclusion, shared resources are protected by a *locking protocol*. If a job J_i needs a resource ℓ_q that is already in use, J_i must wait and incurs *acquisition delay* until its request for ℓ_q is satisfied (*i.e.*, until J_i holds ℓ_q's lock). In this paper, we focus on semaphore protocols, wherein waiting jobs suspend.

B. Multiprocessor Real-Time Scheduling

We consider the general class of *clustered job-level fixed-priority* (JLFP) schedulers. Under a JLFP policy, the priorities of tasks may change over time, but each job is assigned a fixed, unique priority. The two most commonly used JLFP policies are *earliest-deadline first* (EDF) and *fixed-priority* (FP) scheduling. We let $\Upsilon(J_i)$ denote the (fixed) priority of job J_i, where $\Upsilon(J_h) < \Upsilon(J_l)$ indicates that J_h has higher priority than J_l. We assume that priorities are unique (*e.g.*, by breaking any ties in favor of lower-indexed tasks) and transitive (*i.e.*, if $\Upsilon(J_a) < \Upsilon(J_b)$ and $\Upsilon(J_b) < \Upsilon(J_c)$, then $\Upsilon(J_a) < \Upsilon(J_c)$).

Under clustered scheduling the m processors are organized into K disjoint subsets (or clusters) of processors $C_1, \ldots, C_K$, where m_k denotes the number of processors in the k^{th} cluster and $\sum_{k=1}^{K} m_k = m$. Each task is statically assigned to one of the clusters; we let $C(T_i)$ denote the cluster that T_i has been assigned to, define $n_k \triangleq |\{T_i \mid C(T_i) = C_k\}|$, and let $ready(C_k, t)$ denote the set of ready jobs in cluster C_k at time t.

In each cluster, at any point in time the m_k highest-priority ready jobs are selected for scheduling (if that many exist) as determined by the employed JLFP policy, unless the regular prioritization is overruled by a locking protocol's progress mechanism (see Sec. II-D). Jobs may migrate freely within each cluster, but not across cluster boundaries. Clusters may be of non-uniform size and may each employ a different JLFP policy.

Two prominent special cases are *partitioned* scheduling (where $K = m$ and $m_k = 1$ for all C_k) and *global* scheduling (where $K = 1$ and $m_1 = m$). We consider *clustered EDF* (C-EDF) as a representative JLFP policy, and further discuss *global EDF* (G-EDF) and *global FP* (G-FP) scheduling in Sec. III, and *partitioned FP* (P-FP) scheduling in Sec. V. The main result of this paper (Sec. IV), however, applies to any JLFP policy.

C. Priority Inversions, PI-Blocking, and Asymptotic Bounds

Locking protocols give rise to *priority inversions* [11, 22, 24], which intuitively occur if a job that *should* be scheduled (*i.e.*, one among the m_k highest-priority jobs in its assigned cluster) is *not* scheduled (*e.g.*, while waiting for a lock). Priority inversions are problematic as they constitute "blocking" that increases a task's worst-case response time [22, 24]; a real-time locking protocol's primary purpose is to limit such *priority inversion blocking* (*pi-blocking*) so that it can be accounted for by schedulability tests. Pi-blocking includes any locking-related delay that is not anticipated by a schedulability analysis assuming independent jobs, that is, any delay that cannot be attributed to the regular processor demand of higher-priority jobs. Consequently, the exact definition of "priority inversion" depends on the type of the employed schedulability analysis [11].

Schedulability tests are either *suspension-aware* (*s-aware*) or *suspension-oblivious* (*s-oblivious*). An s-aware schedulability test (such as uniprocessor FP response-time analysis [1]) models self-suspensions and locking-related suspensions explicitly, whereas s-oblivious analysis (such as Baruah's G-EDF schedulability test [4]) assumes that jobs are always ready. S-oblivious analysis can still be used in the presence of suspensions, but any suspensions must be pessimistically modeled as processor demand by inflating each job's execution cost by the maximum suspension time prior to applying a schedulability test, which affects the definition of pi-blocking [11].

Def. 1. A job J_i incurs ***s-aware*** (resp., ***s-oblivious***) *pi-blocking* at time t if **(i)** J_i is not scheduled at time t and **(ii)** there are fewer than m_k higher-priority jobs **scheduled** (resp., **pending**) in J_i's assigned cluster $C_k = C(T_i)$ at time t.

The difference between the two types of blocking is illustrated in Fig. 1 for $m_k = 2$. During $[2, 3)$, job J_3 incurs s-aware pi-blocking, but *not* s-oblivious pi-blocking, as there are $m_k = 2$ higher-priority *pending* jobs (which rules out s-oblivious pi-blocking), but only one higher-priority *scheduled* job, namely J_2. Note that it follows from Def. 1 that whenever a job incurs s-oblivious pi-blocking it also incurs s-aware pi-blocking (there cannot be more scheduled jobs than there are pending jobs). As a result, any safe upper bound on s-aware pi-blocking is implicitly also an upper bound on s-oblivious pi-blocking [11].

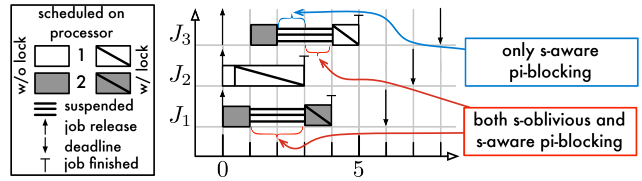


Fig. 1. Example of s-oblivious and s-aware pi-blocking of three jobs sharing one resource on $m_1 = 2$ G-EDF-scheduled processors. J_1 suffers acquisition delay during $[1, 3)$, and since no higher-priority jobs exist it is pi-blocked under either definition. J_3, suspended during $[2, 4)$, suffers pi-blocking under either definition during $[3, 4)$ since it is among the m_1 highest-priority pending jobs, but only s-aware pi-blocking during $[2, 3)$ as J_1 is pending but not ready then.

The focus of this paper is s-aware analysis. We let b_i denote a bound on the maximum total pi-blocking incurred by any J_i.

An "optimal" real-time locking protocol should minimize priority inversions to the extent possible. To study the fundamental limits of real-time synchronization, prior work [7, 11] proposed *maximum pi-blocking* (formally, $\max_{T_i \in \tau}\{b_i\}$) as a blocking complexity metric and established lower bounds of $\Omega(n)$ maximum s-aware (resp., $\Omega(m)$ maximum s-oblivious) pi-blocking. In other words, there exist pathological task sets such that maximum s-aware (resp., s-oblivious) pi-blocking is linear in the number of tasks (resp., number of processors) under *any* semaphore protocol [7, 11]. When stating asymptotic bounds, it is assumed that the number and duration of critical sections and self-suspensions per job are bounded by constants (*i.e.*, $L^{max} = O(1)$, $N_i = O(1)$, and $w_i = O(1)$).

To be asymptotically optimal, a locking protocol must ensure that maximum pi-blocking for *any* task set is *always* within a constant factor of the lower bound, which requires a careful choice of progress mechanism to prevent unbounded pi-blocking without causing "too much" pi-blocking itself.

D. Prior Progress Mechanisms and Optimal Locking Protocols

Several progress mechanisms that prevent unbounded pi-blocking by temporarily overruling the underlying JLFP policy have been developed in the past. Under classic *priority inheritance* [22, 24], which is effective only on uniprocessors and under global scheduling [7, 12], a lock-holder's priority is raised to that of the highest-priority job that it blocks (if any). Under *unrestricted priority boosting* [21, 23], which is also effective under clustered and partitioned scheduling, a lock-holders's priority *unconditionally* exceeds that of non-lock-holding jobs.

In the s-oblivious case, a bound of $O(m)$ maximum pi-blocking is asymptotically optimal, which is achieved by the OMLP [12] and the OMIP [10] under any clustered JLFP scheduler. The OMLP is based on *priority donation* [12], a variant of priority boosting that is suitable only for s-oblivious analysis [7]. The OMIP is based on *migratory priority inheritance*, a simple priority inheritance extension wherein jobs inherit not only the priorities, but also the cluster assignment of blocked jobs.

In the s-aware case, $O(n)$ maximum pi-blocking is asymptotically optimal, which is achieved by the P-FMLP$^+$ [7], but only under partitioned JLFP scheduling. The P-FMLP$^+$ is a fairly simple protocol based on priority boosting, where each resource is protected by a FIFO queue and priority-boosted jobs are scheduled (on each core) in order of non-decreasing lock request times (*i.e.*, jobs executing earlier-issued requests may preempt jobs executing later-issued requests). Key to the analysis of the P-FMLP$^+$ is that local lower-priority jobs are not scheduled (and hence cannot issue requests) while a higher-priority job executes, which, however, is not the case if $m_k > 1$.

Finally, this paper covers the remaining two cases—$O(n)$ maximum s-aware pi-blocking under global and clustered JLFP scheduling—by introducing a novel locking protocol and progress mechanism. We call this new locking protocol the *generalized* FMLP$^+$ because it effectively reduces to the (simpler) P-FMLP$^+$ under partitioned scheduling, in the sense that they generate the same schedule if $m_k = 1$.

Next, we motivate that a new progress mechanism is needed for the generalized FMLP$^+$ by showing that neither priority inheritance nor unrestricted priority boosting is a suitable foundation for asymptotic optimality under global JLFP scheduling.

III. Suboptimal S-Aware PI-Blocking

In prior work [11], it was suggested that the global FMLP [6], which simply combines per-resource FIFO queues with priority inheritance, ensures $O(n)$ maximum s-aware pi-blocking under global scheduling. This is indeed the case under G-EDF with implicit deadlines and if all jobs complete by their deadline (see [7, Ch. 6] for a proof). That is, there indeed exist global JLFP policies and common workloads under which the FMLP is asymptotically optimal under s-aware analysis.

However, there also exist global JLFP policies and workloads under which it is not. For instance, under G-FP scheduling, in the presence of arbitrary deadlines, or if jobs may complete after their deadlines, priority inheritance can give rise to non-optimal maximum s-aware pi-blocking since it may cause lock-holding jobs to preempt higher-priority jobs repeatedly. Further, a similar effect occurs also with unrestricted priority boosting. In general, it is thus impossible to construct semaphore protocols that are asymptotically optimal w.r.t. the entire class of JLFP policies using either progress mechanism.

To show this, we construct a task set τ^ϕ for which there exists an arrival sequence such that an independent job incurs ϕ time units of s-aware pi-blocking under G-EDF or G-FP scheduling on $m = 2$ processors, where ϕ can be chosen arbitrarily (*i.e.*, $\max_{T_i \in \tau^\phi}\{b_i\}$ cannot be bounded in terms of m or n).

Def. 2. Let $\tau^\phi = \{T_1, \ldots, T_4\}$ denote a set of four tasks with parameters as given in Table I that share $n_r = 1$ resource.

Note that $\frac{\max\{p_i\}}{\min\{p_i\}} \approx \phi$ (for large ϕ). This permits an arrival sequence such that a job of T_3 incurs s-aware pi-blocking each time that T_1, T_2, and T_4 release jobs in a certain pattern, which, by design, can occur up to ϕ times while a job of T_3 is pending.

Lemma 1. *Under G-EDF or G-FP scheduling on $m = 2$ processors, if priority inheritance or unrestricted priority boosting is used as a progress mechanism, then $\max_{T_i \in \tau^\phi}\{b_i\} = \Omega(\phi)$.*

Proof: We first consider priority inheritance under G-EDF and construct a legal arrival sequence such that $b_3 = \Omega(\phi)$ for any integer $\phi > 0$ under s-aware analysis. Note that the order in which waiting jobs are queued is irrelevant since the single resource is shared only among two tasks. Suppose jobs of T_4 require ℓ_1 for the entirety of their execution, and that jobs of T_2 require ℓ_1 for the latter half of their execution. Further, suppose each job executes for exactly e_i time units. If T_3 releases its first job at time 0, T_4 at time 0.5, and T_1 and T_2 at time 1, and all tasks release a job periodically every p_i time units, then $J_{3,1}$ (the first job of T_3) incurs ϕ time units of s-aware pi-blocking. The resulting schedule for $\phi = 4$ is shown in Fig. 2(a). By construction, the first ϕ jobs of T_4 have lower priority than $J_{3,1}$, whereas the first ϕ jobs of T_1 and T_2 have higher priority than $J_{3,1}$ (assuming w.l.o.g. that deadline ties are resolved in favor of lower-indexed tasks). As a result, $J_{3,1}$ is preempted whenever a job of T_2 requests the shared resource, thereby incurring one time unit of s-aware pi-blocking for each of the ϕ jobs of T_4 that are released while $J_{3,1}$ is pending.

TABLE I

Task	e_i	p_i	d_i	$N_{i,1}$	$L_{i,1}$
T_1	3	5	5	0	0
T_2	2	5	5	1	1
T_3	$1.5 + 2 \cdot (\phi - 1)$	$1 + 5 \cdot \phi$	$1 + 5 \cdot \phi$	0	0
T_4	1.5	5	$5 + 5 \cdot \phi$	1	1.5

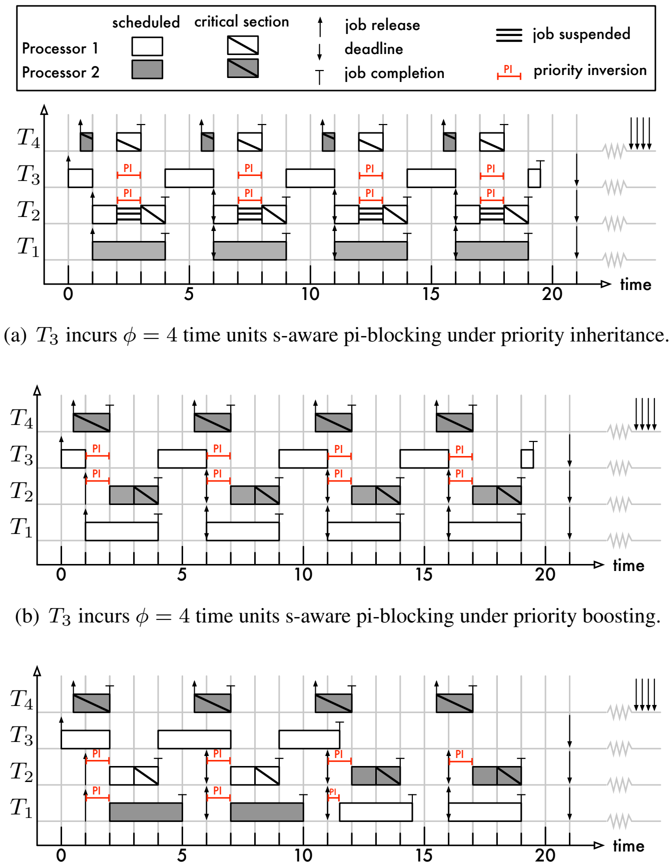
By construction, the schedule of the relevant jobs remains unchanged if G-FP scheduling is assumed and tasks are prioritized in index order (*i.e.*, if T_1 has the highest priority and T_4 the lowest). Hence, $\max_{T_i \in \tau^\phi}\{b_i\} = \Omega(\phi)$ under priority inheritance and either G-EDF or G-FP.

Finally, a similar schedule arises if unrestricted priority boosting is used instead of priority inheritance: if jobs of T_4 are priority-boosted while holding a lock, then $J_{3,1}$ is preempted whenever higher-priority jobs of T_1 are released, as illustrated in Fig. 2(b). As a result, $J_{3,1}$ still incurs s-aware pi-blocking for one time unit for each of the ϕ jobs of T_4 that are released while $J_{3,1}$ is pending. Hence, $\max_{T_i \in \tau^\phi}\{b_i\} = \Omega(\phi)$ under unrestricted priority boosting and either G-EDF or G-FP. ■

In the general case, it is thus not possible to ensure $O(n)$ maximum s-aware pi-blocking when using priority inheritance or unrestricted priority boosting. However, while some s-aware pi-blocking is clearly unavoidable in the depicted schedule, it *is* possible to distribute the pi-blocking among the jobs such that no job accumulates "too much" pi-blocking. Such a schedule is shown in Fig. 2(c). In this example, $J_{3,1}$ incurs no s-aware pi-blocking at all since pi-blocking is incurred only by new jobs that arrive *after* a request is issued. The key observation is that in JLFP schedules pi-blocking is fundamentally tied to preemptions, which occur only if a job is released or resumed. It is thus possible to shift pi-blocking to newly arrived jobs instead of accumulating it in "long-running" jobs such as $J_{3,1}$ We formalize this idea next.

IV. The FIFO Multiprocessor Locking Protocol

In this section, we formally introduce the generalized FMLP$^+$ and "restricted segment boosting," the underlying progress mechanism, and prove them to ensure $O(n)$ maximum s-aware pi-blocking. For the sake of simplicity, we initially assume that critical sections do not contain self-suspensions (jobs may still



(a) T_3 incurs $\phi = 4$ time units s-aware pi-blocking under priority inheritance.

(b) T_3 incurs $\phi = 4$ time units s-aware pi-blocking under priority boosting.

(c) A schedule in which the *per-job* s-aware pi-blocking b_i is indepent of ϕ.

Fig. 2. G-EDF example schedules for $m = 2$ illustrating τ^{ϕ} for $\phi = 4$.

self-suspend outside of critical sections) and that critical sections are non-nested (*i.e.*, jobs request never more than one shared resource at a time). We discuss how to integrate self-suspensions in Sec. IV-F and lift the latter restriction in Sec. IV-G by showing that "restricted segment boosting" seamlessly integrates with Ward and Anderson's asymptotically optimal RNLP [25] for fine-grained nested resource sharing.

We begin by establishing required definitions.

A. Definition of the FMLP$^+$

Central to the FMLP$^+$ is the notion of *job segments*, which are non-overlapping intervals of a job's execution that correspond to critical and non-critical sections. Correspondingly, there are two types of job segments: *independent segments*, during which a job is not using any shared resources, and *request segments*, during which a job requires a shared resource to progress.

Def. 3. An interval $[t_0, t_1]$ is an *independent segment* of a job J_i iff

- J_i is ready and not holding a resource throughout $[t_0, t_1]$,
- J_i is either released or resumed at time t_0, or J_i releases a lock at time $t_0 - 1$ (*i.e.*, just prior to the beginning of the interval), and
- J_i completes, suspends, or issues a lock request at time t_1.

Note that it follows from the above definition that a job is always scheduled at the end of an independent segment since an independent segment ends only when a job suspends, completes, or when it issues a lock request, each of which requires invoking OS services, which in turn a job can do only when it is scheduled.

Complementary to independent segments, request segments denote times during which J_i interacts with the locking protocol.

Def. 4. An interval $[t_0, t_1]$ is a *request segment* of a job J_i iff J_i issues a lock request at time $t_0 - 1$ (*i.e.*, just prior to the beginning of the interval) and releases the lock it requested at time t_1. During $[t_0, t_1]$, J_i is either suspended and waiting to acquire a lock, or ready and holding a lock.

Explicitly separating a job's execution into individual segments allows bounding the pi-blocking incurred during each segment, which at times requires "protecting" certain jobs executing earlier-started segments. To this end, we let $t_r(J_i, t)$ denote the *current segment start time* of J_i, where $t_r(J_i, t) = t_0$ iff $[t_0, t_1]$ is an independent or a request segment of J_i and $t_0 \leq t \leq t_1$. For brevity, we further define the predicate $is(J_i, t)$ to hold iff there exists an independent segment $[t_0, t_1]$ of J_i such that $t_0 \leq t \leq t_1$. Analogously, $rs(J_i, t)$ holds iff $[t_0, t_1]$ is a request segment of J_i and $t_0 \leq t \leq t_1$.

Under the FMLP$^+$ (which will be formally defined in Def. 6), lock-holding jobs become eligible for priority boosting in order of non-decreasing segment start times. We therefore let $boosted(C_k, t)$ denote the lock-holding, ready job in cluster C_k with the earliest segment start time (with ties in segment start time broken arbitrarily but consistently, *e.g.*, in favor of lower-indexed tasks). If no such job exists at time t, then $boosted(C_k, t) = \bot$. Formally, if $boosted(C_k, t) = J_x$, then $J_x \in ready(C_k, t)$, $rs(J_x, t)$, and $t_r(J_x, t) \leq t_r(J_y, t)$ for each $J_y \in \{J_y \mid J_y \in ready(C_k, t) \wedge rs(J_y, t)\}$.

Further, whenever a lock-holding job is priority boosted, certain other jobs must be *co-boosted* to protect them from being repeatedly pi-blocked (*i.e.*, to prevent the accumulation of pi-blocking in particular jobs, as is the case in the examples shown in Sec. III). To this end, we let $cb(J_i, t)$ denote the *co-boosting set* of job J_i at time t, which is the set of higher-priority jobs executing independent segments that started before $t_r(J_i, t)$; formally $cb(J_i, t) = \{J_y \mid is(J_y, t) \wedge \mathsf{Y}(J_y) < \mathsf{Y}(J_i) \wedge t_r(J_y, t) < t_r(J_i, t) \wedge C(J_i) = C(J_y)\}$.

Finally, we let $cb'(J_i, t)$ denote the $m_k - 1$ jobs in $cb(J_i, t)$ with the earliest segment start times (with ties broken arbitrarily), where m_k denotes the number of processors in T_i's cluster (*i.e.*, $C_k = C(T_i)$). If $cb(J_i, t)$ contains only $m_k - 1$ or fewer jobs, then simply $cb'(J_i, t) = cb(J_i, t)$.

With these definitions in place, we are finally ready to define the progress mechanism underlying the generalized FMLP$^+$.

Def. 5. Let $scheduled(C_k, t)$ denote the set of jobs scheduled in cluster C_k at time t. Under *restricted segment boosting*, $scheduled(C_k, t)$ is selected as follows.

1) Let $J_b = boosted(C_k, t)$. If $J_b \neq \bot$, then let $B(C_k, t) = \{J_b\} \cup cb'(J_b, t)$; otherwise, if $J_b = \bot$, let $B(C_k, t) = \emptyset$.
2) Let $H(C_k, t)$ denote the $m_k - |B(C_k, t)|$ highest-priority tasks in $ready(C_k, t) \setminus B(C_k, t)$ (if that many exist; otherwise $H(C_k, t) = ready(C_k, t) \setminus B(C_k, t)$).
3) The set of jobs $scheduled(C_k, t) = B(C_k, t) \cup H(C_k, t)$ is scheduled at time t.

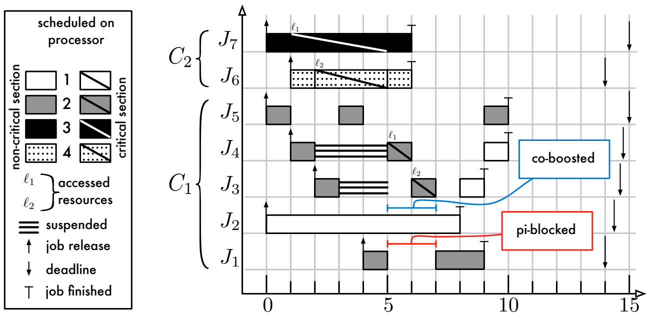


Fig. 3. C-EDF schedule of seven jobs in two two-processor clusters sharing two resources (ℓ_1 and ℓ_2) under the FMLP$^+$. The example is discussed in Sec. IV-B.

In other words, in each cluster C_k and at any point in time t, the set of priority-boosted jobs $B(C_k, t)$ includes at most one lock-holding job J_b, and, if J_b exists, the (up to) $m_k - 1$ highest-priority jobs in J_b's co-boosting set $cb(J_b, t)$. Finally, if $B(C_k, t)$ contains fewer than m_k jobs, then the remaining $m_k - |B(C_k, t)|$ processors are used to service the highest-priority jobs in $ready(C_k, t)$ that are not already scheduled. An example schedule illustrating restricted segment boosting is shown in Fig. 3 and discussed shortly in Sec. IV-B below.

Restricted segment boosting strikes a balance between progress and pi-blocking by forcing the progress of *at least* one lock-holding job, thus preventing unbounded priority inversions, but without disturbing the regular, priority-based schedule "too much" by boosting *at most* one lock-holding job.[2] This provides a strong analytical foundation, which together with the following simple protocol leads to asymptotic optimality.

Def. 6. Under the generalized *FIFO Multiprocessor Locking Protocol* (FMLP$^+$), there is one FIFO queue FQ_q for each resource ℓ_q. Resource requests are satisfied as follows.

1) A scheduled job may request a resource at any time. When a job J_i requests resource ℓ_q, it is enqueued in FQ_q and suspended if FQ_q was previously not empty.
2) J_i holds the resource ℓ_q when it becomes the head of FQ_q. While holding ℓ_q, J_i is eligible for restricted segment boosting, as defined in Def. 5.
3) When J_i releases ℓ_q, it becomes ineligible for restricted segment boosting and is dequeued from FQ_q, and the new head of FQ_q (if any) is resumed.

Next, we briefly illustrate how the FMLP$^+$ works with an example, and then establish its asymptotic optimality.

B. Example Schedule

Fig. 3 shows a C-EDF schedule (with $K = 2$ and $m_1 = m_2 = 2$) of seven jobs sharing two resources under the FMLP$^+$.

We focus on the locking-related events in cluster C_1. At time 2, J_4 suspends because it requested ℓ_1, which is being used by J_7 at the time. J_3 similarly suspends at time 3 when it requests ℓ_2.

At time 5, J_3 and J_4 are resumed simultaneously and co-boosting takes effect. First, note that $boosted(C_1, 5) = J_4$ even though J_3 has higher priority (*i.e.*, an earlier deadline) because $t_r(J_4, 5) = 2 < t_r(J_3, 5) = 3$. Further, $cb(J_4, 5) = \{J_2\}$ as $t_r(J_2, 5) = 0 < t_r(J_3, 5) = 3$ and $\mathsf{Y}(J_2) < \mathsf{Y}(J_3)$, and thus $B(C_1, 5) = \{J_2, J_4\}$ and $H(C_1, 5) = \emptyset$. Note that under *unrestricted* priority boosting, both J_1 and J_2 would have been preempted and incurred pi-blocking, whereas here only J_1 is preempted and J_2 is left undisturbed because it is co-boosted. In contrast, $J_1 \notin cb(J_4, 5)$ since J_1's current segment started at time 4, which is later than J_4's current segment start time, and $J_5 \notin cb(J_4, 5)$ because it has lower priority than J_4. Analogously, J_2 remains scheduled at time 6. Note that because J_2 benefits from co-boosting, J_1 incurs pi-blocking instead.

While it may appear counter-intuitive that the higher-priority job J_1 incurs pi-blocking in place of the lower-priority job J_2, this is deliberate and highlights a key property of the FMLP$^+$ with restricted segment boosting: whenever there is a choice, pi-blocking affects jobs with later segment start times instead of jobs with earlier segment start times.

In general, to expedite the completion of critical sections in lower-priority jobs (*e.g.*, J_3 and J_4 in Fig. 3), inevitably *some jobs* have to incur pi-blocking. By shifting the negative effects of priority boosting to later-arrived jobs, restricted segment boosting prevents the accumulation of pi-blocking in individual "long-running" jobs (Fig. 2(c) is in fact an FMLP$^+$ schedule). This design choice is key to attaining asymptotic optimality, as will become apparent in the following analysis of the FMLP$^+$.

C. Asymptotic Optimality: Proof Overview

We next establish that the FMLP$^+$ in conjunction with restricted segment boosting ensures $O(n)$ maximum s-aware pi-blocking under clustered JLFP scheduling, which matches the known lower bound of $\Omega(n)$ maximum s-aware pi-blocking [11].

In the remainder of this section, let J_i denote an arbitrary job of the task under analysis T_i, and let $C_k = C(T_i)$ denote the cluster in which J_i executes. We first show that s-aware pi-blocking is limited to $n - 1 = O(n)$ critical section lengths in each request segment (Lemma 5), and then show that J_i incurs s-aware pi-blocking for the cumulative duration of at most $n_k - 1 = O(n)$ critical section lengths during each independent segment as well (Lemma 13). Asymptotic optimality follows if the number of segments is bounded by a constant (Theorem 1).

A source of complexity in the proof is that co-boosted jobs may issue lock requests. As a result, it is not immediately clear that J_i is not pi-blocked multiple times by a lower-priority job that manages to repeatedly issue requests while being co-boosted (although this is in fact impossible). Key to the proof is hence Lemma 11, which establishes that, if a lower-priority job J_b causes J_i to incur pi-blocking for (hypothetically) a second time during one of J_i's independent segments, then J_b was *not* among the priority-boosted jobs when it issued its request. Intuitively, the main argument of the proof is that if J_i is not scheduled when J_b executes its critical section, then this implies the presence of higher-priority jobs, which would have already prevented J_b from being scheduled at the time that it issued its request. This property lies at the heart of the FMLP$^+$ and stems from the fact that it resolves all contention strictly in FIFO order.

We begin by bounding pi-blocking during request segments, which are easier to analyze than independent segments.

[2]It is worth emphasizing that Def. 5 does *not* artificially serialize critical sections: $H(C_k, t)$ may contain lock-holding jobs and thus multiple jobs may concurrently execute critical sections (protected by different locks) if they have a sufficiently high priority to be scheduled without being boosted.

D. S-Aware PI-Blocking during Request Segments

In Lemma 5 below, we show that maximum s-aware pi-blocking is limited to $(n-1)$ critical section lengths during each request. In preparation, we first establish three simple lemmas that together encapsulate the observation that, once J_i has "waited long enough," there are no more earlier-issued requests that could delay J_i. We begin by noting that the current segment start time always increases when a lock is released.

Lemma 2. *Let J_b denote a job of task T_b pending at time t. If J_b or an earlier job of T_b unlocks a shared resource at time t_u and $t_u < t$, then $t_r(J_b, t) > t_u$.*

Proof: Recall that $t_r(J_b, t)$ denotes the start time of J_b's current segment. If J_b itself releases a resource at time t_u, then by Defs. 3 and 4 it starts a new segment at time $t_u + 1$, and hence J_b's current segment at time t cannot have started prior to time $t_u + 1$. If J_b is not yet pending at time t_u (*i.e.*, if an earlier job of T_b unlocks a resource at time t_u), then J_b arrives later at some time t_a, where $t_u < t_a$, which implies $t_u < t_a \leq t_r(J_b, t)$. ∎

Next, we observe that any pi-blocking during a request segment implies the existence of another lock-holding job with an earlier-or-equal segment start time.

Lemma 3. *If $rs(J_i, t)$ and J_i incurs s-aware pi-blocking at time t, then there exists a J_x s.th. $rs(J_x, t)$ and $t_r(J_x, t) \leq t_r(J_i, t)$.*

Proof: There are two cases to consider.

Case 1: J_i is ready at time t. Then it holds a resource, but is not scheduled. As exactly one lock-holding job is priority-boosted according to Def. 5, and since jobs are boosted in order of non-decreasing segment start times (recall the definition of $boosted(C_k, t)$), it follows that another lock-holding job with a segment start time no later than $t_r(J_i, t)$ is boosted at time t.

Case 2: J_i is not ready at time t. Then it is waiting for the requested resource, which implies that some other job holds the resource at time t. Since under the FMLP$^+$ resource access is granted in FIFO order, it follows that the job holding the requested resource issued its request no later than J_i. ∎

We observe next that restricted segment boosting guarantees the progress of at least one lock-holding job.

Lemma 4. *If there exists a lock-holding job at time t, then (one of) the lock-holding, ready job(s) with the earliest segment start time (with ties in segment start time broken arbitrarily) progresses towards completion of its critical section at time t.*

Proof: The progress guarantee of restricted segment boosting follows immediately from Def. 5. A job is prevented from progressing towards the completion of its critical section if it is ready but not scheduled. By Def. 5, in each cluster C_k, (one of) the job(s) with the earliest segment start time, namely $boosted(C_k, t)$, is scheduled whenever it is ready. ∎

Finally, with Lemmas 2–4 in place, it is possible to bound the maximum pi-blocking during any individual request segment.

Lemma 5. *Let $[t_0, t_1]$ denote a request segment of J_i. During $[t_0, t_1]$, J_i incurs s-aware pi-blocking for the cumulative duration of at most one critical section per each other task (in any cluster), for a total of at most $n-1$ critical sections.*

Proof: By Lemma 3, J_i incurs s-aware pi-blocking at time $t \in [t_0, t_1]$ only if a lock-holding job with a segment start time no later than $t_r(J_i, t) = t_0$ exists at time t. It follows from Lemma 4 that at least one lock-holding job with a segment start time no later than t_0 progresses towards the completion of its critical section at any time t that J_i is pi-blocked. By Lemma 2, once a task unlocks a resource at some time t_u after time t_0, it has a segment start time later than t_0. Therefore, after a task has completed a critical section at some time t_u after time t_0, it can no longer pi-block J_i during $[t_u, t_1]$. Thus at most one critical section per each task other than T_i causes J_i to incur pi-blocking during $[t_0, t_1]$, for a total of at most $n-1$ critical sections. ∎

This concludes our analysis of request segments. Next, we consider independent segments.

E. S-Aware PI-Blocking during Independent Segments

The maximum pi-blocking incurred by J_i during an independent segment is substantially more challenging to analyze because jobs that issue a request *after* J_i started its independent segment may still pi-block J_i (but only once, as we are going to show in Lemma 12). To begin, we establish two simple lemmas on the conditions necessary for J_i to incur s-aware pi-blocking. In the following discussion, we let $B(C_k, t)$ denote the set of boosted jobs and $H(C_k, t)$ the set of non-boosted jobs selected for scheduling at time t in cluster C_k, as defined in Def. 5.

Lemma 6. *If $is(J_i, t)$ and J_i incurs s-aware pi-blocking at time t, then $boosted(C_k, t) \neq \bot$.*

Proof: Follows from the definition of restricted segment boosting. Recall from Def. 1 that, to incur s-aware pi-blocking at time t, two conditions must be met: (i) J_i must not be scheduled and (ii) fewer than m_k higher-priority jobs are scheduled in J_i's cluster at time t. Suppose no ready, lock-holding job exists at time t in C_k (*i.e.*, $boosted(C_k, t) = \bot$). Then, according to Def. 5, $B(C_k, t) = \emptyset$. Since J_i is ready and, according to (i), not scheduled, this implies that $H(C_k, t) = scheduled(C_k, t)$ contains m_k higher-priority jobs, which contradicts (ii). ∎

Next we establish that any lock-holding, ready job that is priority-boosted in cluster C_k while J_i incurs s-aware pi-blocking has lower priority than J_i.

Lemma 7. *If $is(J_i, t)$, J_i incurs s-aware pi-blocking at time t, and $J_b = boosted(C_k, t)$, then $\Upsilon(J_i) < \Upsilon(J_b)$.*

Proof: By Lemma 6, J_b exists. Suppose J_i does not have a higher priority than J_b (*i.e.*, $\Upsilon(J_b) < \Upsilon(J_i)$ since job priorities are unique). Then, assuming that the job priority order is transitive, each $J_y \in cb(J_b, t)$ has higher priority than J_i since, by the definition of $cb(J_b, t)$, for each such J_y, $\Upsilon(J_y) < \Upsilon(J_b)$ and, by assumption, $\Upsilon(J_b) < \Upsilon(J_i)$. Hence all jobs in $B(C_k, t)$ have higher priority than J_i and, since J_i is not scheduled at time t, all jobs in $H(C_k, t)$ also have higher priority than J_i. Thus there are m_k higher-priority jobs scheduled, which contradicts the assumption that J_i incurs pi-blocking at time t. ∎

Having established that J_i is only pi-blocked when a lower-priority job J_b is priority-boosted, we next analyze in Lemmas 8–12 the conditions that exist when J_b issues its request. In the end, the conditions established in the following lemmas will allow us to conclude that no task can block J_i more than once. In the following, refer to Fig. 4 for an illustration of Lemmas 8–11.

Def. 7. In Lemmas 8–12, let $J_b = \mathit{boosted}(C_k, t)$, $J_b \neq \bot$, and let $t_x = t_r(J_b, t) - 1$ denote the time when J_b issued its request.

First, we establish that any job that is in J_b's co-boosting set at time t is also ready at time t_x and already executing the same independent segment that it is still executing at time t.

Lemma 8. *Define J_b and t_x as in Def. 7. If $J_y \in cb(J_b, t)$, then $is(J_y, t_x)$ and $t_r(J_y, t_x) = t_r(J_y, t)$.*

Proof: By the definition of $cb(J_b, t)$, if $J_y \in cb(J_b, t)$, then $is(J_y, t)$ and $t_r(J_y, t) < t_r(J_b, t) \leq t_r(J_b, t) - 1 = t_x$. From $t_r(J_y, t) \leq t_x$, it follows that each J_y is still executing the same segment at time t that it was executing at time t_x. Hence from Def. 3 we have $is(J_y, t_x)$ and $t_r(J_y, t_x) = t_r(J_y, t)$. ■

Next, we observe that if J_i is in J_b's co-boosting set but not scheduled, then at least m_k jobs are eligible for co-boosting.

Lemma 9. *Define J_b and t_x as in Def. 7. If $J_i \in cb(J_b, t)$, but $J_i \notin cb'(J_b, t)$, then $|cb(J_b, t)| \geq m_k$.*

Proof: Recall that $cb'(J_b, t) \subseteq cb(J_b, t)$ is the set of the (up to) $m_k - 1$ jobs in $cb(J_b, t)$ with the earliest segment starting times. As $J_i \notin cb'(J_b, t)$, it follows that $cb'(J_b, t) \subset cb(J_b, t)$, $|cb'(J_b, t)| = m_k - 1$, and hence $|cb(J_b, t)| \geq m_k$. ■

Based on Lemmas 8 and 9, it is possible to rule out that J_b was scheduled at time t_x due to having a high priority.

Lemma 10. *Define J_b and t_x as in Def. 7. If $J_i \in cb(J_b, t)$, but $J_i \notin cb'(J_b, t)$, then $J_b \notin H(C_k, t_x)$.*

Proof: Recall from Def. 5 that $H(C_k, t_x)$ denotes the set of the (up to) $m_k - |B(C_k, t_x)|$ highest-priority, non-boosted jobs ready at time t_x. Obviously, $|H(C_k, t_x)| + |B(C_k, t_x)| \leq m_k$.

Consider the jobs in $cb(J_b, t)$. By Lemma 8, each job $J_y \in cb(J_b, t)$ is also ready at time t_x. By the definition of $cb(J_b, t)$, each such J_y has a higher priority than J_b: $\Upsilon(J_y) < \Upsilon(J_b)$. Thus, by Def. 5, for J_b to be included in $H(C_k, t_x)$, each such higher-priority $J_y \in cb(J_b, t)$ must be included in $H(C_k, t_x)$ or $B(C_k, t_x)$. However, by Lemma 9, there exist at least m_k such jobs $J_y \in cb(J_b, t)$, which implies $J_b \notin H(C_k, t_x)$. ■

Finally, consider $B(C_k, t_x)$, the set of jobs priority-boosted at time t_x. It follows from Lemma 10 that J_b must be part of $B(C_k, t_x)$ when it issues its request at time t_x. We next establish that this is impossible if T_b releases a lock at a time t_u on or after time $t_r(J_i, t)$—that is, if T_b causes J_i to incur pi-blocking with a *second* critical section—since this implies a lower bound on J_b's current segment start time at time t_x.

Lemma 11. *Define J_b and t_x as in Def. 7. If $J_i \in cb(J_b, t)$, $J_i \notin cb'(J_b, t)$, a job of T_b unlocked a resource at time t_u, and $t_r(J_i, t) \leq t_u < t_x$, then $J_b \notin B(C_k, t_x)$.*

Proof: By contradiction. Suppose $J_b \in B(C_k, t_x)$, and let $J_x = \mathit{boosted}(C_k, t_x)$. According to Def. 5, if $J_x = \bot$, then $B(C_k, t_x) = \emptyset$, so assume otherwise.

As J_b holds a lock at time t that it requested at time t_x, and since lock-holding jobs do not issue further lock requests, J_b does not hold a lock at time t_x, which implies $J_b \neq J_x$. Since by assumption $J_b \in B(C_k, t_x)$, we have $J_b \in cb'(J_x, t_x)$.

Consider the jobs in $cb'(J_b, t)$. By Lemma 8, each job in $cb(J_b, t)$, and hence also each $J_y \in cb'(J_b, t)$, is ready and not holding a lock at time t_x, and, by the definition of $cb(J_b, t)$, each such J_y has higher priority than J_b: $\Upsilon(J_y) < \Upsilon(J_b)$.

By assumption, T_b unlocks a resource at time t_u; it thus follows from Lemma 2 that $t_u < t_r(J_b, t_x)$. And since by assumption $t_r(J_i, t) \leq t_u$, we have $t_r(J_i, t) < t_r(J_b, t_x)$. Further, recall that $cb'(J_b, t)$ denotes the $m_k - 1$ jobs in $cb(J_b, t)$ with the earliest segment start times. Since $J_i \in cb(J_b, t)$, but $J_i \notin cb'(J_b, t)$, we have $t_r(J_y, t) \leq t_r(J_i, t)$ for each $J_y \in cb'(J_b, t)$, and therefore also $t_r(J_y, t) < t_r(J_b, t_x)$.

Thus, each $J_y \in cb'(J_b, t)$ has a higher priority than J_b, has an earlier segment start time than J_b, and is ready and not holding a lock at time t_x (*i.e.*, $is(J_y, t_x)$). Thus, by the definition of $cb'(J_x, t_x)$, if $J_b \in cb'(J_x, t_x)$, then also $J_y \in cb'(J_x, t_x)$ for each $J_y \in cb'(J_b, t)$, which implies $|cb'(J_x, t_x)| \geq |cb'(J_b, t)| + 1$ (since $J_b \notin cb'(J_b, t)$). However, by definition, $|cb'(J_x, t_x)| \leq m_k - 1$ and, since $J_i \notin cb'(J_b, t)$, $|cb'(J_b, t)| = m_k - 1$, which implies $m_k - 1 \geq |cb'(J_x, t_x)| \geq |cb'(J_b, t)| + 1 = (m_k - 1) + 1 = m_k$. Contradiction. ■

Lemma 10 shows that J_b is not among the regularly scheduled jobs at time t_x, and Lemma 11 shows that, under certain conditions, J_b is not priority-boosted at time t_x. Together, they imply that J_b cannot be scheduled at time t_x, which conflicts with the assumption that t_x is the time at which J_b issues the request that blocks J_i at time t. Next, we establish that the conditions required to apply Lemma 11 would in fact be met if J_i would be blocked twice by the same task.

Lemma 12. *If $is(J_i, t)$, J_i incurs s-aware pi-blocking at time t, and $J_b = \mathit{boosted}(C_k, t)$, then no job of task T_b unlocked a resource during $[t_r(J_i, t), t)$.*

Proof: By contradiction. Suppose there exists a time $t_u \in [t_r(J_i, t), t)$ at which a job of T_b unlocked a resource. Let $t_x = t_r(J_b, t)$ denote the time at which J_b issued its request. We are going to show that J_b cannot have been scheduled at time t_x if a job of T_b unlocked a resource at time t_u.

By Lemma 6, J_b exists. By Lemma 7, we have $\Upsilon(J_i) < \Upsilon(J_b)$. Since by initial assumption $t_r(J_i, t) \leq t_u$, and by Lemma 2 $t_u < t_r(J_b, t)$, we further have $t_r(J_i, t) < t_r(J_b, t)$. Hence, by the definition of $cb(J_b, t)$, $J_i \in cb(J_b, t)$.

Since J_i incurs s-aware pi-blocking at time t, it cannot be scheduled at time t, which implies that $J_i \notin cb'(J_b, t)$.

Therefore, by Lemma 10, $J_b \notin H(C_k, t_x)$, and, by Lemma 11, $J_b \notin B(C_k, t_x)$. Since according to Def. 5 $\mathit{scheduled}(C_k, t_x) = B(C_k, t_x) \cup H(C_k, t_x)$, J_b was not scheduled at time t_x and thus cannot have issued its request at time t_x. Contradiction. ■

Lemma 12 implies that no task can block J_i more than once during a single independent segment, which yields the desired $O(n)$ bound on per-segment pi-blocking.

Lemma 13. *Let $[t_0, t_1]$ denote an independent segment of job J_i. During $[t_0, t_1]$, J_i incurs s-aware pi-blocking for the cumulative duration of at most one critical section per each other task in cluster $C_k = C(T_i)$, for a total of $n_k - 1$ critical sections.*

Proof: By Lemma 6, whenever J_i incurs s-aware pi-blocking, a lock-holding job J_b is scheduled, which implies

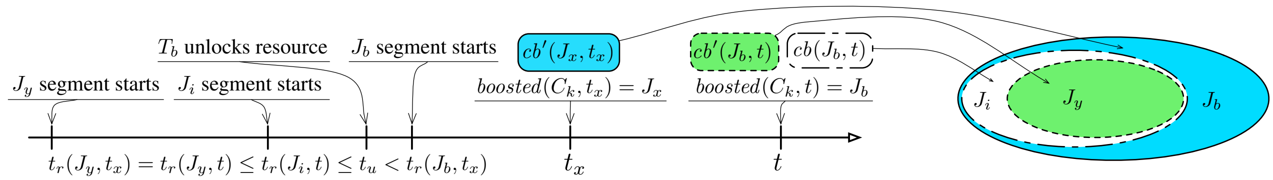


Fig. 4. Illustration of the conditions assumed and established in Lemmas 8–11. At time t, J_i is not scheduled and pi-blocked, J_b is priority-boosted, and J_i is part of J_b's co-boosting set ($J_i \in cb(J_b, t)$), but not among the co-boosted jobs with the $m_k - 1$ earliest segment starting times ($J_i \notin cb'(J_b, t)$). At time t_x, J_b requests the lock for the resource that it uses at time t. *Lemma 8:* each job $J_y \in cb'(J_b, t)$ is executing the same segment at times t and t_x. *Lemma 9:* there exist at least $m_k - 1$ such jobs. *Lemma 10:* at time t_x, there must exist some $J_x \neq J_b$ that is priority-boosted and J_b is included in J_x's co-boosting set $cb(J_x, t_x)$. *Lemma 11:* if a job of task T_b unlocks a resource at time t_u prior to t_x, but not before $t_r(J_i, t)$, each job $J_y \in cb'(J_b, t)$ is also an element of $cb(J_x, t_x)$ since, by the definition of the co-boosting set, each such job J_y has a higher priority than both J_b and J_x (priorities not shown), since $J_i \in cb(J_b, t)$ but $J_i \in cb'(J_b, t)$ implies that each such J_y has a segment start time no later than J_i, and since time t_u establishes a lower bound on the segment start time of J_b at time t_x.

that J_b progresses towards the completion of its critical section. Thus, for more than $n_k - 1$ critical sections to block J_i, some task T_x assigned to C_k needs to block J_i with at least two critical sections during $[t_0, t_1]$. Let $t \in [t_0, t_1]$ denote a point in time at which a job of T_x blocks J_i with a second critical section. Since tasks request at most one lock at a time, this implies that T_x unlocked a resource at some point in time t_u before t and after $t_r(J_i, t) = t_0$, which by Lemma 12 is impossible. ■

Having established a bound on maximum s-aware pi-blocking during independent segments, we are finally ready to state the main result that establishes the FMLP$^+$'s asymptotic optimality.

Theorem 1. *Under clustered JLFP scheduling with the FMLP$^+$, $\max_{T_i \in \tau}\{b_i\} = O(n)$ for any τ.*

Proof: By Lemma 13, during each independent segment, a job J_i of a task assigned to cluster $C_k = C(T_i)$ incurs s-aware pi-blocking for the total duration of at most $n_k - 1 = O(n)$ critical sections. By Lemma 5, during each request segment, J_i incurs s-aware pi-blocking for the total duration of at most $n - 1 = O(n)$ critical sections. Recall from Sec. II that w_i denotes the maximum number of locking-unrelated self-suspensions and N_i the maximum number of requests for any resource. From Defs. 3 and 4, it follows that any job J_i has at most N_i request segments and $1 + w_i + N_i$ independent segments since any two independent segments must be separated by a locking-unrelated self-suspension or a request segment. Thus $b_i \leq N_i \cdot (n-1) \cdot L^{max} + (1 + w_i + N_i) \cdot (n_k - 1) \cdot L^{max} = O(n)$, assuming $w_i + N_i = O(1)$ and $L^{max} = O(1)$ as explained in Sec. II. ■

The lower bound on maximum s-aware pi-blocking is $(n-1)$ critical section lengths per request [11]. Assuming $w_i = 0$, the FMLP$^+$ is hence asymptotically optimal within a factor of

$$\frac{(1 + 2N_i) \cdot (n-1) \cdot L^{max}}{N_i \cdot (n-1) \cdot L^{max}} = 2 + \frac{1}{N_i} \approx 2.$$

In the remainder of this section, we lift the two simplifying assumptions made so far, namely that critical sections do not contain self-suspensions and that critical sections are non-nested.

F. Critical Sections with Self-Suspensions

When synchronizing access to physical resources (*e.g.*, I/O devices), jobs may need to self-suspend during a critical section. For example, a device driver might first acquire a lock serializing access to a device, then prepare buffer contents and request an I/O transaction, self-suspend to wait for the transaction to complete, finally perform cleanup activities when resumed, and only then release the lock. Fortunately, such self-suspensions within critical sections do not invalidate the preceding analysis.

First, note that jobs do not start a new segment when resuming while holding a lock since by Def. 3 a job does not hold a lock during an independent segment, and because by Def. 4 a request segment extends until the lock is released, regardless of whether a job is ready. Second, when a lock-holding job self-suspends, it no longer requires a processor to progress towards completion of its critical section. It therefore does not need to be priority-boosted, and thus does not cause jobs executing independent segments to incur pi-blocking. Thus, provided the maximum self-suspension time is bounded by a constant and appropriately reflected in the maximum critical section length, the preceding analysis remains valid. More precisely, if $L^s_{q,i}$ denotes the maximum self-suspension length of any J_i while using ℓ_q and $L^e_{q,i}$ denotes maximum execution time of any J_i while using ℓ_q (both w.r.t. a single critical section), then $L_{q,i} = L^s_{q,i} + L^e_{q,i}$, but other jobs executing independent segments incur at most $L^e_{q,i}$ pi-blocking due to J_i being priority-boosted.

Next, we consider nested lock requests.

G. Nested Critical Sections

Ward and Anderson showed how to permit fine-grained nesting such that asymptotic optimality is retained with the RNLP [25]. The RNLP consists of two key components, a *token lock* and a *request satisfaction mechanism* (RSM), and is generic in the sense that it can be instantiated with different token locks and RSMs to be asymptotically optimal under various scheduling and analysis approaches. Under s-aware analysis, the RNLP's token lock essentially serves to record a timestamp of an outermost critical section, similar to the segment start time of a request segment in this paper. The RSM ensures progress and determines in which order (nested) requests are satisfied.

Ward and Anderson developed RSMs for spin locks, s-oblivious analysis, and, most relevant to this paper, also for s-aware analysis [25]. In particular, they introduced the *B-RSM*, which employs the P-FMLP$^+$'s notion of priority boosting [7] to ensure $O(n)$ maximum s-aware pi-blocking under partitioned scheduling. However, they also left open the case of clustered scheduling under s-aware analysis due to the lack of a suitable progress mechanism [25]. Since restricted segment boosting, the progress mechanism introduced in this paper, ensures progress while causing only $O(n)$ s-aware pi-blocking itself, it can simply be substituted into Ward and Anderson's B-RSM.

To summarize, nested critical sections can be supported by redefining request segments to correspond to the start and end of *outermost* requests, by adopting the generalized FMLP$^+$'s restricted segment boosting (instead of the P-FMLP$^+$'s unrestricted priority boosting) in the RNLP's B-RSM for s-aware analysis, and by replacing the FMLP$^+$'s simple FIFO queues with the RNLP's queue structure and access rules (which are also based on FIFO queueing). This concludes our discussion of blocking optimality. We present an empirical comparison of the s-aware FMLP$^+$ with prior s-oblivious locking protocols next.

V. Empirical Evaluation

Prior work [8] has shown the P-FMLP$^+$ [7] to be competitive with and—depending on the task set parameters—superior to the two classic locking protocols for P-FP scheduling under s-aware analysis, namely the MPCP [17, 21, 22] and the DPCP [22, 23]. Since the FMLP$^+$ generalizes the P-FMLP$^+$ considered in [8], we compare it instead with two s-oblivious locking protocols for clustered JLFP scheduling: the OMLP [12] and the OMIP [10].

Effective schedulability analysis requires fined-grained blocking bounds that take into account constant factors (*e.g.*, job arrival rates, individual critical section lengths, *etc.*) that are omitted from the asymptotic bound derived in Sec. IV. We derived suitable bounds (provided in Appendix B) using a previously introduced analysis technique based on linear programming [8].

A. Experimental Setup

Our experimental setup closely resembles those used in several prior studies [8, 10, 12]. In short, we considered systems with $m \in \{4, 8, 16, 32\}$ processors and, for each system, generated task sets ranging in size from $n = m$ to $n = 10m$. For a given n, tasks were generated by randomly choosing a period p_i and utilization u_i, and then setting $e_i = p_i \cdot u_i$ (rounding to the next-largest microsecond). Periods were chosen randomly from either a uniform or a log-uniform distribution ranging over $[10ms, 100ms]$; utilizations were chosen from two exponential distributions ranging over $[0, 1]$ with mean 0.1 (*light*) and mean 0.25 (*medium*), and two uniform distributions ranging over $[0.1, 0.2]$ (*light*) and $[0.1, 0.4]$ (*medium*).

Critical sections were generated according to three parameters: the number of resources n_r, the *access probability* p^{acc}, and the *maximum requests parameter* N^{max}. Each of the n_r resources was accessed by a task T_i with probability p^{acc} and, if T_i was determined to access ℓ_q, then $N_{i,q}$ was randomly chosen from $\{1, \ldots, N^{max}\}$, and set to zero otherwise. In our study, we considered $n_r \in \{m/4, m/2, m, 2m\}$, $p^{acc} \in \{0.1, 0.25, 0.5\}$, and set $N^{max} = 5$. For each $N_{i,q} > 0$, the corresponding maximum critical section length $L_{i,q}$ was randomly chosen using three uniform distributions ranging over $[1\mu s, 15\mu s]$ (*short*), $[1\mu s, 100\mu s]$ (*moderate*), and $[5\mu s, 1280\mu s]$ (*long*).

We evaluated each parameter combination under both P-FP (using s-aware response-time analysis [1]) and C-EDF with uniform clusters of size $m_k \in \{2, 4\}$. For s-aware analysis under C-EDF, we used Liu and Anderson's s-aware G-EDF schedulability test [18] on a per-cluster basis; for s-oblivious analysis, we additionally used three s-oblivious G-EDF schedulability tests [2, 4, 5], claiming a task set schedulable if it passed at least one of the tests. We tested at least 400 task sets for each n and each of the 3,456 possible combinations of the listed parameters. All resulting graphs are available online (see Appendix A for download instructions); we focus on major trends in select example graphs due to space constraints.

B. Results

Whether an s-aware locking protocol yields higher schedulability than an s-oblivious locking protocol inevitably depends on the accuracy of the underlying schedulability test (*i.e.*, whether the impact of suspensions is overestimated) and on the tested parameter ranges (*i.e.*, the simpler s-oblivious analysis benefits from m being small). Our results confirm this dependency.

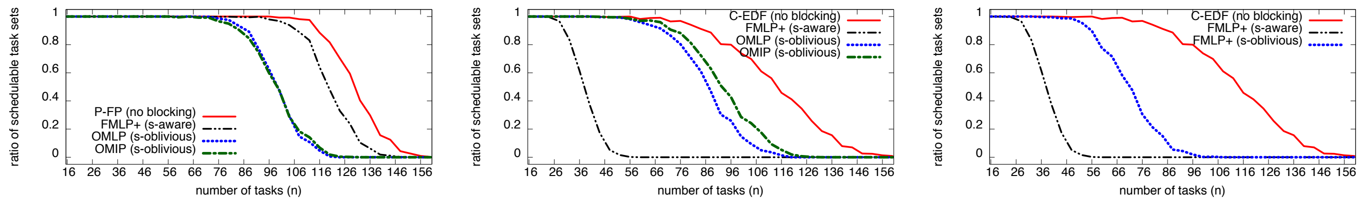
Response-time analysis for uniprocessor fixed-priority scheduling [1] is arguably one of the most accurate s-aware schedulability tests currently known. Consequently, under P-FP scheduling, using locking protocols specifically designed for s-aware analysis can result in substantially higher schedulability than using locking protocols intended for s-oblivious analysis. One such case is shown in Fig. 5(a). The FMLP$^+$ maintains high schedulability until $n \approx 100$, whereas schedulability under the OMIP and the OMLP starts decreasing rapidly already at $n \approx 80$. This is due to the s-oblivious approach of inflating execution times, which results in pessimistic response-time bounds for low-priority tasks with long periods (since each higher-priority task contributes with multiple *inflated* execution costs). In contrast, the more accurate s-aware analysis under the FMLP$^+$ counts only actual execution as interference, which results in tighter response-time bounds in this scenario. While there also exist of course scenarios without discernible differences between the protocols (*e.g.*, if there is only little contention and blocking is not the "schedulability bottleneck") and even scenarios in which the s-oblivious protocols outperform the FMLP$^+$ (*e.g.*, the structure of the OMIP and the OMLP, and s-oblivious analysis in general, are favored if m is small and contention is high), our experiments generally show the P-FP/FMLP$^+$ combination to perform well compared to both s-oblivious protocols.

In contrast, this is not the case if the underlying analysis is Liu and Anderson's s-aware schedulability test for G-EDF [18], which is the best-performing s-aware schedulability test for G-EDF currently available [18]. For example, Fig. 5(b) depicts a scenario where the FMLP$^+$ under s-aware analysis clearly performs significantly worse than both the OMLP and the OMIP.

To confirm that this is due to the underlying analysis of suspensions, and not due to the FMLP$^+$ itself, we also compared the FMLP$^+$ under s-aware analysis with the FMLP$^+$ under s-oblivious analysis. That is, given a task set τ, we first computed for each task T_i a bound b_i on s-aware pi-blocking, and then tested τ for schedulability once with s-aware analysis [18] and once with s-oblivious analysis [2, 4, 5] using the *same* bound.[3]

The results are shown in Fig. 5(c). Interestingly, the FMLP$^+$ under s-oblivious analysis *outperforms* the FMLP$^+$ under s-aware analysis, even though the FMLP$^+$ is asymptotically

[3]Recall from Sec. II-C that any bound on s-aware pi-blocking also bounds s-oblivious pi-blocking because the latter implies the former [11].



(a) Schedulability under P-FP ($m_k = 1$) (b) Schedulability under C-EDF ($m_k = 4$) (c) S-aware vs. s-oblivious analysis of the FMLP$^+$

Fig. 5. Schedulability under the FMLP$^+$, the OMLP, the OMIP, and without blocking under P-FP scheduling and C-EDF scheduling with $m_k = 4$ as a function of n for for $m = 16$, $n_r = 4$ resources, moderate critical sections, $p^{acc} = 0.1$, exponentially light utilizations, and uniformly distributed periods.

optimal under s-aware analysis, but not under s-oblivious analysis. In other words, simply treating the *identical* bound as execution time rather than self-suspension time leads to improved results. This shows that the state-of-the-art analysis of self-suspensions under global scheduling [18] has not yet progressed to the point where self-suspensions due to locking protocols can be efficiently analyzed.

Given the general difficulty of deriving effective multiprocessor schedulability analysis, this is perhaps not too surprising; it does, however, indicate that substantial advances are still required before the analysis of multiprocessor real-time systems can match the current understanding of uniprocessor systems.

VI. Conclusion

Prior to this work, it was not known whether it is possible to construct real-time semaphore protocols that ensure $O(n)$ maximum s-aware pi-blocking under global and clustered JLFP scheduling. We have answered this question positively with the generalized FMLP$^+$, the first asymptotically optimal locking protocol for clustered scheduling under s-aware analysis. Notably, the generalized FMLP$^+$ supports non-uniform cluster sizes, non-uniform JLFP policies, self-suspensions, and can be combined with the RNLP [25] to support nested critical sections.

The generalized FMLP$^+$ uses a number of new techniques. Rather than priority inheritance or unrestricted priority boosting, it relies on restricted segment boosting, a novel progress mechanism that tracks the individual independent and request segments of a job at runtime and imposes a FIFO order w.r.t. segment start times. Perhaps counterintuitively, at most one lock-holding job is boosted in each cluster C_k, but up to $m_k - 1$ other, non-lock-holding jobs may be co-boosted to prevent the accumulation of pi-blocking in individual jobs.

In future algorithmic work, it will be interesting to explore these techniques in the context of reader-writer and k-exclusion protocols [12]. In future systems-oriented work, it will be interesting to assess the practicality of the generalized FMLP$^+$ from the point of view of runtime overheads. While keeping track of segment start times is not problematic (the OS is involved in suspending and resuming jobs and acquiring and releasing semaphores anyway), co-boosting may cause additional preemptions, the overheads of which must be accounted for.

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Appendix

A. Full Set of Results

Due to space constraints, only three representative example graphs, shown in Fig. 5, are discussed in Sec. V. These particular graphs were selected as they clearly reveal the key trends that manifested in our experiments, as discussed in Sec. V. For the sake of completeness and transparency, the complete dataset and all graphs (5,760 in total) are available for inspection online at

- https://www.mpi-sws.org/~bbb/papers/data/fmlp-plus-all-graphs.zip.

B. Fine-Grained Blocking Analysis

In addition to the asymptotic bound derived in Sec. IV, a method for deriving fine-grained blocking bounds that reflect constant factors (such as job arrival rates, per-task maximum critical section length, *etc.*) is required to avoid unnecessary pessimism during schedulability analysis. In recent work [8], we presented a novel approach for obtaining such bounds based on *linear programming* (LP), which we reuse here to analyze blocking under the generalized FMLP$^+$.

At a high level, the LP-based approached works as follows [8]. For each task T_i, a linear program is derived wherein variables called *blocking fractions* represent the pi-blocking incurred by T_i due to critical sections of other tasks. The objective function of the linear program, when maximized, corresponds to a bound on the maximum pi-blocking incurred by any job of T_i in any schedule not shown to be impossible. Linear constraints that encode invariants of the analyzed locking protocol are imposed to rule out impossible schedules.

Two key advantages of the LP-based blocking analysis [8] are that such analysis is easier to reason about (each constraint can be considered in isolation), which enables the derivation of more accurate blocking bounds, and that it results in safe bounds by default (omitting a constraint may result in pessimistic, but not in unsound bounds).

In prior work [8], we have derived LP-based analysis of the partitioned FMLP$^+$, which we extend in the following to the generalized FMLP$^+$. We first introduce necessary definitions.

1) Definitions: Recall that r_x denotes the response-time of task T_x. We let $njobs(T_x, t)$ denote an upper bound on the maximum number of jobs of T_x that execute during any interval of length t. Under the assumed sporadic task model,

$$njobs(T_x, t) \triangleq \left\lceil \frac{t + r_x}{p_x} \right\rceil.$$

(A formal proof of this well-known bound is given in [7, Ch. 5].)

In the following, let J_i denote a job of the task T_i under analysis that incurs maximum pi-blocking.

For each task T_x and each resource ℓ_q, we let $N^i_{x,q}$ denote a bound on the maximum number of requests for ℓ_q issued by jobs of T_x while any J_i is pending, where

$$N^i_{x,q} \triangleq N_{x,q} \cdot njobs(T_x, r_i).$$

We distinguish among three distinct types of pi-blocking, which are defined as follows: a job J_x causes J_i to incur

- *direct* pi-blocking at time t if J_i is suspended because it requested a resource ℓ_q and J_x holds ℓ_q at time t;
- *indirect* pi-blocking at time t if J_i is suspended and waiting for some other job J_a to release a resource and J_a is not scheduled because J_x preempted J_a (*i.e.*, if J_x is priority-boosted and J_a is not co-boosted at time t); and finally
- *preemption* pi-blocking at time t if J_i is ready, not holding a resource, J_x is scheduled in J_i's assigned cluster, and J_x is priority-boosted and J_i is not scheduled (*i.e.*, J_i is not co-boosted).

For convenience, we let $\tau^i \triangleq \tau \setminus \{T_i\}$ denote all tasks other than T_i. For each $T_x \in \tau^i$, for each ℓ_q, and for each $v \in \{1, \ldots, N^i_{x,q}\}$ (*i.e.*, for each request that jobs of T_x may issue while J_i is pending), we define three *blocking fractions* $X^D_{x,q,v} \in [0,1]$, $X^I_{x,q,v} \in [0,1]$, and $X^P_{x,q,v} \in [0,1]$, which resp. correspond to direct, indirect, and preemption pi-blocking.

In the context of a fixed schedule (*e.g.*, a schedule in which J_i incurs maximum pi-blocking), a blocking fraction indicates the fraction of a critical section that actually caused J_i to incur pi-blocking (of a particular type) in relation to the corresponding maximum critical section length [8]. For example, suppose that in a fixed schedule of τ, the third critical section (*i.e.*, $v = 3$)executed by a job of task T_x accessing a resource ℓ_q caused J_i to incur three time units of direct pi-blocking, and that $L_{x,q} = 5$: then $X^D_{x,q,v} = \frac{3}{5}$.

In the context of a linear program, blocking fractions are also referred to as *blocking variables* and are used to express invariants that hold for each schedule not shown to be impossible [8]. For example, if J_i does not access a given resource ℓ_q, then it obviously does not incur direct pi-blocking due to requests for ℓ_q, which can be expressed as $X^D_{x,q,v} = 0$ for each $T_x \in \tau^i$ and each $v \in \{1, \ldots, N^i_{x,q}\}$.

Based on the just-established definitions, we next present a linear maximization problem that bounds the maximum pi-blocking incurred by any J_i in any schedule of τ.

2) Objective function: The objective of the optimizer is to find the maximum pi-blocking across the set of all schedules of τ not shown to be impossible. Recall that b_i denotes the maximum pi-blocking incurred by any J_i. In any schedule of τ, b_i can be expressed by the following linear equation, which serves as the objective function of the linear program [8].

$$b_i \triangleq \sum_{q=1}^{n_r} \sum_{T_x \in \tau^i} \sum_{v=1}^{N^i_{x,q}} \left(X^D_{x,q,v} + X^I_{x,q,v} + X^P_{x,q,v} \right) \cdot L_{x,q,v}$$

In the following, we impose linear constraints that encode invariants of the FMLP$^+$ to rule out impossible schedules.

3) Constraints from prior work: The first two constraints encode properties of blocking fractions and the system topology. These constraints are generic in the sense that they do not depend on the employed locking protocol. As they have previously been derived [8, 9], they are stated here without proof.

Constraint 1 expresses that, at a given point in time, a job J_x causes J_i to incur at most one type of blocking, which follows immediately from the mutually exclusive definitions of the three blocking types.

Constraint 1 (from [9]). *In any schedule of* τ*:*

$$\forall T_x \in \tau^i : \forall \ell_q : \forall v : X^D_{x,q,v} + X^I_{x,q,v} + X^P_{x,q,v} \leq 1.$$

The next constraint reflects the obvious fact that jobs that do not execute in J_i's cluster cannot preempt J_i. For brevity, we let $\tau^r \triangleq \tau \setminus \tau(C(T_i))$ denote the set of remote tasks (*i.e.*, the set of tasks not assigned to T_i's cluster).

Constraint 2 (Constraint 10 in [9]). *In any P-FP schedule of* τ *under a shared-memory semaphore protocol:*

$$\forall T_x \in \tau^r : \sum_{q=1}^{n_r} \sum_{v=1}^{N^i_{x,q}} X^P_{x,q,v} = 0.$$

Next, we reduce the set of schedules not shown to be impossible with additional protocol-specific constraints. Since the generalized FMLP$^+$ generalizes the partitioned FMLP$^+$, which was previously analyzed in [9], several of the constraints in [9] also apply to the generalized FMLP$^+$. In particular, the following three constraints are adopted from [9] and exploit that conflicting requests are serialized in FIFO order, and that jobs are priority-boosted in FIFO order, too.

First, Constraint 3 reflects that, each time that J_i requests a resource, each other task may directly block J_i at most once since conflicting critical sections are serialized in FIFO order.

Constraint 3 (Constraint 12 in [9]). *In any schedule of* τ *under the FMLP*$^+$*:*

$$\forall \ell_q : \forall T_x \in \tau^i : \sum_{v=1}^{N^i_{x,q}} X^D_{x,q,v} \leq N_{i,q}.$$

To exploit the fact that indirect pi-blocking is bounded by the amount of direct contention for shared resources, the next constraint adopted from [9] combines the following two observations:

1) each time that J_i requests a resource, a remote job J_x can directly block or indirectly block J_i with at most one request (since both direct and indirect blocking is only possible if J_i has a segment start time no earlier than J_x); and
2) a remote job J_x can indirectly block J_i only if some other job J_y in J_x's cluster is directly blocking J_i (since indirect blocking requires that a lock-holding, ready job that J_i is waiting for is not scheduled).

Based on these considerations, it is possible to establish the following bound on overall direct and indirect pi-blocking [9].

Constraint 4 (Constraint 13 in [9]). *In any schedule of* τ *under the generalized FMLP*$^+$*:*

$$\forall T_x \in \tau^i : \sum_{q=1}^{n_r} \sum_{v=1}^{N^i_{x,q}} X^D_{x,q,v} + X^I_{x,q,v} \leq \sum_{u=1}^{n_r} \min\left(N_{i,u}, \sum_{T_y \in \tau(C(T_x))} N^i_{y,u} \right)$$

In addition to Constraint 4 above, which bounds both direct and indirect blocking, a second constraint that is applicable only to indirect blocking based on similar reasoning can reduce pessimism if a single T_x is responsible for most of the direct blocking (*e.g.*, if only T_x directly blocks J_i, than it cannot also cause indirect blocking).

Constraint 5 (Constraint 14 in [9]). *In any schedule of* τ *under the generalized FMLP*$^+$*:*

$$\forall T_x \in \tau^i : \sum_{q=1}^{n_r} \sum_{v=1}^{N^i_{x,q}} X^I_{x,q,v} \leq \sum_{u=1}^{n_r} \min\left(N_{i,u}, \sum_{\substack{T_y \in \tau(C(T_x)) \\ T_y \neq T_x}} N^i_{y,u} \right).$$

Next, we derive new constraints applicable to the generalized FMLP$^+$ that have not yet appeared in prior work.

4) New Constraints: We begin by encoding the fact that each other task T_x causes J_i to incur pi-blocking at most once per segment.

Constraint 6. *In any JLFP schedule of* τ *under the generalized FMLP*$^+$*:*

$$\forall T_x \in \tau^i : \sum_{q=1}^{n_r} \sum_{v=1}^{N^i_{x,q}} X^D_{x,q,v} + X^I_{x,q,v} + X^P_{x,q,v} \leq 1 + w_i + 2N_i.$$

Proof: Suppose not. According to Defs. 3 and 4, each time that J_i issues a request, releases a resource, is released, or resumes from a locking-unrelated self-suspension, J_i starts a new segment. Thus, J_i consists of at most $1+w_i+2N_i$ segments. Thus, if the above invariant is violated in some schedule, then it follows from the pigeon-hole principle that during some segment J_i incurred pi-blocking due to more than one request of T_x. By Lemma 5, this is impossible during a request segment, and by Lemma 13, this is also impossible during an independent segment. Contradiction. ■

Next, we establish two constraints that exploit the fact that, under restricted segment boosting, only lower-priority jobs can cause J_i to incur s-aware pi-blocking if J_i is ready. Lemma 7 already establishes this observation for independent segments; an analogous argument applies to request segments as well.

Lemma 14. *Let* $C_k = C(T_i)$ *denote the cluster to which task* T_i *has been assigned. If* $rs(J_i, t)$*,* J_i *is ready and incurs s-aware pi-blocking at time* t*, and* $J_b = boosted(C_k, t)$*, then* $\Upsilon(J_i) < \Upsilon(J_b)$*.*

Proof: Analogous to the proof of Lemma 7. Since J_i is ready at time t, if J_i incurs s-aware pi-blocking at time t, then a lower-priority job J_l is included in $cb'(J_b, t)$ (otherwise, if only higher-priority jobs are included in $cb'(J_b, t)$, then J_i does not incur pi-blocking at time t). By the definition of $cb(J_b, t)$, J_b's co-boosting set include only higher-priority jobs. Thus if $cb'(J_b, t)$ contains a J_l such that $\Upsilon(J_i) < \Upsilon(J_l)$, then $\Upsilon(J_i) < \Upsilon(J_b)$ since $\Upsilon(J_l) < \Upsilon(J_b)$, assuming that the underlying JLFP order is transitive. ■

Lemmas 7 and 14 together imply that preemption pi-blocking is limited by the number of lower-priority jobs that exist in cluster $C(T_i)$ while J_i is pending. To express this observation as a constraint, we introduce the following definition.

Def. 8. We let H_x^i denote a bound on the maximum number of lower-priority jobs of task T_x that exist while J_i is pending.

The exact definition of H_x^i necessarily depends on the underlying JLFP policy. We provide suitable definitions for FP and EDF scheduling.

Def. 9. Under FP scheduling:

$$H_x^i \leq \begin{cases} 0 & \text{if } x < i \\ njobs(T_x, r_i) & \text{otherwise.} \end{cases}$$

Def. 10. Under EDF scheduling:

$$H_x^i \leq \begin{cases} 0 & \text{if } r_i + d_x < d_i \\ njobs(T_x, r_i + d_x - d_i) & \text{otherwise.} \end{cases}$$

With the definition of H_x^i, the bound implied by Lemmas 7 and 14 can be expressed with the following constraint.

Constraint 7. *In any JLFP schedule of τ under the generalized FMLP$^+$:*

$$\forall \ell_q : \forall T_x \in \tau^i : \sum_{v=1}^{N_{x,q}^i} X_{x,q,v}^P \leq H_x^i \cdot N_{x,q}.$$

Proof: Suppose not. Then there exists a schedule in which J_i incurs preemption pi-blocking due to more than $H_x^i \cdot N_{x,q}$ requests issued by jobs of some T_x for some resource ℓ_q. Since each job of T_x issues at most $N_{x,q}$ requests for resource ℓ_q, and since there exist at most H_x^i jobs of T_x that have lower priority than J_i while J_i is pending, this implies that some higher-priority job of T_x caused J_i to incur preemption pi-blocking. By Lemma 7, this is impossible during an independent segment, and by Lemma 14, this is also impossible during a request segment. Contradiction. ∎

This concludes our analysis of the generalized FMLP$^+$. Next, we briefly comment on the special case of singleton clusters (*i.e.*, clusters consisting of only a single processor).

5) The special case of $m_k = 1$: Finally, we consider the special case of $m_k = 1$, that is, when T_i's cluster can be analyzed using uniprocessor schedulability tests. This situation typically arises under partitioned scheduling. However, since the generalized FMLP$^+$ supports arbitrary non-uniform cluster sizes, singleton clusters can also occur in combination with clusters containing more than one processor.

If there is only a single processor in T_i's cluster, local lower-priority tasks cannot execute whenever J_i is ready. Therefore, the number of times that J_i self-suspends or waits to acquire a lock implies a bound on the number of intervals during which local lower-priority tasks have a chance to issue requests that could pi-block. This observation has been previously formalized in [9], from where we adopt the following constraint. For brevity, we let $\tau^{ll} \triangleq \{T_x \mid T_x \in \tau(C(T_i)) \wedge H_x^i > 0\}$ denote the set of local tasks that potentially release lower-priority jobs.

Constraint 8 (analogous to Constraint 11 in [9])**.** *Let $C_k = C(T_i)$ denote T_i's assigned cluster. In any JLFP schedule of τ under the generalized FMLP$^+$ if $m_k = 1$:*

$$\forall T_x \in \tau^{ll} : \sum_{q=1}^{n_r} \sum_{v=1}^{N_{x,q}^i} X_{x,q,v}^D + X_{x,q,v}^I + X_{x,q,v}^P \leq 1 + w_i + N_i.$$

Constraint 8 resembles Constraint 6, but Constraint 6 depends on the number of segments, whereas Constraint 6 depends only on the number of suspensions. Since each lock request causes at most one suspension, but implies the existence of two segments, Constraint 8 is more limiting than Constraint 6.

Finally, uniprocessor schedulability tests distinguish between *local* pi-blocking (*i.e.*, pi-blocking caused by tasks executing on the same processor) and *remote* pi-blocking (*i.e.*, pi-blocking caused by tasks executing on remote processors. The distinction is made because local pi-blocking (which implies that the processor is not idle if the lock-holding task is ready) can be accounted for more accurately than remote pi-blocking (which must be treated like a self-suspension, as the processor may idle). Thus, to achieve best results, it is necessary to derive separate bounds on local and remote pi-blocking, as explained in [9].

In contrast to the uniprocessor case, local and remote pi-blocking is not distinguished under global schedulability analysis (which under clustered scheduling is applied within each cluster), as even "local" blocking carries self-suspension-like effects (*i.e.*, "local" pi-blocking does not imply the absence of idle processors if $m_k > 1$).

6) Implementation: We have implemented the presented LP-based analysis in *SchedCAT*, an open-source schedulability analysis toolkit freely available online at

- https://www.mpi-sws.org/~bbb/projects/schedcat.

SchedCAT uses the well-known *GNU Linear Programming Kit* (GLPK) to solve the generated linear programs. We plan to release our modifications to *SchedCAT*, which are currently available upon request, in a future public release of *SchedCAT*.