Internalizing Relational Parametricity in the Extensional Calculus of Constructions (Technical Appendix)

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1 Type System and Operational Semantics

Our overall system is an explicitly-typed version of the calculus of constructions, extended with an identity type and an elimination rule for equality based on equality reflection.

In Figure 1, we give the syntactic categories of our type system. We present our system with distinct syntactic categories for kinds (ranged over by metavariables κ), types (ranged over by metavariables X, Y, A, B, C, and type variables α, β) and terms (ranged over by metavariables e, and term variables x, y, z). We typically adopt the convention of using A, B, and C for type constructors of arbitrary kind, and X and Y for type constructors of base kind.

The kinds include the base kind of types $*$, the kind of term-indexed types $\Pi x : X.\ \kappa$, and the kind of type-indexed types $\Pi\alpha : \kappa.\ \kappa'$. Our base type constructors include universal quantification $\Pi\alpha : \kappa.\ X$, the dependent function space $\Pi x : X.\ Y$, and the identity type $e_1 =_X e_2$. On top of this, we permit abstracting over both term variables $\lambda x : X.\ A$ and type variables $\lambda\alpha : \kappa.\ A$, with corresponding applications $A\ e$ and $A\ B$. Of course, we can also refer to type variables α in type expressions.

The syntax of terms is also explicitly typed, with both explicitly-typed term $\lambda x : X.\ e$ and type $\lambda\alpha : \kappa.\ e$ lambda-abstractions, and corresponding applications $e\ e'$ and $e\ A$. The witness for equality proofs is the reflexivity term refl. There is no elimination form for this type, since we make use of the equality reflection principle [Martin-Löf(1984)], and therefore do not need an explicit eliminator for equality. We identify a subclass of terms v as values, which we take to be the lambda-terms $\lambda x : X.\ e$ and $\lambda\alpha : \kappa.\ e$, as well as the equality proof term refl.

As an aside, we present a system with distinct syntactic levels rather than as a pure type system [Barendregt(1991)]. Having different syntactic categories and judgements simplifies some theorem statements, but comes at the price of doubling the number of substitution lemmas, since we have to prove substitution properties once each for terms and types.

In Figure 2, we catalog the judgements we use in our system.

We have the judgement $\Gamma\ \mathsf{ok}$, which asserts that a context (consisting of term and type variables) is well-formed, and the judgement $\Gamma \vdash \kappa : \mathsf{kind}$, which describes when a kind is well-formed. Both of these judgements are given in Figure 3. Note that the base case of the well-kinding judgement — $\Gamma \vdash * : \mathsf{kind}$ does not require that the context be well-formed.

This illustrates a general principle in our choice of typing rules. We avoid making a context well-formedness requirement part of the derivations of our system. Instead, we state the well-formedness condition as a precondition to the theorems of our metatheory, and add sufficient premises to ensure that the context well-formedness condition can be derived as needed. For example, the rule for pi-kinds $\Gamma \vdash \Pi x : X.\ \kappa : \mathsf{kind}$ has a premise that $\Gamma \vdash X : *$, which lets us derive the well-formedness of the extended context $\Gamma, x : X$. One further convention that we follow is that our rules have implicit validity premises – if we have a rule ending in $\Gamma \vdash A : \kappa$, then there is an implicit premise that $\Gamma \vdash \kappa : \mathsf{kind}$ (and similarly for other judgements). This simplifies the soundness proof, but since these premises add clutter to the rules we omit them in the display.

In Figure 4, we give the well-kinding judgement $\Gamma \vdash A : \kappa$, which asserts that the type constructor A has the kind κ. The rules here are unsurprising: we have type variables α, abstractions over terms $\lambda x : X.\ A$ and types $\lambda \alpha : \kappa.\ A$, as well as the corresponding applications $A\ e$ and $A\ B$. We also have a few basic rules for forming types (of kind $*$) — polymorphic functions $\Pi \alpha : \kappa.\ X$, dependent functions $\Pi x : X.\ Y$, and the identity type $e_1 =_X e_2$. Finally, we have a conversion rule which says that if A has the kind κ, and $\kappa \equiv \kappa'$, then A also has the kind κ'.

In Figure 5 we give the typing rules for expressions. Variables are looked up in the environment, and we have the expected rules for abstractions and applications over types and terms. There is also a conversion rule for typing terms, and finally we have the rule for equality, which asserts that if e_1 and e_2 are convertible, then refl is a proof of inhabitation of the identity type $e_1 =_X e_2$.

The conversion rules for kinds and types are given in Figure 6. Equivalence of kinds is basically structural – if two kinds have the same shape, and equal type and term components, then they are equal. We model this by giving a substitution principle that says if we substitute equal terms or types into a kind, then the two resulting kinds are equal. Then, we close up under reflexivity, symmetry, and transitivity.

The equality judgement for type constructors is similar. As with kinds, we express the fact that equality is a congruence by giving rules which assert that substituting equal types and terms into a type yields equal results, and we also close up under reflexivity, symmetry, and transitivity. Since kinds have their own notion of equality, we also say that an equality proved at one kind is valid at any equal kind. Then, we add the β and η-rules of the lambda-calculus for the type and term abstractions in the language.

Similarly, we give the equality judgement for expressions in Figure 7. As with kinds and types, we use substitution of equal terms to express that the relation is a congruence, have reflexivity, transitivity, and symmetry rules, and a rule asserting that an equality at one type is also an equality at any equal type. Then, we have the β and η rules for type and term abstractions.

Identity types support a proof-irrelevance principle, called Axiom K [Hofmann and Streicher(1998)], which asserts all proofs of equalities are equal. Furthermore, term equality also contains the equality reflection rule. If $\Gamma \vdash e_p : e =_X e'$, then $\Gamma \vdash e \equiv e' : X$. Note that this rule makes typechecking undecidable, since a well-typing derivation may need to invent equality proofs out of thin air.

In Figure 8, we give the operational semantics of our programming language. This is a standard small-step, call-by-name semantics, with no surprises to it. The reason that we give an operational semantics, rather than (say) a β-convertibility relation, is that full-β does not have to terminate. Since we support the equality reflection principle, we can prove the well-typedness of the Y-combinator in open contexts – so the arbitrary β-reduction of well-typed open terms is not necessarily guaranteed to terminate. However, since the system is (as we will prove) consistent, this means that all *closed* terms do reduce to values. This problem, and our approach to resolving it, are both quite familiar from Nuprl.

Another point worth noting is that our definitional equality is just the $\beta\eta$-theory of the lambda calculus (plus axiom K for identity types). We have not yet included any parametricity properties in our rules for equality. This choice is made for expository purposes.

2 Model

In this section, we describe the model construction we use to interpret the calculus of constructions. Our overall semantics is a realizability model, in which types are interpreted as relations between

$$
\begin{array}{llll}
\kappa & ::= & * \mid \Pi x : X.\, \kappa \mid \Pi \alpha : \kappa.\, \kappa' & \text{Kinds} \\
\\
X, A & ::= & \Pi \alpha : \kappa.\, X \mid \Pi x : X.\, Y \mid e =_X e' & \text{Types} \\
 & \mid & \lambda x : X.\, A \mid A\ e \mid & \\
 & \mid & \lambda \alpha : \kappa.\, A \mid A\ B \mid \alpha & \\
\\
e & ::= & x \mid \lambda x : X.\, e \mid \quad \mid e\ e & \text{Terms} \\
 & \mid & \lambda \alpha : \kappa.\, e \mid e\ A \mid \mathsf{refl} & \\
\\
v & ::= & \lambda x : X.\, e \mid \lambda \alpha : \kappa.\, e \mid \mathsf{refl} & \text{Values} \\
\\
\Gamma & ::= & \cdot \mid \Gamma, x : X \mid \Gamma, \alpha : \kappa & \text{Contexts}
\end{array}
$$

Figure 1: Syntax

$\Gamma\ \mathsf{ok}$	Context Well-formedness
$\Gamma \vdash \kappa : \mathsf{kind}$	Kind Well-formedness
$\Gamma \vdash A : \kappa$	Well-kinding for Type Constructors
$\Gamma \vdash e : X$	Well-typing of Expressions
$\Gamma \vdash \kappa \equiv \kappa' : \mathsf{kind}$	Definitional Equality for Kinds
$\Gamma \vdash A \equiv A' : \kappa$	Definitional Equality for Kinds
$\Gamma \vdash e \equiv e' : X$	Definitional Equality for Kinds
$e \mapsto e'$	Operational Semantics

Figure 2: Summary of Judgments

$$\boxed{\Gamma\ \mathsf{ok}}$$

$$\frac{}{\cdot\ \mathsf{ok}}$$

$$\frac{\Gamma\ \mathsf{ok} \qquad \Gamma \vdash X : *}{\Gamma, x : X\ \mathsf{ok}} \qquad\qquad \frac{\Gamma\ \mathsf{ok} \qquad \Gamma \vdash \kappa : \mathsf{kind}}{\Gamma, \alpha : \kappa\ \mathsf{ok}}$$

$$\boxed{\Gamma \vdash \kappa : \mathsf{kind}}$$

$$\frac{}{\Gamma \vdash * : \mathsf{kind}} \qquad \frac{\Gamma \vdash X : * \qquad \Gamma, x : X \vdash \kappa : \mathsf{kind}}{\Gamma \vdash \Pi x : X.\, \kappa : \mathsf{kind}} \qquad \frac{\Gamma \vdash \kappa : \mathsf{kind} \qquad \Gamma, \alpha : \kappa \vdash \kappa' : \mathsf{kind}}{\Gamma \vdash \Pi \alpha : \kappa.\, \kappa' : \mathsf{kind}}$$

Figure 3: Context and Kind Well-formedness

$$\boxed{\Gamma \vdash A : \kappa}$$

$$\frac{\Gamma \vdash \kappa : \mathsf{kind} \qquad \Gamma, \alpha : \kappa \vdash Y : *}{\Gamma \vdash \Pi\alpha : \kappa.\, Y : *}$$

$$\frac{\Gamma \vdash X : * \qquad \Gamma, x : X \vdash Y : *}{\Gamma \vdash \Pi x : X.\, Y : *} \qquad \frac{\Gamma \vdash e : X \qquad \Gamma \vdash e' : X}{\Gamma \vdash e =_X e' : *}$$

$$\frac{\alpha : \kappa \in \Gamma}{\Gamma \vdash \alpha : \kappa} \qquad \frac{\Gamma \vdash X : * \qquad \Gamma, x : X \vdash A : \kappa}{\Gamma \vdash \lambda x : X.\, A : \Pi x : X.\, \kappa} \qquad \frac{\Gamma \vdash \kappa : \mathsf{kind} \qquad \Gamma, \alpha : \kappa \vdash A : \kappa'}{\Gamma \vdash \lambda\alpha : \kappa.\, A : \Pi\alpha : \kappa.\, \kappa'}$$

$$\frac{\Gamma \vdash A : \Pi x : X.\, \kappa \qquad \Gamma \vdash e : X}{\Gamma \vdash A\ e : [e/x]\kappa} \qquad \frac{\Gamma \vdash A : \Pi\alpha : \kappa.\, \kappa' \qquad \Gamma \vdash A' : \kappa}{\Gamma \vdash A\ A' : [A'/\alpha]\kappa'}$$

$$\frac{\Gamma \vdash A : \kappa' \qquad \Gamma \vdash \kappa \equiv \kappa' : \mathsf{kind}}{\Gamma \vdash A : \kappa}$$

Figure 4: Kinding for Type Constructors

$$\boxed{\Gamma \vdash e : X}$$

$$\frac{x : X \in \Gamma}{\Gamma \vdash x : X} \qquad \frac{\Gamma \vdash e : Y \qquad \Gamma \vdash X \equiv Y : *}{\Gamma \vdash e : X}$$

$$\frac{\Gamma \vdash \kappa : \mathsf{kind} \qquad \Gamma, \alpha : \kappa \vdash e : Y}{\Gamma \vdash \lambda\alpha : \kappa.\, e : \Pi\alpha : \kappa.\, Y} \qquad \frac{\Gamma \vdash e : \Pi\alpha : \kappa.\, Y \qquad \Gamma \vdash A : \kappa}{\Gamma \vdash e\ A : [A/\alpha]Y}$$

$$\frac{\Gamma, x : X \vdash e : Y}{\Gamma \vdash \lambda x : X.\, e : \Pi x : X.\, Y} \qquad \frac{\Gamma \vdash e : \Pi x : X.\, Y \qquad \Gamma \vdash e' : X}{\Gamma \vdash e\ e' : [e'/x]Y}$$

$$\frac{\Gamma \vdash e_1 \equiv e_2 : X}{\Gamma \vdash \mathsf{refl} : e_1 =_X e_2}$$

Figure 5: Expression Typing

$$\boxed{\Gamma \vdash \kappa \equiv \kappa' : \mathsf{kind}}$$

$$\frac{\Gamma \vdash e \equiv e' : X \qquad \Gamma, x : X \vdash \kappa : \mathsf{kind}}{\Gamma \vdash [e/x]\kappa \equiv [e'/x]\kappa : \mathsf{kind}} \qquad \frac{\Gamma \vdash A \equiv A' : \kappa \qquad \Gamma, \alpha : \kappa \vdash \kappa' : \mathsf{kind}}{\Gamma \vdash [A/\alpha]\kappa' \equiv [A'/\alpha]\kappa' : \mathsf{kind}}$$

$$\frac{\Gamma \vdash \kappa_1 \equiv \kappa_1' : \mathsf{kind} \qquad \Gamma, \alpha : \kappa_1 \vdash \kappa_2 \equiv \kappa_2' : \mathsf{kind}}{\Gamma \vdash \Pi\alpha : \kappa_1.\, \kappa_2 \equiv \Pi\alpha : \kappa_1'.\, \kappa_2' : \mathsf{kind}} \qquad \frac{\Gamma \vdash X \equiv X' : * \qquad \Gamma, x : X \vdash \kappa \equiv \kappa' : \mathsf{kind}}{\Gamma \vdash \Pi x : X.\, \kappa \equiv \Pi x : X'.\, \kappa' : \mathsf{kind}}$$

$$\frac{\Gamma \vdash \kappa : \mathsf{kind}}{\Gamma \vdash \kappa \equiv \kappa : \mathsf{kind}} \qquad \frac{\Gamma \vdash \kappa \equiv \kappa' : \mathsf{kind}}{\Gamma \vdash \kappa' \equiv \kappa : \mathsf{kind}} \qquad \frac{\Gamma \vdash \kappa_1 \equiv \kappa_2 : \mathsf{kind} \qquad \Gamma \vdash \kappa_2 \equiv \kappa_3 : \mathsf{kind}}{\Gamma \vdash \kappa_1 \equiv \kappa_3 : \mathsf{kind}}$$

$$\boxed{\Gamma \vdash A \equiv A' : \kappa}$$

$$\frac{\Gamma \vdash e \equiv e' : X \qquad \Gamma, x : X \vdash A : \kappa}{\Gamma \vdash [e/x]A \equiv [e'/x]A : [e/x]\kappa} \qquad \frac{\Gamma \vdash A \equiv A' : \kappa \qquad \Gamma, \alpha : \kappa \vdash B : \kappa'}{\Gamma \vdash [A/\alpha]B \equiv [A'/\alpha]B : [A/\alpha]\kappa'}$$

$$\frac{\Gamma \vdash A \equiv A' : \kappa' \qquad \Gamma \vdash \kappa \equiv \kappa' : \mathsf{kind}}{\Gamma \vdash A \equiv A' : \kappa}$$

$$\frac{\Gamma \vdash A : \kappa}{\Gamma \vdash A \equiv A : \kappa} \qquad \frac{\Gamma \vdash A \equiv A' : \kappa}{\Gamma \vdash A' \equiv A : \kappa} \qquad \frac{\Gamma \vdash A_1 \equiv A_2 : \kappa \qquad \Gamma \vdash A_2 \equiv A_3 : \kappa}{\Gamma \vdash A_1 \equiv A_3 : \kappa}$$

$$\frac{\Gamma \vdash \kappa \equiv \kappa' : \mathsf{kind} \qquad \Gamma, \alpha : \kappa \vdash X \equiv X' : *}{\Gamma \vdash \Pi\alpha : \kappa.\, X \equiv \Pi\alpha : \kappa'.\, X' : *} \qquad \frac{\Gamma \vdash X \equiv X' : \mathsf{kind} \qquad \Gamma, x : X \vdash Y \equiv Y' : *}{\Gamma \vdash \Pi x : X.\, Y \equiv \Pi x : X'.\, Y' : *}$$

$$\frac{\Gamma \vdash \kappa \equiv \kappa' : \mathsf{kind} \qquad \Gamma, \alpha : \kappa \vdash B \equiv B' : \kappa''}{\Gamma \vdash \lambda\alpha : \kappa.\, B \equiv \lambda\alpha : \kappa'.\, B' : \Pi\alpha : \kappa.\, \kappa''} \qquad \frac{\Gamma \vdash X \equiv X' : * \qquad \Gamma, x : X \vdash B \equiv B' : \kappa}{\Gamma \vdash \lambda x : X.\, B \equiv \lambda x : X'.\, B' : \Pi x : X.\, \kappa}$$

$$\frac{\Gamma \vdash C \equiv C' : \Pi\alpha : \kappa.\, \kappa' \qquad \Gamma \vdash A \equiv A' : \kappa}{\Gamma \vdash C\ A \equiv C'\ A' : [A/\alpha]C} \qquad \frac{\Gamma \vdash C \equiv C' : \Pi x : X.\, \kappa \qquad \Gamma \vdash e \equiv e' : X}{\Gamma \vdash C\ e \equiv C'\ e' : [e/x]\kappa}$$

$$\frac{\Gamma \vdash \lambda x : X.\, A : \Pi x : X.\, \kappa \qquad \Gamma \vdash e : X}{\Gamma \vdash (\lambda x : X.\, A)\ e \equiv [e/x]A : [e/x]\kappa} \qquad \frac{\Gamma \vdash \lambda\alpha : \kappa.\, A : \Pi\alpha : \kappa.\, \kappa' \qquad \Gamma \vdash A' : \kappa'}{\Gamma \vdash (\lambda\alpha : \kappa.\, A)\ A' \equiv [A'/\alpha]A : [A'/\alpha]\kappa'}$$

$$\frac{\Gamma, x : X \vdash A\ x \equiv A'\ x : \kappa \qquad \Gamma \vdash A : \Pi x : X.\, \kappa \qquad \Gamma \vdash A' : \Pi x : X.\, \kappa}{\Gamma \vdash A \equiv A' : \Pi x : X.\, \kappa}$$

$$\frac{\Gamma, \alpha : \kappa \vdash A\ \alpha \equiv A'\ \alpha : \kappa' \qquad \Gamma \vdash A : \Pi\alpha : \kappa.\, \kappa' \qquad \Gamma \vdash A' : \Pi\alpha : \kappa.\, \kappa'}{\Gamma \vdash A \equiv A' : \Pi\alpha : \kappa.\, \kappa'}$$

Figure 6: Kind and Type Equality

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : X}$$

$$\frac{\Gamma \vdash e_p : e =_X e'}{\Gamma \vdash e \equiv e' : X} \qquad \frac{\Gamma \vdash e \equiv e' : Y \qquad \Gamma \vdash X \equiv Y : *}{\Gamma \vdash e \equiv e' : X}$$

$$\frac{\Gamma \vdash e_0 \equiv e_0' : Y \qquad \Gamma, x : Y \vdash e : X}{\Gamma \vdash [e_0/x]e \equiv [e_0'/x]e : [e_0/x]X} \qquad \frac{\Gamma \vdash A \equiv A' : \kappa \qquad \Gamma, \alpha : \kappa \vdash e : X}{\Gamma \vdash [A/\alpha]e \equiv [A'/\alpha]e : [A/\alpha]X}$$

$$\frac{\Gamma \vdash e : X}{\Gamma \vdash e \equiv e : X} \qquad \frac{\Gamma \vdash e \equiv e' : X}{\Gamma \vdash e' \equiv e : X} \qquad \frac{\Gamma \vdash e_1 \equiv e_2 : X \qquad \Gamma \vdash e_2 \equiv e_3 : X}{\Gamma \vdash e_1 \equiv e_3 : X}$$

$$\frac{\Gamma \vdash \lambda\alpha : \kappa.\ e : \Pi\alpha : \kappa.\ X \qquad \Gamma \vdash A : \kappa}{\Gamma \vdash (\lambda\alpha : \kappa.\ e)A \equiv [A/\alpha]e : [A/\alpha]X}$$

$$\frac{\Gamma, \alpha : \kappa \vdash e\ \alpha \equiv e'\ \alpha : Y \qquad \Gamma \vdash e : \Pi\alpha : \kappa.\ Y \qquad \Gamma \vdash e' : \Pi\alpha : \kappa.\ Y}{\Gamma \vdash e \equiv e' : \Pi\alpha : \kappa.\ Y}$$

$$\frac{\Gamma \vdash \lambda x : X.\ e : \Pi x : X.\ Y \qquad \Gamma \vdash e' : X}{\Gamma \vdash (\lambda x : X.\ e)\ e' \equiv [e'/x]e : [e'/x]Y}$$

$$\frac{\Gamma, x : X \vdash e\ x \equiv e'\ x : Y \qquad \Gamma \vdash e : \Pi x : X.\ Y \qquad \Gamma \vdash e' : \Pi x : X.\ Y}{\Gamma \vdash e \equiv e' : \Pi x : X.\ Y}$$

$$\frac{\Gamma \vdash e : e_1 =_X e_2 \qquad \Gamma \vdash e' : e_1 =_X e_2}{\Gamma \vdash e \equiv e' : e_1 =_X e_2}$$

$$\frac{\Gamma \vdash \kappa \equiv \kappa' : \mathsf{kind} \qquad \Gamma, \alpha : \kappa \vdash e \equiv e' : Y}{\Gamma \vdash \lambda\alpha : \kappa.\ e \equiv \lambda\alpha : \kappa'.\ e' : \Pi\alpha : \kappa.\ \Pi\alpha : \kappa.\ Y} \qquad \frac{\Gamma \vdash X \equiv X' : * \qquad \Gamma, x : X \vdash e \equiv e' : Y}{\Gamma \vdash \lambda x : X.\ e \equiv \lambda x : X'.\ e' : \Pi x : X.\ Y}$$

$$\frac{\Gamma \vdash e \equiv e' : \Pi\alpha : \kappa.\ Y \qquad \Gamma \vdash A \equiv A' : \kappa}{\Gamma \vdash e\ A \equiv e'\ A' : [A/\alpha]Y} \qquad \frac{\Gamma \vdash t \equiv t' : \Pi x : X.\ Y \qquad \Gamma \vdash e \equiv e' : X}{\Gamma \vdash t\ e \equiv t'\ e' : [e/x]Y}$$

Figure 7: Expression Equality

$$\boxed{e \mapsto e'}$$

$$\frac{e_1 \mapsto e_1'}{e_1\ e_2 \mapsto e_1'\ e_2} \qquad \frac{}{(\lambda x : X.\ e)\ e' \mapsto [e'/x]e} \qquad \frac{e \mapsto e'}{e\ X \mapsto e'\ X} \qquad \frac{}{(\lambda\alpha : \kappa.\ e)\ X \mapsto [X/\alpha]e}$$

Figure 8: Operational Semantics

closed expressions e. However, since the syntactic types appearing within expressions are computationally irrelevant, we simplify matters by working with relations over **Exp**, the set of equivalence classes of closed expressions modulo differences in syntactic types. That is, in the model, we consider $\lambda x : X.\ e = \lambda x : Y.\ e$, $\lambda\alpha : \kappa.\ e = \lambda\alpha : \kappa'.\ e$, and $e\ A = e\ B$, for *arbitrary* X, Y, κ, κ', A, B. This is analogous to building the model with type-erased terms, and we will sometimes write $\_$ in place of an irrelevant type annotation/argument.

First, we describe what quasi-PERs are, and then we describe how we interpret each of the judgements of our type theory – contexts, kinds, types, terms, and equalities.

2.1 Quasi-PERs

A *quasi-PER* (also known as a "difunctional relation", or "zig-zag complete relation"), is a relaxation of the concept of a partial equivalence relation to the asymmetric case. Formally, they are defined as follows:

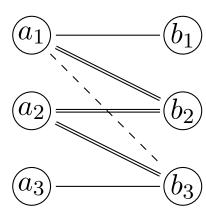
Definition 1. (Quasi-PER) Given sets A and B, a *quasi-PER* $R \subseteq A \times B$ is a relation such that for all $a, a' \in A$ and $b, b' \in B$, if $(a, b) \in R$ and $(a', b') \in R$ and $(a', b) \in R$, then $(a, b') \in R$.

In Figure 9, we give a diagram of the quasi-PER condition, which should illustrate why they are also called "zig-zag complete relations" — if there is a "zig" between two pairs of related points, then there must also be a "zag".

For our purposes, quasi-PERs are interesting because their asymmetry lets us prove representation independence results (i.e., we can relate different representations of a datatype), without losing the possibility of using the logical relation to *define* equality of datatypes.

To understand why, suppose that we have a type X, which is interpreted by a relation $R \subseteq A \times B$. Furthermore, suppose that we have three terms of type X, e_1, e_2 and e_3, with the property that $e_1 = e_2$ and $e_2 = e_3$ according to the equational theory of the language.

To show transitivity, we will need the following two properties. First, we will need a version of the fundamental theorem of logical relations, to tell us that each e_i is interpreted by $(a_i, b_i) \in R$. Secondly, we will need the soundness of the equality rules to tell us that $e_1 = e_2$ means $(a_1, b_2) \in R$ and that $e_2 = e_3$ means that $(a_2, b_3) \in R$. Then, the fact that R is a quasi-PER implies that (a_1, b_3) is also in R. Below, we illustrate how the zig-zag condition implies $(a_1, b_3) \in R$, by doubling the three lines that we use to reach the conclusion.



Just as with ordinary PERs, quasi-PERs are closed under arbitrary intersections, but also like PERs they are not closed under unions. However, since the intersection equips quasi-PERs with a complete semilattice structure, we can define the join of a set of quasi-PERs as the least quasi-PER containing the union. We can also define the join directly, as the "zig-zag" closure, where we add the pairs necessary to ensure that the zig-zag condition holds. The construction closely resembles the join on partial equivalence relations, and just as with PERs, existential or union types defined using the join only support a weak (unpack-style) elimination form, rather than a projective elimination.

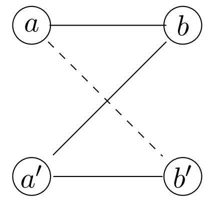


Figure 9: Quasi-PER

Notation If Q is a quasi-PER, then we will write $\bar{e} \in Q$ to denote a pair $(e, e') \in Q$. For $(a, b) \in Q$ and $(a', b') \in Q$, we will write $(a, b) \sim_Q (a', b')$ if $(a, b') \in Q$ and/or $(a', b) \in Q$. (We say "and/or" because "and" is equivalent to "or" here thanks to zig-zag completeness). We will also suppress the subscript and write $\overline{e_1} \sim \overline{e_2}$ when Q is obvious from context.

3 Semantics

We interpret relations with a set of mutually-recursive semantic interpretation functions, which we describe below. Given the high degree of mutual recursion, there is unfortunately no way to describe the interpretations without some degree of forward reference.

In order to minimize the number of forward references, we introduce some auxilliary semantic objects that will let us prove some basic structural theorems without having to do a simultaneous induction with the soundness of the whole semantics.

3.1 Contexts

The interpretation of the $\Gamma\ \mathsf{ok}$ judgement is the *set of grounding parallel substitutions* which satisfy it. We give the interpretation in Figure 10.

The interpretation of an empty context is just an empty substitution, and the interpretation of the context $\Gamma, x : X\ \mathsf{ok}$ is an element of $[\![\Gamma\ \mathsf{ok}]\!]$, together with a pair $\bar{e}$ of closed terms from the interpretation of $\Gamma \vdash X : *$. The interpretation of the context $\Gamma, \alpha : \kappa\ \mathsf{ok}$ is an element of $[\![\Gamma\ \mathsf{ok}]\!]$, together with a *triple* (A, A', R). Here, A and A' are closed syntactic types, and R is the semantic interpretation of the type. Note that there are no well-formedness constraints on the types: we do not need them, since the operational semantics never examines a type constructor, and the relation R carries all the necessary semantic constraints.

In Figure 10, we also define a notion of equivalence $\gamma \sim_\Gamma \gamma'$ on contexts. This relation says that the relation part of type variables must be equal (ignoring the syntactic type constructors), and that pairs of terms $\overline{e_1}/x$ and $\overline{e_2}/x$ must lie in the same equivalence class of the relation. This definition does form a quasi-PER, but actually proving that fact can only be done after the proof of soundness of the interpretation of types and kinds. This is due to the fact that the definition is "biased" — note that the choice of quasi-PER to interpret types and kinds comes from the left-hand side. By construction, this choice will not matter, since all of our semantic functions will be invariant under this equivalence, but we cannot show that yet.

In Figure 11, we give some notations on contexts that we will use in the sequel. An element γ contains left- and right-bindings for each of the variables in it, and γ_1 is the left projection of the environment, and γ_2 is the right projection of the environment. We write $\gamma(e)$ to indicate the pair of terms we get from the left and right substitutions of γ applied to e.

3.2 Kinds

We give the interpretation of derivations of the kind judgement in Figure 12. We begin by giving a "pre-interpretation", which interprets kinds less precisely than we will ultimately want, but whose presence simplifies our well-definedness arguments since $\|\kappa\|$ is defined without reference to the context.

The full interpretation of kind $[\![\Gamma \vdash \kappa : \mathsf{kind}]\!]$, on the other hand, *is* relative to a context γ. The interpretation of the base kind $\Gamma \vdash * : \mathsf{kind}$ is a slight restriction of the set of quasi-PERs on expressions,

$$\textsc{Cand} \triangleq \left\{ R \in \textsc{Qper}(\mathrm{Exp}, \mathrm{Exp}) \;\middle|\; \begin{array}{l} \forall (e_1, e_2) \in R.\ e_1 \downarrow \wedge\ e_2 \downarrow\ \wedge \\ \forall (e_1, e_2) \in R, (e_1', e_2') \in \mathrm{Exp}^2.\ e_1 \leftrightarrow^* e_1' \wedge e_2 \leftrightarrow^* e_2' \implies (e_1', e_2') \in R \end{array} \right\}$$

Namely, we restrict ourselves to quasi-PERs over *terminating* expressions (or more precisely, equivalence classes of such expressions modulo differences in syntactic types), and further require that quasi-PERs be closed under expansion and reduction.

The interpretation of the higher kind $\Gamma \vdash \Pi\alpha : \kappa.\ \kappa' : \mathsf{kind}$ is morally a currying of the interpretation $\Gamma, \alpha : \kappa \vdash \kappa' : \mathsf{kind}$. In particular, it is the set of functions $\|\kappa\| \to \|\kappa'\|$, such that (1) we ignore the syntactic part of an argument triple $(\overline{X}, R)$, and (2) on any argument $R \in [\![\Gamma \vdash \kappa : \mathsf{kind}]\!]\ \gamma$, the result is in $[\![\Gamma, \alpha : \kappa \vdash \kappa' : \mathsf{kind}]\!]\ (\gamma, (\overline{A}, R)/\alpha)$. On anything outside the dependent domain, we require the result to be a fixed element of the pre-interpretation.

Similarly, the interpretation of the higher kind $\Gamma \vdash \Pi x : X.\ \kappa' : \mathsf{kind}$ is a subset of the currying of the interpretation $\Gamma, x : X \vdash \kappa' : \mathsf{kind}$. However, in this case, we require that the type constructor return the same answer for all equivalent $\bar{e} \sim_X \overline{e'}$.

3.3 Type Constructors

In Figure 13, we give the interpretation of the type constructors of our language, as a function that takes a derivation and returns an element of the appropriate semantic kind. The first line of the definition says that the interpretation of a derivation $\Gamma \vdash \alpha : \kappa$ proceeds by looking up α in the environment argument γ, and returning the relation component of the triple.

The interpretation of a lambda-abstraction $\lambda\alpha : \kappa.\ A$ is just a function that takes an argument in κ, and returns the result of interpreting A in an extended environment. Likewise, a type constructor application $A\ B$ takes the meaning of A, and passes it the syntax and semantics of B. Similarly, a term abstraction $\lambda x : X.\ A$ just returns a function which takes a pair $\bar{e}$, and returns the interpretation of A in an extended environment, and the application $A\ e$ passes the substitution instance of $\gamma(e)$ to the interpretation of A.

When an equality rule is used, we simply interpret the subderivation and return that as our answer. The presence of equalities is why we interpret full typing derivations. As a result, however, we will need to do a coherence proof on our semantic interpretations, though the fact that the equality rules do nothing interesting means the coherence proof is easy.

Next, we give the interpretations of types. The kinding interpretation requires that each type be a candidate relation, and so we define each type as a relation on expressions. The interpretation of the function type $\Pi x : X.\ Y$ is the set of expressions that take related pairs of arguments in

$$
\begin{array}{lcl}
[\![\cdot\ \mathsf{ok}]\!] & = & \{\langle\rangle\} \\
[\![\Gamma, x : X\ \mathsf{ok}]\!] & = & \{(\gamma, \bar{e}/x) \mid \gamma \in [\![\Gamma\ \mathsf{ok}]\!] \wedge \bar{e} \in [\![\Gamma \vdash X : *]\!]\ \gamma\} \\
[\![\Gamma, \alpha : \kappa\ \mathsf{ok}]\!] & = & \left\{(\gamma, (\overline{A}, R)/\alpha) \;\middle|\; \gamma \in [\![\Gamma\ \mathsf{ok}]\!] \wedge (\overline{A}, R) \in (\mathrm{Type} \times \mathrm{Type}) \times [\![\Gamma \vdash \kappa : \mathsf{kind}]\!]\ \gamma\right\}
\end{array}
$$

$$
\begin{array}{lcl}
\langle\rangle \sim_{\cdot} \langle\rangle & \iff & \text{always} \\
(\gamma_1, \overline{e_1}/x) \sim_{(\Gamma, x:X)} (\gamma_2, \overline{e_2}/x) & \iff & \gamma_1 \sim_\Gamma \gamma_2 \wedge \overline{e_1} \sim_{[\![\Gamma \vdash X:*]\!]\ \gamma_1} \overline{e_2} \\
(\gamma_1, (\overline{A_1}, R_1)/\alpha) \sim_{(\Gamma, \alpha:\kappa)} (\gamma_2, (\overline{A_2}, R_2)/\alpha) & \iff & \gamma_1 \sim_\Gamma \gamma_2 \wedge R_1 = R_2
\end{array}
$$

Figure 10: Environment Semantics

X to related pairs of expressions in the relation for Y, in the context extended by the argument pair. This is basically the same as the usual rule for function types in logical relations, minimally adjusted to support dependency. Likewise, the interpretation of the polymorphic type $\Pi\alpha : \kappa.\ X$ says that a pair of terms is in the relation, if for every relation R in the kind κ, the bodies are related at the expression relation for X, in the environment augmented with R for α.

The interpretation for the identity type $e_1 =_X e_2$ is of a pair of terms reducing to $\{(\mathsf{refl}, \mathsf{refl})\}$ when e_1 and e_2 are related, and is the empty set otherwise. Observe that refl is a proof *only* when the equality holds, and that because the identity type is interpreted by a relation containing at most one pair, our model ensures it satisfies axiom K.

3.4 Other Judgements

Since we are building a term model, we do not need to give an explicit interpretation of the expression typing or equality derivations. We will establish that we got these rules right as part of the proof of the soundness theorem, when we prove the fundamental lemma of logical relations and show that the syntactic equality judgement is sound with respect to the semantic expression relation.

As a result, at this point in the paper, we have not yet established that our definition is actually well-defined. The reason is that the structure of a dependent type theory means that the well-definedness of our semantics is mutually inductive with the actual soundness property for the type theory.

4 Examples

In the following, we assume that the context Γ is well-formed, and that any context $\gamma \in [\![\Gamma\ \mathsf{ok}]\!]$, so that we can appeal to the fundamental property. Also, given a QPER Q, we define $Q^\dagger$ to be its closure under reduction and expansions.

4.1 Empty Type

Define $0 \triangleq \Pi\alpha : *.\ \alpha$.

Lemma 1. *(Uninhabitation of 0)* $[\![\Gamma \vdash 0 : *]\!]\ \gamma$ *is not inhabited.*

$$
\begin{array}{lcl}
(\langle\rangle)_1 & = & \langle\rangle \\
(\gamma, (e_1, e_2)/x)_1 & = & \gamma_1, e_1/x \\
(\gamma, ((A_1, A_2), R)/\alpha)_1 & = & \gamma_1, A_1/\alpha \\
\\
(\langle\rangle)_2 & = & \langle\rangle \\
(\gamma, (e_1, e_2)/x)_2 & = & \gamma_2, e_2/x \\
(\gamma, ((A_1, A_2), R)/\alpha)_2 & = & \gamma_2, A_2/\alpha \\
\\
\gamma(e) & = & (\gamma_1(e), \gamma_2(e)) \\
\gamma(A) & = & (\gamma_1(A), \gamma_2(A)) \\
\gamma(\kappa) & = & (\gamma_1(\kappa), \gamma_2(\kappa))
\end{array}
$$

Figure 11: Notation

$$
\begin{array}{lcl}
\|*\| & = & \mathrm{Rel}(\mathrm{Exp}, \mathrm{Exp}) \\
\|\Pi\alpha : \kappa.\ \kappa'\| & = & (\mathrm{Type}^2 \times \|\kappa\|) \to \|\kappa'\| \\
\|\Pi x : X.\ \kappa\| & = & (\mathrm{Exp} \times \mathrm{Exp}) \to \|\kappa\| \\
\\
!_* & = & \emptyset \\
!_{\Pi x:X.\ \kappa} & = & \lambda\bar{e} \in \mathrm{Exp}^2.\ !_\kappa \\
!_{\Pi\alpha:\kappa.\ \kappa'} & = & \lambda(\overline{A}, R) \in \mathrm{Type}^2 \times \|\kappa\|.\ !_{\kappa'}
\end{array}
$$

$$
\begin{array}{lcl}
[\![\Gamma \vdash * : \mathsf{kind}]\!]\ \gamma & = & \textsc{Cand} \\
[\![\Gamma \vdash \Pi\alpha : \kappa.\ \kappa' : \mathsf{kind}]\!]\ \gamma & = & \left\{ T \in \|\Pi\alpha : \kappa.\ \kappa'\| \;\middle|\; \begin{array}{l} \forall \overline{A}, \overline{B}, R \in \|\kappa\|\ .\ T(\overline{A}, R) = T(\overline{B}, R)\ \wedge \\ \forall \overline{A}, R \in [\![\Gamma \vdash \kappa : \mathsf{kind}]\!]\ \gamma. \\ \quad T(\overline{A}, R) \in [\![\Gamma, \alpha : \kappa \vdash \kappa' : \mathsf{kind}]\!]\ (\gamma, (\overline{A}, R)/\alpha)\ \wedge \\ \forall \overline{A}, R \notin [\![\Gamma \vdash \kappa : \mathsf{kind}]\!]\ \gamma.\ T(\overline{A}, R) = !_{\kappa'} \end{array} \right\} \\
[\![\Gamma \vdash \Pi x : X.\ \kappa : \mathsf{kind}]\!]\ \gamma & = & \mathsf{let}\ \hat{X} = [\![\Gamma \vdash X : *]\!]\ \gamma\ \mathsf{in} \\
 & & \left\{ R \in \|\Pi x : X.\ \kappa\| \;\middle|\; \begin{array}{l} \forall \bar{e}, \overline{e'} \in \hat{X}.\ \bar{e} \sim_{\hat{X}} \overline{e'} \implies R\ \bar{e} = R\ \overline{e'}\ \wedge \\ \forall \bar{e} \in \hat{X}.\ R\ \bar{e} \in [\![\Gamma, x : X \vdash \kappa : \mathsf{kind}]\!]\ (\gamma, \bar{e}/x)\ \wedge \\ \forall \bar{e} \notin \hat{X}. R\ \bar{e} = !_\kappa \end{array} \right\}
\end{array}
$$

Figure 12: Kind Semantics

$$
\begin{array}{lcl}
[\![\Gamma, \alpha : \kappa, \Gamma' \vdash \alpha : \kappa]\!] \ (\gamma, (\overline{A}, R)/\alpha, \gamma') & = & R \\
[\![\Gamma \vdash \lambda\alpha : \kappa.\ A : \Pi\alpha : \kappa.\ \kappa']\!] \ \gamma & = & \lambda(\overline{B}, R). \begin{cases} [\![\Gamma, \alpha : \kappa \vdash A : \kappa']\!] \ (\gamma, (\overline{B}, R)/\alpha) & \text{if } R \in [\![\Gamma \vdash \kappa : \mathsf{kind}]\!] \ \gamma \\ !_{\kappa'} & \text{otherwise} \end{cases} \\
[\![\Gamma \vdash A\ B : [B/\alpha]\kappa']\!] \ \gamma & = & [\![\Gamma \vdash A : \Pi\alpha : \kappa.\ \kappa']\!] \ \gamma \ (\gamma(B), [\![\Gamma \vdash B : \kappa]\!] \ \gamma) \\
[\![\Gamma \vdash \lambda x : X.\ A : \Pi x : X.\ \kappa]\!] \ \gamma & = & \lambda\bar{e}. \begin{cases} [\![\Gamma, x : X \vdash A : \kappa]\!] \ (\gamma, \bar{e}/x) & \text{if } \bar{e} \in [\![\Gamma \vdash X : *]\!] \ \gamma \\ !_{\kappa} & \text{otherwise} \end{cases} \\
[\![\Gamma \vdash A\ e : [e/x]\kappa]\!] \ \gamma & = & [\![\Gamma \vdash A : \Pi x : X.\ \kappa]\!] \ \gamma\ \gamma(e) \\
[\![\Gamma \vdash A : \kappa]\!] \ \gamma & = & [\![\Gamma \vdash A : \kappa']\!] \ \gamma \ (\text{when } \Gamma \vdash \kappa \equiv \kappa' : \mathsf{kind})
\end{array}
$$

$$
\begin{array}{l}
[\![\Gamma \vdash \Pi x : X.\ Y : *]\!] \ \gamma = \\
\qquad \left\{ (e_1, e_1') \;\middle|\; \begin{array}{l} e_1 \downarrow \wedge\ e_1' \downarrow \wedge \\ \forall\, (e_2, e_2') \in [\![\Gamma \vdash X : *]\!] \ \gamma.\ (e_1\ e_2, e_1'\ e_2') \in [\![\Gamma, x : X \vdash Y : *]\!] \ (\gamma, (e_2, e_2')/x) \end{array} \right\} \\
[\![\Gamma \vdash \Pi\alpha : \kappa.\ X : *]\!] \ \gamma = \\
\qquad \left\{ (e, e') \;\middle|\; \begin{array}{l} e \downarrow \wedge\ e' \downarrow \wedge \\ \forall A, A', R \in [\![\Gamma \vdash \kappa : \mathsf{kind}]\!] \ \gamma.\, (e\ A, e'\ A') \in [\![\Gamma, \alpha : \kappa \vdash X : *]\!] \ (\gamma, (A, A', R)/\alpha) \end{array} \right\} \\
[\![\Gamma \vdash e_1 =_X e_2 : *]\!] \ \gamma = \\
\qquad \{(e, e') \mid e \mapsto^* \mathsf{refl} \wedge e' \mapsto^* \mathsf{refl} \wedge (\gamma_1(e_1), \gamma_2(e_2)) \in [\![\Gamma \vdash X : *]\!] \ \gamma\}
\end{array}
$$

Figure 13: Type Constructor Semantics

Proof. Assume $(e, e') \in [\![\Gamma \vdash 0 : *]\!] \ \gamma$. Now, consider $(e\ 0, e'\ 0)$, instantiated with the empty relation. Hence $(e\ 0, e'\ 0) \in [\![\Gamma, \alpha : * \vdash \alpha : *]\!] \ (\gamma, (\bar{0}, \emptyset)/\alpha) = \emptyset$. This is a contradiction. Hence (e, e') cannot be in $[\![\Gamma \vdash 0 : *]\!] \ \gamma$. □

4.2 Coproducts

Define $A + B \triangleq \Pi\alpha : *.\ (A \to \alpha) \to (B \to \alpha) \to \alpha$. Define $\mathsf{inl} : A \to A + B = \lambda a.\ \lambda\alpha, l, r.\ l\ a$, and define $\mathsf{inr} : A \to A + B = \lambda b.\ \lambda\alpha, l, r.\ r\ b$.

Lemma 2. *(Reducing eta-expanded Church sums) If $(e, e') \in [\![\Gamma \vdash A + B : *]\!] \ \gamma$, then either $e\ (A + B)$ inl inr shares a reduct with $\mathsf{inl}\ t$ for some t and $e'\ (A + B)$ inl inr shares a reduct with $\mathsf{inl}\ t'$ for some t', or $e\ (A + B)$ inl inr shares a reduct with $\mathsf{inr}\ t$ for some t and $e'\ (A + B)$ inl inr shares a reduct with $\mathsf{inr}\ t'$ for some t'.*

Proof. Assume $(e, e') \in [\![\Gamma \vdash A + B : *]\!] \ \gamma$.

Now, note that $\mathsf{inl}(e)$ and $\mathsf{inr}(e')$ are not β-equivalent for any e and e', and hence do not share any reducts. Furthermore, note that $\mathsf{inl}(e) \mapsto \lambda\alpha, k_1, k_2.\ k_1\ e$ and $\mathsf{inr}(e) \mapsto \lambda\alpha, k_1, k_2.\ k_2\ e$. Define $\underline{\mathsf{inl}(e)}$ as $\lambda\alpha, k_1, k_2.\ k_1\ e$ and $\underline{\mathsf{inr}(e)}$ as $\lambda\alpha, k_1, k_2.\ k_2\ e$. Hence the relation:

$$
R = \begin{array}{l} \left\{ (\underline{\mathsf{inl}(e)}, \underline{\mathsf{inl}(e')}) \;\middle|\; (e, e') \in [\![\Gamma \vdash A : *]\!] \ \gamma \right\}^{\dagger} \\ \cup \\ \left\{ (\underline{\mathsf{inr}(e)}, \underline{\mathsf{inr}(e')}) \;\middle|\; (e, e') \in [\![\Gamma \vdash B : *]\!] \ \gamma \right\}^{\dagger} \end{array}
$$

is a candidate relation. □

Now, apply $A + B$ to e and e' choosing the relation R, so that $(e\ (A+B), e'\ (A+B)) \in (\hat{A} \to R) \to (\hat{B} \to R) \to R$, where $\hat{A} = [\![\Gamma \vdash A : *]\!]\ \gamma$ and $\hat{B} = [\![\Gamma \vdash B : *]\!]\ \gamma$. Now, note that $(\mathsf{inl}, \mathsf{inl}) \in \hat{A} \to R$ and $(\mathsf{inr}, \mathsf{inr}) \in \hat{B} \to R$.

Hence $(e\ (A+B)\ \mathsf{inl}\ \mathsf{inr}, e'\ (A+B)\ \mathsf{inl}\ \mathsf{inr}) \in R$. Hence the conclusion follows.

Lemma 3. *(Eta-expanding sums) If $(e, e') \in [\![\Gamma \vdash A + B : *]\!]\ \gamma$, then $(e\ (A+B)\ \mathsf{inl}\ \mathsf{inr}, e\ (A+B)\ \mathsf{inl}\ \mathsf{inr}) \sim (e, e')$.*

Proof. Assume we have $\overline{C}, R, \bar{l} \in \hat{A} \to R$, and $\bar{r} \in \hat{B} \to R$. It suffices to show that $(e\ C\ l\ r, e'\ (A+B)\ \mathsf{inl}\ \mathsf{inr}\ C'\ l'\ r') \in R$. Now, consider the QPER:

$$S = \left\{(t, t') \mid (t, t'\ C'\ l'\ r') \in R\right\}$$

That S is a reduction-closed QPER follows from the fact that R is itself a reduction-closed QPER.

Now, we will try to show that $(e\ C\ l\ r, e'\ (A+B)\ \mathsf{inl}\ \mathsf{inr}) \in S$. Since $(e, e') \in [\![\Gamma \vdash A + B : *]\!]\ \gamma$, we can apply $(C, A+B)$ and S to it, to get: $(e\ C, e'\ (A+B)) \in (\hat{A} \to S) \to (\hat{B} \to S) \to S$. So we need to show that $(l, \mathsf{inl}) \in \hat{A} \to S$ and $(r, \mathsf{inr}) \in \hat{B} \to S$.

To show $(l, \mathsf{inl}) \in \hat{A} \to S$, we need to show that for all $(a, a') \in \hat{A}$, $(l\ a, l'\ a') \in S$. So it suffices to show that $(l\ a, \mathsf{inl}\ C'\ l'\ r'\ a') \in \hat{A} \to R$. By reduction, we know that the right-hand-side reduces to $l'\ a'$. So it suffices to show that $(l\ a, l'\ a') \in R$. Since by assumption $(l, l') \in \hat{A} \to R$ and $(a, a') \in \hat{A}$, we know that $(l\ a, l'\ a') \in R$.

Symmetrically, $(r, \mathsf{inr}) \in \hat{B} \to S$. □

4.3 Natural Numbers

Define $\mathbb{N} \triangleq \Pi\alpha : *.\ \alpha \to (\alpha \to \alpha) \to \alpha$, $z : \mathbb{N} \triangleq \lambda\alpha, i, f.\ i$, and $s : \mathbb{N} \to \mathbb{N} = \lambda n.\ \lambda\alpha, i, f.\ f(n\ \alpha\ i\ f)$.

Lemma 4. *(Normal forms of eta-expanded Church numerals) If $(n, n') \in [\![\mathbb{N}]\!]$, then $n\ \mathbb{N}\ z\ s \leftrightarrow^* s^k(z) \leftrightarrow^* n'\ \mathbb{N}\ z\ s$ for some k.*

Proof. Assume $(n, n') \in [\![\mathbb{N}]\!]$. Next, note that $s^k(z) \leftrightarrow^* s^j(z)$ iff $j = k$. So if $\hat{k}$ is the β-normal form of $s^k(z)$, then the relation $R = \left\{(\hat{k}, \hat{k}) \,\middle|\, k \in \mathbb{N}\right\}^{\dagger}$ is a QPER, since the union of QPERs with no overlaps is a QPER. Hence $(n\ \mathbb{N}, n'\ \mathbb{N}) \in R \to (R \to R) \to R$. Note that $(z, z) \in R$, and that $(s, s) \in R \to R$. Hence $(n\ \mathbb{N}\ z\ s, n'\ \mathbb{N}\ z\ s) \in R$. Hence $(n\ \mathbb{N}\ z\ s, n'\ \mathbb{N}\ z\ s) \sim_R (s^k(z), s^k(z))$ for some k. □

Lemma 5. *(Eta-expanding Church numerals) If $(n, n') \in [\![\mathbb{N}]\!]$, then $(n, n') \sim (n\ \mathbb{N}\ z\ s, n'\ \mathbb{N}\ z\ s)$.*

Proof. Assume $(n, n') \in [\![\mathbb{N}]\!]$. It follows immediately that $(n\ \mathbb{N}\ z\ s, n'\ \mathbb{N}\ z\ s) \in [\![\mathbb{N}]\!]$. So it suffices to show that $(n, n'\ \mathbb{N}\ z\ s) \in [\![\mathbb{N}]\!]$.

Assume we have $\overline{A}$, R, $\bar{i} \in R$ and $\overline{f} \in R \to R$. We want $(n\ A\ i\ f, n'\ \mathbb{N}\ z\ s\ A'\ i'\ f') \in R$. Now, consider the QPER

$$S = \left\{(e, e') \mid (e, e'\ A'\ i'\ f') \in R\right\}$$

Again, as with coproducts, that the comprehension is a reduction-closed QPER follows from the fact that R is a reduction-closed QPER.

Now, we want to (n, n') with $\overline{A}$ and S, and then show $(n\ A\ i\ f, n'\ \mathbb{N}\ z\ s) \in S$. Hence we have to show that $(i, z) \in S$ and $(f, s) \in S \to S$.

To show that $(i, z) \in S$, we need to show that $(i, z\ A'\ i'\ f') \in R$. But by reduction, we know that $z\ A'\ i'\ f$ reduces to i', and so it suffices to show that $(i, i') \in R$, which we have by hypothesis.

Next, we want to show that $(f, s) \in S \to S$. Assume that we have $(e, e') \in S$. So now we want to show that $(f\ e, s\ e') \in S$. So, it suffices to show that $(f\ e, s\ e'\ A'\ i'\ f') \in R$. By reduction, it suffices to show that $(f\ e, f'(e'\ A'\ i'\ f')) \in R$. We know that $(f, f') \in R \to R$, and since $(e, e') \in S$, we know that $(e, e'\ A'\ i'\ f') \in R$, and so the conclusion follows. $\square$

4.4 Dependent Records

- Define $\Sigma x : X.\ Y \triangleq \Pi\alpha : *.\ (\Pi x : X.\ Y \to \alpha) \to \alpha$.
- Define $\mathsf{pair} : \Pi x : X.\ Y \to \Sigma x : X.\ Y = \lambda x : X, y : Y.\ \lambda\alpha, k.\ k\ x\ y$
- Define $\mathsf{fst}\ :\ (\Sigma x : X.\ Y) \to X = \lambda p.\ p\ X\ (\lambda x.\ \lambda y.\ x)$
- Define $\mathsf{snd}\ :\ \Pi p : (\Sigma x : X.\ Y).\ [\mathsf{fst}\, p/x]Y = \lambda p.\ p\ (\Sigma x : X.\ Y)\ \mathsf{pair}\ ([\mathsf{fst}\, p/x]Y)\ (\lambda x.\ \lambda y.\ y)$

  Note that snd is not syntactically well-typed!

Lemma 6. *(Reducing eta-expanded pairs) If $(p, p') \in [\![\Gamma \vdash \Sigma x : X.\ Y : *]\!]\ \gamma$, then $(p\ [\Sigma x : X.\ Y]\ \mathsf{pair}, p'\ [\Sigma x : X.\ Y]\ \mathsf{pair}) \leftrightarrow^* (\mathsf{pair}\ u\ t, \mathsf{pair}\ u'\ t')$ where $(u, u') \in [\![\Gamma \vdash X : *]\!]\ \gamma$ and $(t, t') \in [\![\Gamma, x : X \vdash Y : *]\!]\ (\gamma, (u, u')/x)$.*

Proof. Assume $(p, p') \in [\![\Gamma \vdash \Sigma x : X.\ Y : *]\!]\ \gamma$. Now, consider the relation:

$$S \triangleq \left\{ (\lambda\alpha : *, k.\ k\ u\ t, \lambda\alpha : *, k.\ k\ u'\ t')\ \middle|\ \begin{array}{l} \bar{u} \in [\![\Gamma \vdash X : *]\!]\ \gamma \wedge \\ \bar{t} \in [\![\Gamma, x : X \vdash Y : *]\!]\ (\gamma, \bar{u}/x) \end{array} \right\}^{\dagger}$$

This is evidently a reduction-closed QPER.

So $(p\ (\Sigma x : X.\ Y), p'\ (\Sigma x : X.\ Y)) \in [\![\Gamma, \alpha : * \vdash (\Pi x : X.\ Y \to \alpha) \to \alpha : *]\!]\ (\gamma, (\gamma(\Sigma x : X.\ Y), S)/\alpha)$.
Now we will show that $(\mathsf{pair}, \mathsf{pair}) \in [\![\Gamma, \alpha : * \vdash \Pi x : X.\ Y \to \alpha : *]\!]\ (\gamma, (\gamma(\Sigma x : X.\ Y), S)/\alpha)$.
Assume we have $\bar{u} \in [\![\Gamma \vdash X : *]\!]\ \gamma$ and $\bar{t} \in [\![\Gamma, x : X \vdash Y : *]\!]\ (\gamma, \bar{u}/x)$.
Now we want to show that $(\mathsf{pair}\ u\ t, \mathsf{pair}\ u'\ t') \in S$.
However, note that $\mathsf{pair}\ u\ t$ reduces to $\lambda\alpha, k.\ k\ u\ t$ and $\mathsf{pair}\ u'\ t'$ reduces to $\lambda\alpha, k.\ k\ u'\ t'$.
Hence $(\mathsf{pair}\ u\ t, \mathsf{pair}\ u'\ t') \in S$.
Hence $(\mathsf{pair}, \mathsf{pair}) \in [\![\Gamma, \alpha : * \vdash \Pi x : X.\ Y \to \alpha : *]\!]\ (\gamma, (\gamma(\Sigma x : X.\ Y), S)/\alpha)$.
Hence $(p\ [\Sigma x : X.\ Y]\ \mathsf{pair}, p'\ [\Sigma x : X.\ Y]\ \mathsf{pair}) \in S$.
Hence there are $\bar{u} \in [\![\Gamma \vdash X : *]\!]\ \gamma$ and $\bar{t} \in [\![\Gamma, x : X \vdash Y : *]\!]\ (\gamma, \bar{u}/x)$ such that $\overline{p\ [\Sigma x : X.\ Y]\ \mathsf{pair}}$ is convertible with $\overline{\mathsf{pair}\ u\ t}$.

$\square$

Lemma 7. *(Eta for pairs)*
If $(p, p') \in [\![\Gamma \vdash \Sigma x : X.\ Y : *]\!]\ \gamma$, then $(p, p') \sim (p\ (\Sigma x : X.\ Y)\ \mathsf{pair}, p'\ (\Sigma x : X.\ Y)\ \mathsf{pair})$.

Proof. Assume $(p, p') \in [\![\Gamma \vdash \Sigma x : X.\ Y : *]\!]\ \gamma$.
It suffices to show that $(p, p'\ (\Sigma x : X.\ Y)\ \mathsf{pair}) \in [\![\Gamma \vdash \Sigma x : X.\ Y : *]\!]\ \gamma$.
Assume $\overline{A}$, R, $\overline{f} \in [\![\Gamma, \alpha : * \vdash \Pi x : X.\ Y \to \alpha : *]\!]\ (\gamma, (\overline{A}, R)/\alpha)$.
Consider the relation

$$S \triangleq \left\{ (e, e')\ \middle|\ (e, e'\ A'\ f') \in R \right\}$$

This is a reduction-closed QPER because R is.
Hence $(p\ A, p'\ (\exists x : X.\ Y)) \in [\![\Gamma, \alpha : * \vdash (\Pi x : X.\ Y \to \alpha) \to \alpha : *]\!]\ (\gamma, ((A, \exists x : X.\ Y), S)/\alpha)$.
Now we need to show that $(f, \mathsf{pair}) \in [\![\Gamma, \alpha : * \vdash \Pi x : X.\ Y \to \alpha : *]\!]\ (\gamma, ((A, \exists x : X.\ Y), S)/\alpha)$.
Assume $\bar{u} \in [\![\Gamma, \alpha : * \vdash X : *]\!]\ (\gamma, ((A, \exists x : X.\ Y), S)/\alpha)$
and $\bar{t} \in [\![\Gamma, \alpha : *, x : X \vdash Y : *]\!]\ (\gamma, ((A, \exists x : X.\ Y), S)/\alpha, \bar{u}/x)$.
So we need to show that $(f\ u\ t, \mathsf{pair}\ u'\ t') \in S$.
So we need to show that $(f\ u\ t, \mathsf{pair}\ u'\ t'\ A'\ f') \in R$.
Note $\mathsf{pair}\ u'\ t'\ A'\ f'$ reduces to $f'\ u'\ t'$.
So we need to show that $(f\ u\ t, f'\ u'\ t') \in R$.
Note that $\bar{u} \in [\![\Gamma \vdash X : *]\!]\ \gamma$ and $\bar{t} \in [\![\Gamma, x : X \vdash Y : *]\!]\ (\gamma, \bar{u}/x)$.
Hence $\bar{u} \in [\![\Gamma, \alpha : * \vdash X : *]\!]\ (\gamma, (\overline{A}, R)/\alpha)$
and $\bar{t} \in [\![\Gamma, \alpha : *, x : X \vdash Y : *]\!]\ (\gamma, (\overline{A}, R)/\alpha, \bar{u}/x)$.
Since $\overline{f} \in [\![\Gamma, \alpha : * \vdash \Pi x : X.\ Y \to \alpha : *]\!]\ (\gamma, (\overline{A}, R)/\alpha)$, it follows that $(f\ u\ t, f'\ u'\ t') \in R$.
Therefore $(f\ u\ t, \mathsf{pair}\ u'\ t'\ A'\ f') \in R$.
Therefore $(f\ u\ t, \mathsf{pair}\ u'\ t') \in S$.
Therefore $(f, \mathsf{pair}) \in [\![\Gamma, \alpha : * \vdash \Pi x : X.\ Y \to \alpha : *]\!]\ (\gamma, ((A, \exists x : X.\ Y), S)/\alpha)$.
Therefore $(p\ A\ f, p'\ (\exists x : X.\ Y)\ \mathsf{pair}) \in S$.
Therefore $(p\ A\ f, p'\ (\exists x : X.\ Y)\ \mathsf{pair}\ A'\ f') \in R$.
Hence $(p, p'\ (\Sigma x : X.\ Y)\ \mathsf{pair}) \in [\![\Gamma \vdash \Sigma x : X.\ Y : *]\!]\ \gamma$. □

Lemma 8. *(Semantic well-typedness for* snd*)* $(\mathsf{snd}, \mathsf{snd}) \in [\![\Gamma \vdash \Pi q : (\Sigma x : X.\ Y).\ [\mathsf{fst}\, q/x]Y : *]\!]\ \gamma$.

Proof. Assume $\bar{p} = (p, p') \in [\![\Gamma \vdash \Sigma x : X.\ Y : *]\!]\ \gamma$.
By Lemma 6, there exist $(u, u') \in [\![\Gamma \vdash X : *]\!]\ \gamma$ and $(t, t') \in [\![\Gamma, x : X \vdash Y : *]\!]\ (\gamma, (u, u')/x)$ such that $(p\ \_\ \mathsf{pair}, p'\ \_\ \mathsf{pair}) \leftrightarrow^* (\mathsf{pair}\ u\ t, \mathsf{pair}\ u'\ t')$ and by the eta-rule $(p, p') \sim (\mathsf{pair}\ u\ t, \mathsf{pair}\ u'\ t')$.
Therefore:

$$\begin{array}{rcl} (\mathsf{snd}\, p, \mathsf{snd}\, p') & \mapsto^* & (p\ \_\ \mathsf{pair}\ \_\ (\lambda xy.\ y), p'\ \_\ \mathsf{pair}\ \_\ (\lambda xy.\ y)) \\ & \leftrightarrow^* & ((\mathsf{pair}\ u\ t)\ \_\ (\lambda xy.\ y), (\mathsf{pair}\ u'\ t')\ \_\ (\lambda xy.\ y)) \\ & \mapsto^* & (t, t') \end{array}$$

Hence it suffices to show that $(t, t') \in [\![\Gamma, q : \Sigma x : X.\ Y \vdash [\mathsf{fst}\, q/x]Y : *]\!]\ (\gamma, \bar{p}/q)$.

Let $\gamma' = \gamma, \bar{u}/x, \bar{t}/y$.
Consider the set $[\![\Gamma, q : \Sigma x : X.\ Y \vdash [\mathsf{fst}\, q/x]Y : *]\!]\ (\gamma, \bar{p}/q)$.
By stability, this equals $[\![\Gamma, q : \Sigma x : X.\ Y \vdash [\mathsf{fst}\, q/x]Y : *]\!]\ (\gamma, \overline{\mathsf{pair}\ u\ t}/q)$.
By weakening, this equals $[\![\Gamma, x : X, y : Y, q : \Sigma x : X.\ Y \vdash [\mathsf{fst}\, q/x]Y : *]\!]\ (\gamma', \overline{\mathsf{pair}\ u\ t}/q)$.
This equals $[\![\Gamma, x : X, y : Y, q : \Sigma x : X.\ Y \vdash [\mathsf{fst}\, q/x]Y : *]\!]\ (\gamma', \gamma'(\mathsf{pair}\ x\ y)/q)$.
By substitution, this is $[\![\Gamma, x : X, y : Y \vdash [\mathsf{fst}\,(\mathsf{pair}\ x\ y)/x]Y : *]\!]\ \gamma'$.
Since the logical relation respects β-equivalence, this is $[\![\Gamma, x : X \vdash Y : *]\!]\ (\gamma, \bar{u}/x)$.
But we know $(t, t') \in [\![\Gamma, x : X \vdash Y : *]\!]\ (\gamma, \bar{u}/x)$, so we are done. □

Lemma 9. *(Projective eta for* Σ*-types) If* $(p, p') \in [\![\Gamma \vdash \Sigma x : X.\ Y : *]\!]\ \gamma$*,*
then $(p, p') \sim (\mathsf{pair}\ (\mathsf{fst}\, p)\ (\mathsf{snd}\, p), \mathsf{pair}\ (\mathsf{fst}\, p')\ (\mathsf{snd}\, p'))$*.*

Proof. It suffices to show $(p, \mathsf{pair}\ (\mathsf{fst}\, p')\ (\mathsf{snd}\, p')) \in [\![\Gamma \vdash \Sigma x : X.\ Y : *]\!]\ \gamma$.
We have $(u, u') \in [\![\Gamma \vdash X : *]\!]\ \gamma$ and $(t, t') \in [\![\Gamma, x : X \vdash Y : *]\!]\ (\gamma, (u, u')/x)$
such that $(p, p') \sim (\mathsf{pair}\ u\ t, \mathsf{pair}\ u'\ t')$.
By the semantic well-typing of fst and snd, we know that

1. $(\mathsf{fst}\, p, \mathsf{fst}\, p') \sim (\mathsf{fst}\, (\mathsf{pair}\ u\ t), \mathsf{fst}\, (\mathsf{pair}\ u'\ t'))$
2. $(\mathsf{snd}\, p, \mathsf{snd}\, p') \sim (\mathsf{snd}\, (\mathsf{pair}\ u\ t), \mathsf{snd}\, (\mathsf{pair}\ u'\ t'))$

Note that $\mathsf{fst}\, (\mathsf{pair}\ u\ t)$ reduces to u and $\mathsf{fst}\, (\mathsf{pair}\ u'\ t')$ reduces to u'.
Note that $\mathsf{snd}\, (\mathsf{pair}\ u\ t)$ reduces to t and $\mathsf{snd}\, (\mathsf{pair}\ u'\ t')$ reduces to t'.
Hence by closure under reduction $(\mathsf{fst}\, p, \mathsf{fst}\, p') \sim (u, u')$.
Hence by closure under reduction $(\mathsf{snd}\, p, \mathsf{snd}\, p') \sim (t, t')$.
Therefore $(\mathsf{pair}\ u\ t, \mathsf{pair}\ u'\ t') \sim (\mathsf{pair}\ (\mathsf{fst}\, p)\ (\mathsf{snd}\, p), \mathsf{pair}\ (\mathsf{fst}\, p')\ (\mathsf{snd}\, p'))$.
Therefore $(p, p') \sim (\mathsf{pair}\ (\mathsf{fst}\, p)\ (\mathsf{snd}\, p), \mathsf{pair}\ (\mathsf{fst}\, p')\ (\mathsf{snd}\, p'))$.

□

4.5 Induction for Natural Numbers

Projective records make it convenient to support the induction principle for natural numbers. That is, we want to show that the type

$$\Pi P : \mathbb{N} \to *.\ P(z) \to (\Pi n : \mathbb{N}.\ P(n) \to P(s\ n)) \to \Pi n : \mathbb{N}.\ P(n)$$

is inhabited, where $\mathbb{N}$ is the Church encoding of the naturals, and z and s are the Church zero and successor. To demonstrate this, we will show that the term ind

$$\mathit{iter} \triangleq \lambda P, i, f, n.\ n\ (\Sigma x : \mathbb{N}.\ P(x))\ (\mathsf{pair}\ z\ i)\ (\lambda p.\ \mathsf{pair}\ (s\ (\mathsf{fst}\, p))\ (f\ (\mathsf{fst}\, p)\ (\mathsf{snd}\, p)))$$

$$\mathit{ind} \triangleq \lambda P, i, f, n.\ \mathsf{snd}\, (\mathit{iter}\ P\ i\ f\ n)$$

is related to itself at this type.

Proof. First, note that $\mathit{iter} : \Pi P : \mathbb{N} \to *.\ P(z) \to (\Pi n : \mathbb{N}.\ P(n) \to P(s\ n)) \to \mathbb{N} \to \Sigma x : \mathbb{N}.\ P(x)$.

Now, we'll show that for any predicate Q, and $\bar{i} \in [\![P : \mathbb{N} \to * \vdash P(z) : *]\!]\ \ ((\_\,, Q)/P)$, and $\bar{f} \in [\![P : \mathbb{N} \to * \vdash \Pi x : \mathbb{N}.\ P(x) \to P(s\ x) : *]\!]\ \ ((\_\,, Q)/P)$, and $\bar{n} \in [\![\mathbb{N}]\!]$, where $\bar{n}$ are both the same Church numeral, we have that $\overline{\mathit{iter}\ \_\ i\ f\ n} \sim \overline{\mathsf{pair}\ n\ t} \in [\![P : \mathbb{N} \to * \vdash \Sigma x : \mathbb{N}.\ P(x) : *]\!]\ \ ((\_\,, Q)/P)$ for some $\bar{t} \in Q(\bar{n})$. We proceed by induction on the structure of the Church numerals:

- Case $n = z = \lambda \alpha, b, r.\ b$.

  In this case, we know that $\overline{\mathit{iter}\ \_\ i\ f\ n}$ reduces to $\overline{(z, i)}$. Since relations are closed under reduction, the conclusion follows.

- Case $n = s(k) \mapsto^* \lambda \alpha, b, r.\ r(k\ \alpha\ b\ r)$.

  In this case, we know that $\overline{\mathit{iter}\ \_\ i\ f\ n}$

$$\begin{array}{ll}
\mapsto^* & \overline{(\lambda \alpha, b, r.\ r(k\ \alpha\ b\ r))\ _\ (z, i)\ (\lambda p.\ \mathsf{pair}\ (s\ (\mathsf{fst}\, p))\ (f\ (\mathsf{fst}\, p)\ (\mathsf{snd}\, p)))} \\
\mapsto^* & \overline{(\lambda p.\ \mathsf{pair}\ (s\ (\mathsf{fst}\, p))\ (f\ (\mathsf{fst}\, p)\ (\mathsf{snd}\, p)))\ (k\ _\ (z, i)\ (\lambda p.\ \mathsf{pair}\ (s\ (\mathsf{fst}\, p))\ (f\ (\mathsf{fst}\, p)\ (\mathsf{snd}\, p))))} \\
\sim & \overline{(\lambda p.\ \mathsf{pair}\ (s\ (\mathsf{fst}\, p))\ (f\ (\mathsf{fst}\, p)\ (\mathsf{snd}\, p)))\ (\mathsf{pair}\ k\ t)} \qquad \text{for some } \bar{t} \in Q(\bar{k}) \\
\mapsto^* & \overline{\mathsf{pair}\ (s\ (\mathsf{fst}\, (\mathsf{pair}\ k\ t)))\ (f\ (\mathsf{fst}\, (\mathsf{pair}\ k\ t))\ (\mathsf{snd}\, (\mathsf{pair}\ k\ t)))} \\
\sim & \overline{\mathsf{pair}\ n\ (f\ k\ t)}
\end{array}$$

  On the third line, we (a) appeal to induction, and (b) make use of the fact that $\overline{\lambda p.\ \mathsf{pair}\ (s\ (\mathsf{fst}\, p))\ (f\ (\mathsf{fst}\, p)\ (\mathsf{snd}\, p))} \in [\![(\Sigma x : \mathbb{N}.\ P(x)) \to (\Sigma x : \mathbb{N}.\ P(x))]\!]\, ((\_\,, Q)/P)$, and so applying it to related pairs of arguments yields a related pair of results. This fact follows by a straightforward congruence argument given the semantic well-typedness of $\overline{f}$, pair, fst, and snd. The last line follows from the semantic well-typing of pair and $\overline{f}$ and $\bar{t}$, which yields that $\overline{f\ k\ t} \in Q(\bar{n})$, together with basic reasoning about β-reduction.

Since every pair $\bar{n} \in [\![\mathbb{N}]\!]$ is itself equivalent to a Church numeral paired with itself, it follows that for arbitrary $\bar{n}$, there are $\bar{t} \in Q(\bar{n})$ such that $\overline{iter\ \_\ i\ f\ n} \sim \overline{\mathsf{pair}\ n\ t}$.

Then we know that $\overline{ind\ \_\ i\ f\ n}$

$$\begin{array}{ll} \mapsto^* & \overline{\mathsf{snd}\,(iter\ \_\ i\ f\ n)} \\ \sim & \overline{\mathsf{snd}\,(\mathsf{pair}\ n\ t)} \\ \mapsto^* & \bar{t} \end{array}$$

On the second line, we know that $\bar{t} \in Q(\bar{n})$, which is the semantic type we need.

□

4.6 Existential Types

Define:

- $\exists\alpha : \kappa.\ X \triangleq \Pi\beta : *.\ (\Pi\alpha : \kappa.\ X \to \beta) \to \beta.$
- $\mathsf{pack} : \Pi\alpha : \kappa.\ X \to \exists\alpha : \kappa.\ X = \lambda\alpha, x.\ \lambda\beta, k.\ k\ \alpha\ x$

Lemma 10. *(Normal forms of eta-expanded existentials) If $(e, e') \in [\![\Gamma \vdash \exists\alpha : \kappa.\ X : *]\!]\ \gamma$, then $e\ \_\ \mathsf{pack} \leftrightarrow^* \mathsf{pack}\ \_\ t$ and $e'\ \_\ \mathsf{pack} \leftrightarrow^* \mathsf{pack}\ \_\ t'$ for some t and t'.*

Proof. Assume $(e, e') \in [\![\Gamma \vdash \exists\alpha : \kappa.\ X : *]\!]\ \gamma$.
Now, consider the following QPER:

$$R = \left\{(\lambda\alpha, k.\ k\ \_\ t, \lambda\alpha, k.\ k\ \_\ t')\ \middle|\ \top\right\}^{\dagger}$$

The comprehension is clearly a QPER on values, and so the closure is a reduction-closed QPER.

Hence $(e\ \_, e'\ \_) \in [\![\Gamma, \beta : * \vdash (\Pi\alpha : \kappa.\ X \to \beta) \to \beta : *]\!]\ (\gamma, (\_, R)/\beta)$.
Now, we want to show that $(\mathsf{pack}, \mathsf{pack}) \in [\![\Gamma, \beta : * \vdash \Pi\alpha : \kappa.\ X \to \beta : *]\!]\ (\gamma, (\_, R)/\beta)$.
Assume $T \in [\![\Gamma, \beta : * \vdash \kappa : \mathsf{kind}]\!]\ (\gamma, (\_, R)/\beta)$,
and $\bar{t} \in [\![\Gamma, \beta : *, \alpha : \kappa \vdash X : *]\!]\ (\gamma, (\_, R)/\beta, (\_, T)/\alpha)$.
Then $\mathsf{pack}\ \_\ t \mapsto^* \lambda\alpha, k.\ k\ \_\ t$ and $\mathsf{pack}\ \_\ t' \mapsto^* \lambda\alpha, k.\ k\ \_\ t'$.
Hence $(\mathsf{pack}\ \_\ t, \mathsf{pack}\ \_\ t') \in R$.
Hence $(\mathsf{pack}, \mathsf{pack}) \in [\![\Gamma, \beta : * \vdash \Pi\alpha : \kappa.\ X \to \beta : *]\!]\ (\gamma, (\_, R)/\beta)$.
Hence $(e\ \_\ \mathsf{pack}, e'\ \_\ \mathsf{pack}) \in R$.
Hence by construction of R, there must exist some t and t' such that
$e\ \_\ \mathsf{pack} \leftrightarrow^* \mathsf{pack}\ \_\ t$ and $e'\ \_\ \mathsf{pack} \leftrightarrow^* \mathsf{pack}\ \_\ t'$. □

Lemma 11. *(Eta-rule for existentials) If $(e, e') \in [\![\Gamma \vdash \exists\alpha : \kappa.\ X : *]\!]\ \gamma$, then $(e, e') \sim (e\ \_\ \mathsf{pack}, e'\ \_\ \mathsf{pack})$.*

Proof. Assume $(e, e') \in [\![\Gamma \vdash \exists\alpha : \kappa.\ X : *]\!]\ \gamma$.
It suffices to show $(e, e'\ \_\ \mathsf{pack}) \in [\![\Gamma \vdash \exists\alpha : \kappa.\ X : *]\!]\ \gamma$.
Assume $R \in [\![\Gamma \vdash * : \mathsf{kind}]\!]\ \gamma$ and $(k, k') \in [\![\Gamma, \beta : * \vdash \Pi\alpha : \kappa.\ X \to \beta : *]\!]\ (\gamma, (\_, R)/\beta)$.
It remains to show that $(e\ \_\ k, e'\ \_\ \mathsf{pack}\ \_\ k') \in R$.
Consider the relation:

$$S = \left\{ (\hat{e}, \hat{e'}) \;\middle|\; (\hat{e}, \hat{e'}\ \_\ k') \in R \right\}$$

Again, this is a reduction-closed QPER because R is.
Instantiating the original assumption, we obtain
$(e\ \_, e'\ \_) \in [\![\Gamma, \beta : * \vdash (\Pi\alpha : \kappa.\ X \to \beta) \to \beta : *]\!]\ (\gamma, (\_, S))/\beta)$.
Now we'll show $(k, \mathsf{pack}) \in [\![\Gamma, \beta : * \vdash \Pi\alpha : \kappa.\ X \to \beta : *]\!]\ (\gamma, (\_, S))/\beta)$,
which will give us that $(e\ \_\ k, e'\ \_\ \mathsf{pack}) \in S$, from which the goal follows.
Assume $T \in [\![\Gamma, \beta : * \vdash \kappa : \mathsf{kind}]\!]\ (\gamma, (\_, S)/\beta)$
and $(t, t') \in [\![\Gamma, \beta : *, \alpha : \kappa \vdash X : *]\!]\ (\gamma, (\_, S)/\beta, (\_, T)/\alpha)$.
It suffices to show $(k\ \_\ t, \mathsf{pack}\ \_\ t') \in S$.
So it suffices to show $(k\ \_\ t, \mathsf{pack}\ \_\ t'\_k') \in R$.
By reduction, it suffices to show $(k\ \_\ t, k'\ \_\ t') \in R$.
Since $\beta \notin \mathrm{FV}(\kappa)$, weakening ensures that
$T \in [\![\Gamma \vdash \kappa : \mathsf{kind}]\!]\ \gamma = [\![\Gamma, \beta : * \vdash \kappa : \mathsf{kind}]\!]\ (\gamma, (\_, R)/\beta)$.
Similarly, since $\beta \notin \mathrm{FV}(X)$, weakening ensures that
$(t, t') \in [\![\Gamma, \alpha : \kappa \vdash X : *]\!]\ (\gamma, (\_, T)/\alpha) = [\![\Gamma, \beta : *, \alpha : \kappa \vdash X : *]\!]\ (\gamma, (\_, R)/\beta, (\_, T)/\alpha)$.
Thus, since $(k, k') \in [\![\Gamma, \beta : * \vdash \Pi\alpha : \kappa.\ X \to \beta : *]\!]\ (\gamma, (\_, R)/\beta)$,
it follows that $(k\ \_\ t, k'\ \_\ t') \in R$.

□

Lemma 12. *(Existential equality) If* $(e, e') \in [\![\Gamma \vdash \exists\alpha : \kappa.\ X : *]\!]\ \gamma$*, then there exist*

1. $A, A' \in \mathrm{Type}$,
2. $R \in [\![\Gamma \vdash \kappa : \mathsf{kind}]\!]\ \gamma$
3. $(t, t') \in [\![\Gamma, \alpha : \kappa \vdash X : *]\!]\ (\gamma, ((A, A'), R)/\alpha)$

such that $(e, e') \sim_{[\![\Gamma \vdash \exists\alpha:\kappa.\ X:*]\!]\ \gamma} (\mathsf{pack}\ A\ t, \mathsf{pack}\ A'\ t')$.

Proof. First consider the pair (e, e'), and the application $(e\ \_,\ e'\ \_)$. Starting from $(e, e') \in [\![\Gamma \vdash \exists\alpha : \kappa.\ X : *]\!]\ \gamma$, we instantiate the type abstraction on both sides and choose the relational interpretation of the abstract type to be the following, defined by a QPER join:

$$S = \bigsqcup_{R \in [\![\kappa]\!]\ \gamma} \left\{ \overline{\lambda\beta, k.\ k\ A\ e} \;\middle|\; \overline{A} \in \mathrm{Type}^2 \wedge \bar{e} \in [\![\Gamma, \alpha : \kappa \vdash X : *]\!]\ (\gamma, (\overline{A}, R)/\alpha) \right\}^{\dagger}$$

For each $R \in [\![\kappa]\!]\ \gamma$, it is clear that the comprehension is a QPER on values. Hence the reduction-closure for each QPER yields a new QPER. Then the join of QPERs is also a QPER, and the join of reduction-closed QPERs is obviously itself reduction-closed.

Hence $(e\ \_, e'\ \_) \in [\![\Gamma, \beta : * \vdash (\Pi\alpha : \kappa.\ X \to \beta) \to \beta : *]\!]\ (\gamma, (\_, S)/\beta)$.
Now we will show that $(\mathsf{pack}, \mathsf{pack}) \in [\![\Gamma, \beta : * \vdash \Pi\alpha : \kappa.\ X \to \beta : *]\!]\ (\gamma, (\_, S)/\beta)$.
Assume $\overline{A} \in \mathrm{Type}^2, R \in [\![\Gamma, \beta : * \vdash \kappa : \mathsf{kind}]\!]\ (\gamma, (\_, S)/\beta)$,

and $\bar{t} \in [\![\Gamma, \beta : *, \alpha : \kappa \vdash X : *]\!]\ (\gamma, (\_, S)/\beta, (\overline{A}, R)/\alpha)$.
By weakening, since $\beta \notin \mathrm{FV}(X)$, we have $\bar{t} \in [\![\Gamma, \alpha : \kappa \vdash X : *]\!]\ (\gamma, (\overline{A}, R)/\alpha)$.
By construction of S, then, $\overline{\mathsf{pack}\, A\, t} \in S$.
So $(\mathsf{pack}, \mathsf{pack}) \in [\![\Gamma, \beta : * \vdash \Pi\alpha : \kappa.\ X \to \beta : *]\!]\ (\gamma, (\_, S)/\beta)$.

Hence $(e\ \_\ \mathsf{pack},\ e'\ \_\ \mathsf{pack}) \in S$.
By Lemma 10, they must be interconvertible with $(\mathsf{pack}\ \_\ t, \mathsf{pack}\ \_\ t')$ for some t and t',
and thus $(\mathsf{pack}\ \_\ t, \mathsf{pack}\ \_\ t') \in S$.

Ideally, we would like to use the fact that $(\mathsf{pack}\ \_\ t, \mathsf{pack}\ \_\ t') \in S$ to conclude that there is an R such that $(t, t') \in [\![\Gamma, \alpha : \kappa \vdash X : *]\!]\ (\gamma, (\_, R)/\alpha)$. However, the QPER-join adds elements that are not in the union, so this does not immediately follow.

We will show that if $\overline{\mathsf{pack}\ \_\ t} \in S_n$ then there is a $\overline{\mathsf{pack}\ \_\ s} \sim \overline{\mathsf{pack}\ \_\ t}$ such that there is a relation $R \in [\![\Gamma \vdash \kappa : \mathsf{kind}]\!]\, \gamma$ and $\bar{s}$ such that $\bar{s} \in [\![\Gamma, \alpha : \kappa \vdash X : *]\!]\ (\gamma, (\_, R)/\alpha)$.

- If $\overline{\mathsf{pack}\ \_\ t} \in S_0$:
  The result is immediate; we choose $\overline{\mathsf{pack}\ \_\ t}$.

- If $\overline{\mathsf{pack}\ \_\ t} \in S_{k+1}$:
  There are $\mathsf{pack}\ \_\ s, \mathsf{pack}\ \_\ s'$ such that
  1. $(\mathsf{pack}\ \_\ t, \mathsf{pack}\ \_\ s') \in S_k$
  2. $(\mathsf{pack}\ \_\ s, \mathsf{pack}\ \_\ t') \in S_k$
  3. $(\mathsf{pack}\ \_\ s, \mathsf{pack}\ \_\ s') \in S_k$

  By induction on $(\mathsf{pack}\ \_\ s, \mathsf{pack}\ \_\ s') \in S_k$, we know
  there is a $\overline{\mathsf{pack}\ \_\ r} \in [\![\Gamma, \alpha : \kappa \vdash X : *]\!]\ (\gamma, (\_, R)/\alpha)$
  such that $\overline{\mathsf{pack}\ \_\ r} \sim \overline{\mathsf{pack}\ \_\ s}$.
  Since $\overline{\mathsf{pack}\ \_\ t} \sim \overline{\mathsf{pack}\ \_\ s}$, we know $\overline{\mathsf{pack}\ \_\ t} \sim \overline{\mathsf{pack}\ \_\ r}$.

Because we know $\overline{\mathsf{pack}\ \_\ t} \in S$, we know there is an n such that $\overline{\mathsf{pack}\ \_\ t} \in S_n$.
Hence we can use the lemma to derive R and $\bar{s}$ such that $\bar{s} \in [\![\Gamma, \alpha : \kappa \vdash X : *]\!]\ (\gamma, (\_, R)/\alpha)$ and $R \in [\![\Gamma \vdash \kappa : \mathsf{kind}]\!]\, \gamma$ and $\overline{\mathsf{pack}\ \_\ t} \sim \overline{\mathsf{pack}\ \_\ s}$. So $\overline{e\ \_\ \mathsf{pack}} \sim \overline{\mathsf{pack}\ \_\ s}$.
Now, implicitly here we have been reasoning about $\sim_S$, but a standard representation independence argument shows that $S \subseteq [\![\Gamma \vdash \exists\alpha : \kappa.\ X : *]\!]\, \gamma$, and thus that $\overline{e\ \_\ \mathsf{pack}} \sim_{[\![\Gamma \vdash \exists\alpha:\kappa.\ X:*]\!]\gamma} \overline{\mathsf{pack}\ \_\ s}$.
Thus, by Lemma 11, we can conclude that $(e, e') \sim_{[\![\Gamma \vdash \exists\alpha:\kappa.\ X:*]\!]\gamma} (\mathsf{pack}\ \_\ s, \mathsf{pack}\ \_\ s')$. □

4.7 Quotient Types

While not an application of parametricity in the sense of theorems for free [Wadler(1989)], we can also show the realizability of quotient types [Hofmann(1995)] in our semantics. Quotient types, like their name suggests, are a way of defining new types by taking an existing type, and quotienting it by an equivalence relation.

To do this, we first define the auxilliary predicate Eq_X, which formalizes the notion of an equivalence relation. This is a predicate on relations of kind $X \to X \to *$, defined as follows:

$$\begin{array}{rcl} \mathrm{Eq}_X(R) & \triangleq & \Pi x : X.\ R\ x\ x\ \times \\ & & \Pi x : X, y : X.\ R\ x\ y \leftrightarrow R\ y\ x\ \times \\ & & \Pi x : X, y : X, z : X.\ R\ x\ y \to R\ y\ z \to R\ x\ z \end{array}$$

Next, we can show the realizability of the following datatype:

$$\begin{array}{rcl} X/R & \triangleq & \exists\beta : *, \\ & & \Sigma inj : X \to \beta. \\ & & \Sigma app : \Pi\gamma : *. \, \Pi f : X \to \gamma. \\ & & \qquad (\Pi a : X, a' : X. \\ & & \qquad\quad R\ a\ a' \to f\ a =_\gamma f\ a') \\ & & \qquad \to (\beta \to \gamma). \\ & & \Pi a : X, a' : X.\ R\ a\ a' \to inj(a) =_\beta inj(a') \times \\ & & \Pi\gamma.\ \Pi f, pf, x.\ app\ \gamma\ f\ pf\ (inj\ x) =_\gamma f\ x \end{array}$$

What we are doing is defining an existential type, such that if X is a type and R is an equivalence relation on it, we return a new type β and two operations *inj* and *app*.

The *inj* is the injection into the quotient type. It takes an X, and returns a β, with the property that if a and a' are related by R, then $inj\ a = inj\ a'$. The *app* function then lifts any function f from $X \to \gamma$ into one on $\beta \to \gamma$, provided that f respects the equivalence relation R. The last two lines give the equational theory of the quotient type. First, if a and a' are related by R, then $inj\ a = inj\ a'$. Second, if we lift a function f to operate on quotients, and we pass it the argument $inj\ x$, then the application of the lifted function should equal $f\ x$.

Proof. (Sketch) The proof of the soundness of the axiom is quite easy. Essentially, we just need to define the following relation:

$$S = \left\{ (v_1, v_2') \;\middle|\; \begin{array}{l} \exists v_1', v_2, \bar{q}. \\ \quad (v_1, v_1') \in X \wedge (v_2, v_2') \in X \wedge \\ \quad \bar{q} \in R\ (v_1, v_1')\ (v_2, v_2') \end{array} \right\}$$

Now, we can define the operators $\mathsf{inj} = \lambda x : X.\ x$ and $\mathsf{app} = \lambda\gamma : *.\ \lambda f : X \to \gamma, pf : \ldots, x : X.\ f\ x$. Given these, we can then show the realizability of the term:

$$\mathsf{pack}\, X\ \mathsf{pair\ inj}\ (\mathsf{pair\ app}\ (\mathsf{pair}\ (\lambda a, a', r.\ \mathsf{refl})\ (\lambda\gamma.\ \lambda f, pf, x.\ \mathsf{refl})))$$

paired with itself at the witness relation S. Note that this term is not well-typed in the syntactic system, but that it does inhabit the appropriate semantic type. □

In terms of the operational semantics of the underlying realizers, quotienting is a no-op. Just as an ML programmer might expect, we do not need to perform any changes of representation to protect the invariant of the quotient type — data abstraction is enough.

5 Soundness

Our main theorem is a consistency proof of our semantics. By an induction over derivations, we show that every well-typed expression lies in the expression relation. As a result, we know that the system is consistent: every closed term reduces to a value, and hence empty types are not inhabited.

5.1 Structural Properties

Note that this interpretation is defined purely on the syntax of kinds, and makes no use of the context. So we can then define the pre-interpretation of contexts, which we call the set of *pre-contexts*:

$$\begin{array}{lcl} \|\cdot\| & = & \{\langle\rangle\} \\ \|\Gamma, x : X\| & = & \left\{(\gamma, \bar{e}/x) \;\middle|\; \gamma \in \|\Gamma\| \wedge \bar{e} \in \mathrm{Exp}^2\right\} \\ \|\Gamma, \alpha : \kappa\| & = & \left\{(\gamma, (\overline{A}, R)/\alpha) \;\middle|\; \gamma \in \|\Gamma\| \wedge \overline{A} \in \mathrm{Type}^2 \wedge R \in \|\kappa\|\right\} \end{array}$$

Since the pre-interpretation is defined solely on syntax, we can prove the following two lemmas about it:

Lemma 13 (Kind Pre-interpretations Ignore Term Substitutions)**.** *For all kind κ and terms e, $\|\kappa\| = \|[e/x]\kappa\|$.*

Lemma 14 (Kind Pre-interpretations Ignore Type Substitutions)**.** *For all kind κ and types A, $\|\kappa\| = \|[A/\alpha]\kappa\|$.*

This implies the following trivial coherence property.

Theorem 1 (Kind Coherence)**.** *If $\Gamma \vdash \kappa \equiv \kappa' : \mathsf{kind}$, then $\|\kappa\| = \|\kappa'\|$.*

Once we have this property in place, we can prove the following well-formedness conditions on the context, kind, and type judgements.

Theorem 2 (Well-Definedness)**.**

1. *If $D :: \Gamma\ \mathsf{ok}$, then $[\![D :: \Gamma\ \mathsf{ok}]\!] \in \mathcal{P}(\|\Gamma\|)$.*
2. *If $D :: \Gamma \vdash \kappa : \mathsf{kind}$, then $[\![D :: \Gamma \vdash \kappa : \mathsf{kind}]\!] \in \|\Gamma\| \to \mathcal{P}(\|\kappa\|)$.*
3. *If $D :: \Gamma \vdash A : \kappa$, then $[\![D :: \Gamma \vdash A : \kappa]\!] \in \|\Gamma\| \to \|\kappa\|$.*

Now that we know that we have a well-formed definition, we can prove coherence property for kinds and types.

Theorem 3 (Coherence of Types and Kinds)**.**

1. *If $D :: \Gamma \vdash \kappa : \mathsf{kind}$ and $D' :: \Gamma \vdash \kappa : \mathsf{kind}$ and $\gamma \in \|\Gamma\|$, then $[\![D]\!]\ \gamma = [\![D']\!]\ \gamma$.*
2. *If $D :: \Gamma \vdash A : \kappa$ and $D' :: \Gamma \vdash A : \kappa'$ and $\gamma \in \|\Gamma\|$, then $[\![D]\!]\ \gamma = [\![D']\!]\ \gamma$.*

This immediately implies the following corollary:

Corollary 1 (Coherence of Environment Interpretation)**.** *If $D :: \Gamma\ \mathsf{ok}$ and $D' :: \Gamma\ \mathsf{ok}$, then $[\![D :: \Gamma\ \mathsf{ok}]\!] = [\![D' :: \Gamma\ \mathsf{ok}]\!]$.*

Now, we can prove weakening.

Theorem 4 (Weakening of Kinds and Types)**.** *We have that:*

1. *If $D :: \Gamma_0, \Gamma_2 \vdash \kappa : \mathsf{kind}$ then there exists $D' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash \kappa : \mathsf{kind}$ such that for all $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and γ_1 such that $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$, $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.*

2. *If* $D :: \Gamma_0, \Gamma_2 \vdash A : \kappa$ *then there exists* $D' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash A : \kappa$ *such that for all* $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ *and* γ_1 *such that* $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$, $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

Theorem 5 (Substitution for Pre-Contexts)**.** *We have that:*

1. *If* $\Gamma \vdash e : X$, *and* $(\gamma, \gamma(e)/x, \gamma') \in \|\Gamma, x : X, \Gamma'\|$, *then* $(\gamma, \gamma') \in \|\Gamma, [e/x]\Gamma'\|$.
2. *If* $\Gamma \vdash A : \kappa$, *and* $(\gamma, (\gamma(A), R)/\alpha, \gamma') \in \|\Gamma, \alpha : \kappa, \Gamma'\|$, *then* $(\gamma, \gamma') \in \|\Gamma, [A/\alpha]\Gamma'\|$.

Theorem 6 (Substitution of Terms)**.** *Suppose that* $\Gamma \vdash e : X$ *and* $(\gamma, \gamma(e)/x, \gamma') \in \|\Gamma, x : X, \Gamma'\|$. *Then:*

1. *For all* $D :: \Gamma, x : X, \Gamma' \vdash \kappa_0 : \mathsf{kind}$, *there exists* $D' :: \Gamma, [e/x]\Gamma' \vdash [e/x]\kappa_0 : \mathsf{kind}$ *such that* $[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.
2. *For all* $D :: \Gamma, x : X, \Gamma' \vdash C : \kappa_0$, *there exists* $D' :: \Gamma, [e/x]\Gamma' \vdash [e/x]C : [e/x]\kappa_0$ *such that* $[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.

Theorem 7 (Substitution of Types)**.** *Suppose that* $\Gamma \vdash A : \kappa$ *and* $(\gamma, (\gamma(A), [\![D_1]\!]\ \gamma)/\alpha, \gamma') \in \|\Gamma, \alpha : \kappa, \Gamma'\|$. *Then:*

1. *For all* $D :: \Gamma, \alpha : \kappa, \Gamma' \vdash \kappa_0 : \mathsf{kind}$, *there exists* $D' :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]\kappa_0 : \mathsf{kind}$ *such that* $[\![D]\!]\ (\gamma, (\gamma(A), [\![D_1]\!]\ \gamma)/\alpha, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.
2. *For all* $D :: \Gamma, \alpha : \kappa, \Gamma' \vdash C : \kappa_0$, *there exists* $D' :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]C : [A/\alpha]\kappa_0$ *such that* $[\![D]\!]\ (\gamma, (\gamma(A), [\![D_1]\!]\ \gamma)/\alpha, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.

Theorem 8 (Fundamental Property)**.** *We have that:*

1. *If* $D :: \Gamma \vdash \kappa : \mathsf{kind}$, *then for all* $\gamma, \gamma' \in [\![D_0 :: \Gamma\ \mathsf{ok}]\!]$ *such that* $\gamma \sim \gamma'$, $[\![D]\!]\ \gamma = [\![D]\!]\ \gamma'$.
2. *If* $D :: \Gamma \vdash A : \kappa$, *then for all* $\gamma, \gamma' \in [\![D_0 :: \Gamma\ \mathsf{ok}]\!]$ *such that* $\gamma \sim \gamma'$, $[\![D]\!]\ \gamma = [\![D]\!]\ \gamma'$.
3. *If* $D :: \Gamma \vdash e : X$ *then for all* $D_1 :: \Gamma \vdash X : *$ *and* $\gamma, \gamma' \in [\![D_0 :: \Gamma\ \mathsf{ok}]\!]$ *such that* $\gamma \sim \gamma'$, $\gamma(e) \sim_{[\![D_1]\!]\ \gamma} \gamma'(e)$.
4. *If* $D :: \Gamma \vdash A : \kappa$, *then for all* $\gamma \in [\![D_0 :: \Gamma\ \mathsf{ok}]\!]$, $[\![D]\!]\ \gamma \in [\![D_1 :: \Gamma \vdash \kappa : \mathsf{kind}]\!]\ \gamma$.
5. *If* $D :: \Gamma \vdash \kappa \equiv \kappa' : \mathsf{kind}$, *then for all* $\gamma \in [\![D_0 :: \Gamma\ \mathsf{ok}]\!]$, $D_1 :: \Gamma \vdash \kappa : \mathsf{kind}$ *and* $D_2 :: \Gamma \vdash \kappa : \mathsf{kind}$, $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$.
6. *If* $D :: \Gamma \vdash A \equiv A' : \kappa$, *then for all* $\gamma \in [\![D_0 :: \Gamma\ \mathsf{ok}]\!]$, $D_1 :: \Gamma \vdash A : \kappa$ *and* $D_2 :: \Gamma \vdash A' : \kappa$, $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$.
7. *If* $D :: \Gamma \vdash e_1 \equiv e_2 : X$, *then for all* $\gamma \in [\![D_0 :: \Gamma\ \mathsf{ok}]\!]$, $D_1 :: \Gamma \vdash X : *$, $\gamma(e_1) \sim_{[\![D_1]\!]\ \gamma} \gamma(e_2)$.

6 Proofs

Lemma 13 (Kind Pre-interpretations Ignore Term Substitutions)**.** *For all kind κ and terms e, $\|\kappa\| = \|[e/x]\kappa\|$.*

Proof. This follows by induction on κ.

- Case $\kappa = *$:
  Immediate.
- Case $\kappa = \Pi y : Y.\ \kappa_1$:
  By definition, $[e/x]\Pi y : Y.\ \kappa_1 = \Pi y : [e/x]Y.\ [e/x]\kappa_1$.
  By definition, $\|\Pi y : Y.\ \kappa_1\| = (\text{Exp} \times \text{Exp}) \to \|\kappa_1\|$.
  By definition, $\|\Pi y : [e/x]Y.\ [e/x]\kappa_1\| = (\text{Exp} \times \text{Exp}) \to \|[e/x]\kappa_1\|$.
  By induction, we know that $\|\kappa_1\| = \|[e/x]\kappa_1\|$.
  Hence $\|\Pi y : Y.\ \kappa_1\| = \|\Pi y : [e/x]Y.\ [e/x]\kappa_1\|$.
- Case $\kappa = \Pi\beta : \kappa_1.\ \kappa_2$:
  By definition, $[e/x]\Pi\beta : \kappa_1.\ \kappa_2 = \Pi\beta : [e/x]\kappa_2.\ [e/x]\kappa_2$.
  By definition, $\|\Pi\beta : \kappa_1.\ \kappa_2\| = \|\kappa_1\| \to \|\kappa_2\|$.
  By definition, $\|\Pi\beta : [e/x]\kappa_1.\ [e/x]\kappa_2\| = \|[e/x]\kappa_1\| \to \|[e/x]\kappa_2\|$.
  By induction, we know that $\|\kappa_1\| = \|[e/x]\kappa_1\|$.
  By induction, we know that $\|\kappa_2\| = \|[e/x]\kappa_2\|$.
  Hence $\|\Pi\beta : \kappa_1.\ \kappa_2\| = \|\Pi y : [e/x]Y.\ [e/x]\kappa_2\|$.

□

Lemma 14 (Kind Pre-interpretations Ignore Type Substitutions)**.** *For all kind κ and types A, $\|\kappa\| = \|[A/\alpha]\kappa\|$.*

Proof. This follows by induction on κ.

- Case $\kappa = *$:
  Immediate.
- Case $\kappa = \Pi y : Y.\ \kappa_1$:
  By definition, $[A/\alpha]\Pi y : Y.\ \kappa_1 = \Pi y : [A/\alpha]Y.\ [A/\alpha]\kappa_1$.
  By definition, $\|\Pi y : Y.\ \kappa_1\| = (\text{Exp} \times \text{Exp}) \to \|\kappa_1\|$.
  By definition, $\|\Pi y : [A/\alpha]Y.\ [A/\alpha]\kappa_1\| = (\text{Exp} \times \text{Exp}) \to \|[A/\alpha]\kappa_1\|$.
  By induction, we know that $\|\kappa_1\| = \|[A/\alpha]\kappa_1\|$.
  Hence $\|\Pi y : Y.\ \kappa_1\| = \|\Pi y : [A/\alpha]Y.\ [A/\alpha]\kappa_1\|$.
- Case $\kappa = \Pi\beta : \kappa_1.\ \kappa_2$:
  By definition, $[A/\alpha]\Pi\beta : \kappa_1.\ \kappa_2 = \Pi\beta : [A/\alpha]\kappa_2.\ [A/\alpha]\kappa_2$.
  By definition, $\|\Pi\beta : \kappa_1.\ \kappa_2\| = \|\kappa_1\| \to \|\kappa_2\|$.
  By definition, $\|\Pi\beta : [A/\alpha]\kappa_1.\ [A/\alpha]\kappa_2\| = \|[A/\alpha]\kappa_1\| \to \|[A/\alpha]\kappa_2\|$.
  By induction, we know that $\|\kappa_1\| = \|[A/\alpha]\kappa_1\|$.
  By induction, we know that $\|\kappa_2\| = \|[A/\alpha]\kappa_2\|$.
  Hence $\|\Pi\beta : \kappa_1.\ \kappa_2\| = \|\Pi y : [A/\alpha]Y.\ [A/\alpha]\kappa_2\|$.

□

Theorem 1 (Kind Coherence)**.** *If* $\Gamma \vdash \kappa \equiv \kappa' : \mathsf{kind}$*, then* $\|\kappa\| = \|\kappa'\|$*.*

Proof. This proof is by induction on the equality derivation.

- Case $\Gamma \vdash [e/x]\kappa \equiv [e'/x]\kappa : \mathsf{kind}$: (term substitution)
  This follows from the fact that kind pre-interpretations ignore term substitutions.

- Case $\Gamma \vdash [A/\alpha]\kappa \equiv [A'/\alpha]\kappa : \mathsf{kind}$: (type substitution)
  This follows from the fact that kind pre-interpretations ignore type substitutions.

- Case $\Gamma \vdash \kappa \equiv \kappa : \mathsf{kind}$: (reflexivity):
  This follows since $\|\kappa\| = \|\kappa\|$.

- Case $\Gamma \vdash \kappa \equiv \kappa' : \mathsf{kind}$: (symmetry):
  By inversion, we know $\Gamma \vdash \kappa' \equiv \kappa : \mathsf{kind}$.
  Hence by induction, $\|\kappa'\| = \|\kappa\|$, which we sought.

- Case $\Gamma \vdash \kappa \equiv \kappa'' : \mathsf{kind}$: (transitivity):
  By inversion, we know $\Gamma \vdash \kappa \equiv \kappa' : \mathsf{kind}$.
  By inversion, we know $\Gamma \vdash \kappa' \equiv \kappa'' : \mathsf{kind}$.
  By induction, $\|\kappa\| = \|\kappa'\|$.
  By induction, $\|\kappa'\| = \|\kappa''\|$.
  Hence $\|\kappa\| = \|\kappa''\|$.

□

Theorem 2 (Well-Definedness)**.**

1. *If* $D :: \Gamma\ \mathsf{ok}$*, then* $[\![D :: \Gamma\ \mathsf{ok}]\!] \in \mathcal{P}(\|\Gamma\|)$*.*

2. *If* $D :: \Gamma \vdash \kappa : \mathsf{kind}$*, then* $[\![D :: \Gamma \vdash \kappa : \mathsf{kind}]\!] \in \|\Gamma\| \to \mathcal{P}(\|\kappa\|)$*.*

3. *If* $D :: \Gamma \vdash A : \kappa$*, then* $[\![D :: \Gamma \vdash A : \kappa]\!] \in \|\Gamma\| \to \|\kappa\|$*.*

Proof. We proceed by induction on the relevant derivations.

1.
   - Case $\cdot\ \mathsf{ok}$:
     $[\![\Gamma\ \mathsf{ok}]\!] = \{\langle\rangle\}$.
     $\|\cdot\| = \{\langle\rangle\}$.
     Hence $\{\langle\rangle\} \in \mathcal{P}(\{\langle\rangle\})$.
   - Case $\Gamma, x : X\ \mathsf{ok}$:
     By inversion, we know $\Gamma\ \mathsf{ok}$ and $\Gamma \vdash X : *$.
     By induction, $[\![\Gamma\ \mathsf{ok}]\!] \in \mathcal{P}(\|\Gamma\|)$.
     By induction, $[\![\Gamma \vdash X : *]\!] \in \|\Gamma\| \to \|*\|$.
     By definition, $[\![\Gamma, x : X\ \mathsf{ok}]\!] = \{(\gamma, \bar{e}/x) \mid \gamma \in [\![\Gamma\ \mathsf{ok}]\!] \wedge \bar{e} \in [\![\Gamma \vdash X : *]\!]\ \gamma\}$.
     By definition, $\|\Gamma, x : X\| = \|\Gamma\| \times \mathrm{Exp}^2$.
     So we want to show $[\![\Gamma, x : X\ \mathsf{ok}]\!] \subseteq \|\Gamma\| \times \mathrm{Exp}^2$.
     Assume $(\gamma, \bar{e}/x)$ such that $\gamma \in [\![\Gamma\ \mathsf{ok}]\!] \wedge \bar{e} \in [\![\Gamma \vdash X : *]\!]\ \gamma$.

From induction hypothesis, we know we know $\gamma \in \|\Gamma\|$.
Hence from $\|\Gamma \vdash X : *\| \ \gamma \in \mathcal{P}(\|*\|)$.
Hence $\|\Gamma \vdash X : *\| \ \gamma \subseteq \text{Exp}^2$.
Hence $\bar{e} \in \text{Exp}^2$.
Hence $[\![\Gamma, x : X \ \mathsf{ok}]\!] \subseteq \|\Gamma\| \times \text{Exp}^2$.

- Case $\Gamma, \alpha : \kappa \ \mathsf{ok}$:
  By inversion, we know $\Gamma \ \mathsf{ok}$ and $\Gamma \vdash \kappa : \mathsf{kind}$.
  By induction, $[\![\Gamma \ \mathsf{ok}]\!] \in \mathcal{P}(\|\Gamma\|)$.
  By induction, $[\![\Gamma \vdash \kappa : \mathsf{kind}]\!] \in \|\Gamma\| \to \mathcal{P}(\|\kappa\|)$.
  By definition, $[\![\Gamma, \alpha : \kappa \ \mathsf{ok}]\!] = \{(\gamma, (\overline{A}, R)/\alpha) \mid \gamma \in [\![\Gamma \ \mathsf{ok}]\!] \wedge \overline{A} \in \text{Type}^2 \wedge R \in [\![\Gamma \vdash \kappa : \mathsf{kind}]\!] \ \gamma\}$.
  By definition, $\|\Gamma, \alpha : \kappa\| = \|\Gamma\| \times (\text{Type}^2 \times \|\kappa\|)$.
  So we want to show $[\![\Gamma, \alpha : \kappa \ \mathsf{ok}]\!] \subseteq \|\Gamma\| \times (\text{Type}^2 \times \|\kappa\|)$.
  Assume $(\gamma, (\overline{A}, R)/\alpha)$ such that $\gamma \in [\![\Gamma \ \mathsf{ok}]\!] \wedge \overline{A} \in \text{Type}^2 \wedge R \in [\![\Gamma \vdash \kappa : \mathsf{kind}]\!] \ \gamma$.
  From induction hypothesis, we know we know $[\![\Gamma \ \mathsf{ok}]\!] \subseteq \|\Gamma\|$.
  Hence $\gamma \in \|\Gamma\|$.
  By induction, we know $[\![\Gamma \vdash \kappa : \mathsf{kind}]\!] \ \gamma \subseteq \|\kappa\|$.
  Hence $R \in \|\kappa\|$.
  Hence $(\gamma, (\overline{A}, R)/\alpha) \in \|\Gamma, \alpha : \kappa\|$.
  Hence $[\![\Gamma, \alpha : \kappa \ \mathsf{ok}]\!] \subseteq \|\Gamma\| \times (\text{Type}^2 \times \|\kappa\|)$.

2.
- Case $\Gamma \vdash * : \mathsf{kind}$:
  By definition, $[\![\Gamma \vdash * : \mathsf{kind}]\!] = \lambda\gamma.\ \textsc{Cand}$.
  We want to show this is in $\|\Gamma\| \to \mathcal{P}(\|*\|)$.
  Assume $\gamma \in \|\Gamma\|$.
  Hence $[\![\Gamma \vdash * : \mathsf{kind}]\!] \ \gamma = \textsc{Cand}$.
  We want to show $\textsc{Cand} \in \mathcal{P}(\|*\|) = \mathcal{P}(\text{Rel}(\text{Exp}, \text{Exp}))$.
  This is equivalent to showing $\textsc{Cand} \subseteq \text{Rel}(\text{Exp}, \text{Exp})$.
  Assume $R \in \textsc{Cand}$.
  By definition, $R \in \textsc{Qper}(\text{Exp}, \text{Exp})$.
  Hence $R \in \text{Rel}(\text{Exp}, \text{Exp})$.
  Hence $\textsc{Cand} \subseteq \text{Rel}(\text{Exp}, \text{Exp})$.
- Case $D :: \Gamma \vdash \Pi x : X.\ \kappa : \mathsf{kind}$:
  We want to show $[\![D :: \Gamma \vdash \Pi x : X.\ \kappa : \mathsf{kind}]\!] \in \|\Gamma\| \to \mathcal{P}(\|\Pi x : X.\ \kappa\|)$.
  Assume we have $\gamma \in \|\Gamma\|$.
  By definition, $[\![D]\!] \ \gamma \subseteq \|\Pi x : X.\ \kappa\|$.
- Case $D :: \Gamma \vdash \Pi\alpha : \kappa_1.\ \kappa_2 : \mathsf{kind}$:
  We want to show $[\![D :: \Gamma \vdash \Pi\alpha : \kappa_1.\ \kappa_2 : \mathsf{kind}]\!] \in \|\Gamma\| \to \mathcal{P}(\|\Pi\alpha : \kappa_1.\ \kappa_2\|)$.
  Assume we have $\gamma \in \|\Gamma\|$.
  By definition, $[\![D]\!] \ \gamma \subseteq \|\Pi\alpha : \kappa_1.\ \kappa_2\|$.

3.
- Case $D :: \Gamma \vdash \alpha : \kappa$:
  Assume we have $\gamma \in \|\Gamma\|$.
  Then $[\![D :: \Gamma \vdash \alpha : \kappa]\!] \ \gamma = \gamma(\alpha)$.
  By inversion, we know that $\alpha : \kappa \in \Gamma$.
  Hence $\gamma(\alpha) \in \|\kappa\|$.

- Case $D :: \Gamma \vdash \lambda\alpha : \kappa_1.\ A : \Pi\alpha : \kappa_1.\ \kappa_2$:
  By inversion, we have $D' :: \Gamma, \alpha : \kappa_1 \vdash A : \kappa_2$.
  By induction, for all $(\gamma, (\overline{A}, R)/\alpha) \in \|\Gamma, \alpha : \kappa_1\|$, $[\![D']\!]\ (\gamma, (\overline{A}, R)/\alpha) \in \|\kappa\|$.
  Assume we have $\gamma \in \|\Gamma\|$.

  We want to show $[\![D]\!]\ \gamma \in \|\Pi\alpha : \kappa_1.\ \kappa_2\|$.
  Equivalently, $[\![D]\!] \in (\text{Type}^2 \times \|\kappa_1\|) \to \|\kappa_2\|$.
  Assume $(\overline{A}, R) \in \text{Type}^2 \times \|\kappa_1\|$.
  Then $(\gamma, (\overline{A}, R)/\alpha) \in \|\Gamma, \alpha : \kappa_1\|$.
  Now consider whether $R \in [\![\Gamma \vdash \kappa_1 : \mathsf{kind}]\!]\ \gamma$.

  - If $R \in [\![\Gamma \vdash \kappa_1 : \mathsf{kind}]\!]\ \gamma$:
    Then $[\![D]\!]\ \gamma\ (\overline{A}, R) = [\![D']\!]\ (\gamma, (\overline{A}, R)/\alpha)$.
    By induction, this is in $\|\kappa_2\|$.
  - If $R \notin [\![\Gamma \vdash \kappa_1 : \mathsf{kind}]\!]\ \gamma$:
    Then $[\![D]\!]\ \gamma\ (\overline{A}, R) = !_{\kappa_2}$.
    So this is in $\|\kappa_2\|$.

  Hence $[\![D]\!]\ \gamma \in \|\kappa_1\| \to \|\kappa_2\|$.
  Hence $[\![D]\!]\ \gamma \in \|\Pi\alpha : \kappa_1.\ \kappa_2\|$.
- Case $D :: \Gamma \vdash B\ A : [A/\alpha]\kappa_2$:
  By inversion, $D_1 :: \Gamma \vdash B : \Pi\alpha : \kappa_1.\ \kappa_2$ and $D_2 :: \Gamma \vdash A : \kappa_1$.
  By induction, for all $\gamma \in \|\Gamma\|$, $[\![D_1]\!]\ \gamma \in \|\Pi\alpha : \kappa_1.\ \kappa_2\|$.
  By induction, for all $\gamma \in \|\Gamma\|$, $[\![D_2]\!]\ \gamma \in \|\kappa_1\|$.
  Assume we have $\gamma \in \|\Gamma\|$.
  Then $[\![D]\!]\ \gamma = [\![D_1]\!]\ \gamma\ (\gamma(A), [\![D_2]\!]\ \gamma)$.
  Note $\|\Pi\alpha : \kappa_1.\ \kappa_2\| = (\text{Type}^2 \times \|\kappa_1\|) \to \|\kappa_2\|$.
  Hence $[\![D]\!]\ \gamma \in \|\kappa_2\|$.
  Since kind pre-interpretations ignore type substitutions, $\|[A/\alpha]\kappa_2\| = \|\kappa_2\|$.
  Hence $[\![D]\!]\ \gamma \in \|[A/\alpha]\kappa_2\|$.
- Case $D :: \Gamma \vdash \lambda x : X.\ A : \Pi x : X.\ \kappa$:
  By inversion, we have $D' :: \Gamma, x : X \vdash A : \kappa$.
  By induction, for all $(\gamma, \bar{e}/x) \in \|\Gamma, x : X\|$, $[\![D']\!]\ (\gamma, \bar{e}/x) \in \|\kappa\|$.
  Assume we have $\gamma \in \|\Gamma\|$.
  We want to show $[\![D]\!]\ \gamma \in \|\Pi x : X.\ \kappa\|$.
  Equivalently, $[\![D]\!]\ \gamma \in \text{Exp}^2 \to \|\kappa\|$.
  Assume $\bar{e} \in \text{Exp}^2$.
  Then $(\gamma, \bar{e}/x) \in \|\Gamma, x : X\|$.
  Now consider whether $\bar{e} \in [\![\Gamma \vdash X : *]\!]\ \gamma$:

  - If $\bar{e} \in [\![\Gamma \vdash X : *]\!]\ \gamma$:
    Then $[\![D]\!]\ \gamma\ \bar{e} = [\![D']\!]\ (\gamma, \bar{e}/x)$.
    By induction, this is in $\|\kappa\|$.
  - If $\bar{e} \notin [\![\Gamma \vdash X : *]\!]\ \gamma$:
    Then $[\![D]\!]\ \gamma\ \bar{e} = !_{\kappa}$.
    But this is in $\|\kappa\|$.

Hence $[\![D]\!]\ \gamma \in \text{Exp}^2 \to \|\kappa\|$.
Hence $[\![D]\!]\ \gamma \in \|\Pi x : X.\ \kappa\|$.

- Case $D :: \Gamma \vdash A\ e : [e/x]\kappa$:
  By inversion, we have $D_1 :: \Gamma \vdash A : \Pi x : X.\ \kappa$ and $\Gamma \vdash e : X$ and $\Gamma \vdash X : *$.
  Assume we have $\gamma \in \|\Gamma\|$.
  By induction, we know that $[\![D_1]\!]\ \gamma \in \|\Pi x : X.\ \kappa\|$.
  Hence $[\![D_1]\!]\ \gamma \in \text{Exp}^2 \to \|\kappa\|$.
  Hence we know that $[\![D_1]\!]\ \gamma\ \gamma(e) \in \|\kappa\|$.
  Since kind pre-interpretations ignore term substitutions, $\|[e/x]\kappa\| = \|\kappa\|$.
  Hence $[\![D_1]\!]\ \gamma\ \gamma(e) \in \|[e/x]\kappa\|$.
  By definition, $[\![D]\!]\ \gamma = [\![D_1]\!]\ \gamma\ \gamma(e)$.
  Hence $[\![D]\!]\ \gamma \in \|[e/x]\kappa\|$.
- Case $D :: \Gamma \vdash \Pi x : X.\ Y : *$.
  Assume we have $\gamma \in \|\Gamma\|$.
  By definition, we know that $[\![D]\!]\ \gamma$ is a subset of Exp^2.
  Hence $[\![D]\!]\ \gamma \in \text{Rel}(\text{Exp}, \text{Exp})$.
  Hence $[\![D]\!]\ \gamma \in \|*\|$.
- Case $D :: \Gamma \vdash \Pi \alpha : \kappa.\ Y : *$.
  Assume we have $\gamma \in \|\Gamma\|$.
  By definition, we know that $[\![D]\!]\ \gamma$ is a subset of Exp^2.
  Hence $[\![D]\!]\ \gamma \in \text{Rel}(\text{Exp}, \text{Exp})$.
  Hence $[\![D]\!]\ \gamma \in \|*\|$.
- Case $D :: \Gamma \vdash e =_X e' : *$.
  Assume we have $\gamma \in \|\Gamma\|$.
  By definition, we know that $[\![D]\!]\ \gamma$ is a subset of Exp^2.
  Hence $[\![D]\!]\ \gamma \in \text{Rel}(\text{Exp}, \text{Exp})$.
  Hence $[\![D]\!]\ \gamma \in \|*\|$.
- Case $D :: \Gamma \vdash A : \kappa$. (Equality case)
  By inversion, we have $D_1 :: \Gamma \vdash A : \kappa'$ and $D_2 :: \Gamma \vdash \kappa : \kappa'\mathsf{kind}$.
  Assume we have $\gamma \in \|\Gamma\|$.
  By induction, we have $[\![D]\!]\ \gamma \in \|\kappa'\|$.
  By coherence of kind equality, we know that $\|\kappa\| = \|\kappa'\|$.
  Hence $[\![D]\!]\ \gamma \in \|\kappa\|$.

□

Theorem 3 (Coherence of Types and Kinds)**.**

1. *If* $D :: \Gamma \vdash \kappa : \mathsf{kind}$ *and* $D' :: \Gamma \vdash \kappa : \mathsf{kind}$ *and* $\gamma \in \|\Gamma\|$*, then* $[\![D]\!]\ \gamma = [\![D']\!]\ \gamma$*.*
2. *If* $D :: \Gamma \vdash A : \kappa$ *and* $D' :: \Gamma \vdash A : \kappa'$ *and* $\gamma \in \|\Gamma\|$*, then* $[\![D]\!]\ \gamma = [\![D']\!]\ \gamma$*.*

Proof. We proceed by simultaneous mutual induction on the derivations of well-formedness of kind and well-kindedness of type constructors.

1. • Case $D :: \Gamma \vdash * : \mathsf{kind}$ and $D' :: \Gamma \vdash * : \mathsf{kind}$.
Since there is only one rule for $\Gamma \vdash * : \mathsf{kind}$, we know $D = D'$.
Hence for all $\gamma \in \|\Gamma\|$, $[\![D]\!]\ \gamma = [\![D']\!]\ \gamma$.
   • Case $D :: \Gamma \vdash \Pi x : X.\ \kappa : \mathsf{kind}$ and $D' :: \Gamma \vdash \Pi x : X.\ \kappa : \mathsf{kind}$.
By inversion on D, we get $D_1 :: \Gamma \vdash X : *$ and $D_2 :: \Gamma, x : X \vdash \kappa : \mathsf{kind}$.
By inversion on D', we get $D'_1 :: \Gamma \vdash X : *$ and $D'_2 :: \Gamma, x : X \vdash \kappa : \mathsf{kind}$.
By mutual induction on D_1 and D'_1, we get $[\![D_1]\!] = [\![D'_1]\!]$.
By induction on D_2 and D'_2, we get $[\![D_2]\!] = [\![D'_2]\!]$.
Then by inspection of the kind semantics (Figure 12), $[\![D]\!] = [\![D']\!]$.
   • Case $D :: \Gamma \vdash \Pi\alpha : \kappa_1.\ \kappa_2 : \mathsf{kind}$ and $D' :: \Gamma \vdash \Pi\alpha : \kappa_1.\ \kappa_2 : \mathsf{kind}$.
By inversion on D, we get $D_1 :: \Gamma \vdash \kappa_1 : \mathsf{kind}$ and $D_2 :: \Gamma, \alpha : \kappa_1 \vdash \kappa_2 : \mathsf{kind}$.
By inversion on D', we get $D'_1 :: \Gamma \vdash \kappa_1 : \mathsf{kind}$ and $D'_2 :: \Gamma, \alpha : \kappa_1 \vdash \kappa_2 : \mathsf{kind}$.
By induction on D_1 and D'_1, we get $[\![D_1]\!] = [\![D'_1]\!]$.
By induction on D_2 and D'_2, we get $[\![D_2]\!] = [\![D'_2]\!]$.
Then by inspection of the kind semantics (Figure 12), $[\![D]\!] = [\![D']\!]$.

2. In this proof, we first consider whether D or D' ends in the use of an equality rule.

   • $D :: \Gamma \vdash A : \kappa$ and ends in an equality rule.
By inversion, we have $D_1 :: \Gamma \vdash A : \kappa_1$ and $D_2 :: \Gamma \vdash \kappa : \kappa_1$.
Then, by induction on D_1 and D', we know that $[\![D_1]\!] = [\![D']\!]$.
But by definition, $[\![D]\!] = [\![D_1]\!]$.
Hence $[\![D]\!] = [\![D']\!]$.

   • $D' :: \Gamma \vdash A : \kappa$ and ends in an equality rule.
By inversion, we have $D'_1 :: \Gamma \vdash A : \kappa_1$ and $D'_2 :: \Gamma \vdash \kappa : \kappa_1$.
Then, by induction on D_1 and D', we know that $[\![D]\!] = [\![D'_1]\!]$.
But by definition, $[\![D']\!] = [\![D'_1]\!]$.
Hence $[\![D]\!] = [\![D']\!]$.

   Now, we can consider the cases where neither D and D' ends in an equality rule.

   • Case $D :: \Gamma \vdash \alpha : \kappa$ and $D' :: \Gamma \vdash \alpha : \kappa'$:
Assume $\gamma \in \|\Gamma\|$.
By definition, $[\![D]\!]\ \gamma = [\![D']\!]\ \gamma = \gamma(\alpha)$.
   • Case $D :: \Gamma \vdash \lambda\alpha : \kappa_1.\ A : \Pi\alpha : \kappa_1.\ \kappa_2$ and $D' :: \Gamma \vdash \lambda\alpha : \kappa_1.\ A : \Pi\alpha : \kappa_1.\ \kappa'_2$:
By inversion, $D_1 :: \Gamma, \alpha : \kappa_1 \vdash A : \kappa_2$.
By inversion, $D'_1 :: \Gamma, \alpha : \kappa_1 \vdash A : \kappa'_2$.
Assume $\gamma \in \|\Gamma\|$.
By definition,

$$[\![D]\!]\ \gamma = \lambda(\overline{B}, R) \in \mathrm{Type}^2 \times \|\kappa_1\|. \begin{cases} [\![D_1]\!]\ (\gamma, (\overline{B}, R)/\alpha) & \text{if } R \in [\![\Gamma \vdash \kappa_1 : \mathsf{kind}]\!]\ \gamma \\ !_{\kappa_2} & \text{otherwise} \end{cases}$$

By definition,

$$[\![D']\!]\ \gamma = \lambda(\overline{B}, R) \in \mathrm{Type}^2 \times \|\kappa_1\|. \begin{cases} [\![D'_1]\!]\ (\gamma, (\overline{B}, R)/\alpha) & \text{if } R \in [\![\Gamma \vdash \kappa_1 : \mathsf{kind}]\!]\ \gamma \\ !_{\kappa_2} & \text{otherwise} \end{cases}$$

By induction, $[\![D_1]\!] = [\![D'_1]\!]$.
Hence $[\![D]\!]\ \gamma = [\![D']\!]\ \gamma$.
Hence $[\![D]\!] = [\![D']\!]$.

- Case $D :: \Gamma \vdash \lambda x : X.\ A : \Pi x : X.\ \kappa_2$ and $D' :: \Gamma \vdash \lambda x : X.\ A : \Pi x : X.\ \kappa'_2$:
By inversion, $D_1 :: \Gamma, x : X \vdash A : \kappa_2$.
By inversion, $D'_1 :: \Gamma, x : X \vdash A : \kappa'_2$.
Assume $\gamma \in \|\Gamma\|$.
By definition,

$$[\![D]\!]\ \gamma = \lambda \bar{e} \in \mathrm{Exp}^2. \begin{cases} [\![D_1]\!]\ \bar{e}/x) & \text{if } \bar{e} \in [\![\Gamma \vdash X : *]\!]\ \gamma \\ !_{\kappa_2} & \text{otherwise} \end{cases}$$

By definition,

$$[\![D']\!]\ \gamma = \lambda \bar{e} \in \mathrm{Exp}^2. \begin{cases} [\![D'_1]\!]\ \bar{e}/x) & \text{if } \bar{e} \in [\![\Gamma \vdash X : *]\!]\ \gamma \\ !_{\kappa_2} & \text{otherwise} \end{cases}$$

By induction, $[\![D_1]\!] = [\![D'_1]\!]$.
Hence $[\![D]\!]\ \gamma = [\![D']\!]\ \gamma$.
Hence $[\![D]\!] = [\![D']\!]$.

- Case $D :: \Gamma \vdash B\ A : [A/\alpha]\kappa_2$ and $D' :: \Gamma \vdash B\ A : [A/\alpha]\kappa'_2$:
By inversion, $D_1 :: \Gamma \vdash B : \Pi\alpha : \kappa_1.\ \kappa_2$ and $D_2 :: \Gamma \vdash A : \kappa_1$.
By inversion, $D'_1 :: \Gamma \vdash B : \Pi\alpha : \kappa'_1.\ \kappa'_2$ and $D'_2 :: \Gamma \vdash A : \kappa'_1$.
By induction, $[\![D_1]\!] = [\![D'_1]\!]$.
By induction, $[\![D_2]\!] = [\![D'_2]\!]$.
Assume $\gamma \in \|\Gamma\|$.
By definition, $[\![D]\!]\ \gamma = [\![D_1]\!]\ \gamma\ (\gamma(A), [\![D_2]\!]\ \gamma)$.
By definition, $[\![D']\!]\ \gamma = [\![D'_1]\!]\ \gamma\ (\gamma(A), [\![D'_2]\!]\ \gamma)$.
Hence $[\![D]\!]\ \gamma = [\![D']\!]\ \gamma$.

- Case $D :: \Gamma \vdash B\ e : [e/X]\kappa_2$ and $D' :: \Gamma \vdash B\ e : [e/X]\kappa'_2$:
By inversion, $D_1 :: \Gamma \vdash B : \Pi X : X.\ \kappa_2$ and $D_2 :: \Gamma \vdash e : X$ and $D_3 :: \Gamma \vdash X : *$.
By inversion, $D'_1 :: \Gamma \vdash B : \Pi X : \kappa'_1.\ \kappa'_2$ and $D'_2 :: \Gamma \vdash e : \kappa'_1$ and $D'_3 :: \Gamma \vdash X : *$.
By induction, $[\![D_1]\!] = [\![D'_1]\!]$.
Assume $\gamma \in \|\Gamma\|$.
By definition, $[\![D]\!]\ \gamma = [\![D_1]\!]\ \gamma\ \gamma(e)$.
By definition, $[\![D']\!]\ \gamma = [\![D'_1]\!]\ \gamma\ \gamma(e)$.
Hence $[\![D]\!]\ \gamma = [\![D']\!]\ \gamma$.

- Case $D :: \Gamma \vdash \Pi x : X.\ Y : *$ and $D' :: \Gamma \vdash \Pi x : X.\ Y : *$:
By inversion, $D_1 :: \Gamma \vdash X : *$ and $D_2 :: \Gamma, x : X \vdash Y : *$.
By inversion, $D'_1 :: \Gamma \vdash X : *$ and $D'_2 :: \Gamma, x : X \vdash Y : *$.
By induction, $[\![D_1]\!] = [\![D'_1]\!]$.
By induction, $[\![D_2]\!] = [\![D'_2]\!]$.
Then by inspection of the type semantics (Figure 13), $[\![D]\!] = [\![D']\!]$.

- Case $D :: \Gamma \vdash \Pi\alpha : \kappa.\ Y : *$ and $D' :: \Gamma \vdash \Pi\alpha : \kappa.\ Y : *$:
By inversion, $D_1 :: \Gamma \vdash \kappa : \mathsf{kind}$ and $D_2 :: \Gamma, \alpha : \kappa \vdash Y : *$.

By inversion, $D'_1 :: \Gamma \vdash \kappa : \mathsf{kind}$ and $D'_2 :: \Gamma, \alpha : \kappa \vdash Y : *$.
By mutual induction, $[\![D_1]\!] = [\![D'_1]\!]$.
By induction, $[\![D_2]\!] = [\![D'_2]\!]$.
Then by inspection of the type semantics (Figure 13), $[\![D]\!] = [\![D']\!]$.

- Case $D :: \Gamma \vdash e =_X e' : *$ and $D' :: \Gamma \vdash e =_X e' : *$:
  By inversion, $D_1 :: \Gamma \vdash X : *$ and $D_2 :: \Gamma \vdash e : X$ and $D_3 :: \Gamma \vdash e' : X$.
  By inversion, $D'_1 :: \Gamma \vdash X : *$ and $D'_2 :: \Gamma \vdash e : X$ and $D'_3 :: \Gamma \vdash e' : X$.
  By induction, $[\![D_1]\!] = [\![D'_1]\!]$.
  Then by inspection of the type semantics (Figure 13), $[\![D]\!] = [\![D']\!]$.

□

Corollary 1 (Coherence of Environment Interpretation)**.** *If* $D :: \Gamma\ \mathsf{ok}$ *and* $D' :: \Gamma\ \mathsf{ok}$*, then* $[\![D :: \Gamma\ \mathsf{ok}]\!] = [\![D' :: \Gamma\ \mathsf{ok}]\!]$.

Proof. This follows by simultaneous induction on the derivations of D and D'.

- Case $D :: \cdot\ \mathsf{ok}$ and $D' :: \cdot\ \mathsf{ok}$.
  In this case $[\![D]\!] = [\![D']\!] = \{\langle\rangle\}$.

- Case $D :: \Gamma, x : X\ \mathsf{ok}$ and $D' :: \Gamma, x : X\ \mathsf{ok}$.
  By inversion, we have $D_1 :: \Gamma\ \mathsf{ok}$ and $D_2 :: \Gamma \vdash X : *$.
  By inversion, we have $D'_1 :: \Gamma\ \mathsf{ok}$ and $D'_2 :: \Gamma \vdash X : *$.
  By induction, we know $[\![D_1]\!] = [\![D'_1]\!]$.
  By well-definedness, we know $[\![D_1]\!] \subseteq \|\Gamma\|$.
  Hence for each $\gamma \in [\![D_1]\!]$, we know $\gamma \in \|\Gamma\|$.
  By type coherence, for each $\gamma \in [\![D_1]\!]$, it follows that $[\![D_2]\!]\ \gamma = [\![D'_2]\!]\ \gamma$.
  Hence by inspection of the semantics of environment (Figure 10), $[\![D]\!] = [\![D']\!]$.

- Case $D :: \Gamma, \alpha : \kappa\ \mathsf{ok}$ and $D' :: \Gamma, \alpha : \kappa\ \mathsf{ok}$.
  By inversion, we have $D_1 :: \Gamma\ \mathsf{ok}$ and $D_2 :: \Gamma \vdash \kappa : \mathsf{kind}$.
  By inversion, we have $D'_1 :: \Gamma\ \mathsf{ok}$ and $D'_2 :: \Gamma \vdash \kappa : \mathsf{kind}$.
  By induction, we know $[\![D_1]\!] = [\![D'_1]\!]$.
  By well-definedness, we know $[\![D_1]\!] \subseteq \|\Gamma\|$.
  Hence for each $\gamma \in [\![D_1]\!]$, we know $\gamma \in \|\Gamma\|$.
  By kind coherence, for each $\gamma \in [\![D_1]\!]$, it follows that $[\![D_2]\!]\ \gamma = [\![D'_2]\!]\ \gamma$.
  Hence by inspection of the semantics of environment (Figure 10), $[\![D]\!] = [\![D']\!]$.

□

Theorem 4 (Weakening of Kinds and Types)**.** *We have that:*

1. *If* $D :: \Gamma_0, \Gamma_2 \vdash \kappa : \mathsf{kind}$ *then there exists* $D' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash \kappa : \mathsf{kind}$ *such that for all* $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ *and* γ_1 *such that* $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$, $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

2. *If* $D :: \Gamma_0, \Gamma_2 \vdash A : \kappa$ *then there exists* $D' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash A : \kappa$ *such that for all* $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ *and* γ_1 *such that* $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$, $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

Proof. We prove this by mutual induction on the derivations of kinds and types.

1. • Case $\Gamma \vdash * : \mathsf{kind}$:
Assume $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$.
Now, note that $[\![\Gamma_0, \Gamma_2 \vdash * : \mathsf{kind}]\!]\ (\gamma_0, \gamma_2) = \mathrm{Rel}(\mathrm{Exp}, \mathrm{Exp})$.
By rule, we have $D' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash * : \mathsf{kind}$.
By definition, $[\![D' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash * : \mathsf{kind}]\!]\ (\gamma_0, \gamma_1, \gamma_2) = \mathrm{Rel}(\mathrm{Exp}, \mathrm{Exp})$.
Hence $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

 • Case $D :: \Gamma \vdash \Pi x : X.\ \kappa_2 : \mathsf{kind}$:
By inversion, we have $D_1 :: \Gamma_0, \Gamma_2 \vdash X : *$ and $D_2 :: \Gamma_0, \Gamma_2, x : X \vdash \kappa_2 : \mathsf{kind}$.

By mutual induction, we have $D_1' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash X : *$ such that
for all $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$,
$[\![D_1]\!]\ (\gamma_0, \gamma_2) = [\![D_1']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

By induction, we have $D_2' :: \Gamma_0, \Gamma_1, \Gamma_2, x : X \vdash \kappa_2 : \mathsf{kind}$ such that
for all $\gamma_0, \gamma_1, \gamma_2'$ such that $(\gamma_0, \gamma_2') \in \|\Gamma_0, (\Gamma_2, x : X)\|$ and $(\gamma_0, \gamma_1, \gamma_2') \in \|\Gamma_0, \Gamma_1, \Gamma_2, x : X\|$,
$[\![D_1]\!]\ (\gamma_0, \gamma_2') = [\![D_1']\!]\ (\gamma_0, \gamma_1, \gamma_2')$.

Assume $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$.
By inspection of the kind semantics, $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

 • Case $D :: \Gamma \vdash \Pi\alpha : \kappa_1.\ \kappa_2 : \mathsf{kind}$:
By inversion, we have $D_1 :: \Gamma_0, \Gamma_2 \vdash X : *$ and $D_2 :: \Gamma_0, \Gamma_2, \alpha : \kappa_1 \vdash \kappa_2 : \mathsf{kind}$.

By induction, we have $D_1' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash X : *$ such that
for all $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$,
$[\![D_1]\!]\ (\gamma_0, \gamma_2) = [\![D_1']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

By induction, we have $D_2' :: \Gamma_0, \Gamma_1, \Gamma_2, \alpha : \kappa_1 \vdash \kappa : \mathsf{kind}$ such that
for all $\gamma_0, \gamma_1, \gamma_2'$ such that $(\gamma_0, \gamma_2') \in \|\Gamma_0, (\Gamma_2, \alpha : \kappa_1)\|$ and $(\gamma_0, \gamma_1, \gamma_2') \in \|\Gamma_0, \Gamma_1, \Gamma_2, \alpha : \kappa_1\|$,
$[\![D_1]\!]\ (\gamma_0, \gamma_2') = [\![D_1']\!]\ (\gamma_0, \gamma_1, \gamma_2')$.

Assume $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$.
By inspection of the kind semantics, $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

2. • Case $D :: \Gamma_0, \Gamma_2 \vdash A : \kappa$: (equality rule)
By inversion, we know $D_1 :: \Gamma_0, \Gamma_2 \vdash A : \kappa'$ and $D_2 :: \Gamma_0, \Gamma_2 \vdash \kappa \equiv \kappa' : \mathsf{kind}$.
By syntactic weakening, $D_2' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash \kappa \equiv \kappa' : \mathsf{kind}$.

By induction, we have $D_1' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash A : \kappa'$ such that
for all $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$,
$[\![D_1]\!]\ (\gamma_0, \gamma_2) = [\![D_1']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

By equality rule with D_2', we have $D' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash A : \kappa$.
Assume $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$.
By semantics, $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D_1]\!]\ (\gamma_0, \gamma_2)$.

By semantics, $[\![D']\!]\ (\gamma_0, \gamma_1, \gamma_2) = [\![D'_1]\!]\ (\gamma_0, \gamma_1, \gamma_2)$.
Hence $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

- Case $D :: \Gamma_0, \Gamma_2 \vdash \alpha : \kappa$:
  By inversion, we know that $\alpha : \kappa \in \Gamma_0, \Gamma_2$.
  By rule, we know $D' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash \alpha : \kappa$.
  Assume $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$.
  By definition, $[\![D]\!]\ (\gamma_0, \gamma_2) = (\gamma_0, \gamma_2)(\alpha)$.
  By definition, $[\![D']\!]\ (\gamma_0, \gamma_1, \gamma_2) = (\gamma_0, \gamma_1, \gamma_2)(\alpha)$.
  Since $\alpha \notin \Gamma_1$, $(\gamma_0, \gamma_1, \gamma_2)(\alpha) = (\gamma_0, \gamma_2)(\alpha)$.
  So $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.
- Case $D :: \Gamma_0, \Gamma_2 \vdash B\ A : [A/\alpha]\kappa_2$:
  By inversion, $D_1 :: \Gamma_0, \Gamma_2 \vdash B : \Pi\alpha : \kappa_1.\ \kappa_2$ and $D_2 :: \Gamma_0, \Gamma_2 \vdash A : \kappa_1$.

  By induction, we have $D'_1 :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash B : \Pi\alpha : \kappa_1.\ \kappa_2$ such that
  for all $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$,
  $[\![D_1]\!]\ (\gamma_0, \gamma_2) = [\![D'_1]\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

  By induction, we have $D'_2 :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash A : \kappa_1$ such that
  for all $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$,
  $[\![D_2]\!]\ (\gamma_0, \gamma_2) = [\![D'_2]\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

  Hence by rule on D'_1 and D'_2, we have $D' :: \Gamma, \Gamma_1, \Gamma_2 \vdash B\ A : [A/\alpha]\kappa_2$.
  Assume $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$.
  By definition, $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D_1]\!]\ (\gamma_0, \gamma_2)\ ((\gamma_0, \gamma_2)(A), [\![D_2]\!]\ (\gamma_0, \gamma_2))$.
  Hence $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D'_1]\!]\ (\gamma_0, \gamma_1, \gamma_2)\ ((\gamma_0, \gamma_2)(A), [\![D'_2]\!]\ (\gamma_0, \gamma_1, \gamma_2))$.
  Since $\mathrm{FV}(A) \cap \mathrm{dom}(\Gamma_1) = \emptyset$, we know $(\gamma_0, \gamma_2)(A) = (\gamma_0, \gamma_1, \gamma_2)(A)$.
  Hence $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D'_1]\!]\ (\gamma_0, \gamma_1, \gamma_2)\ ((\gamma_0, \gamma_1, \gamma_2)(A), [\![D'_2]\!]\ (\gamma_0, \gamma_1, \gamma_2))$.
  From the definition, $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.
- Case $D :: \Gamma_0, \Gamma_2 \vdash B\ e : [e/x]\kappa_2$:
  By inversion, $D_1 :: \Gamma_0, \Gamma_2 \vdash B : \Pi x : X.\ \kappa_2$ and $D_2 :: \Gamma_0, \Gamma_2 \vdash e : X$
  and $D_3 :: \Gamma_0, \Gamma_2 \vdash X : *$.

  By induction, we have $D'_1 :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash B : \Pi x : X.\ \kappa_2$ such that
  for all $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$,
  $[\![D_1]\!]\ (\gamma_0, \gamma_2) = [\![D'_1]\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

  By syntactic weakening, we have $D'_2 :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash e : X$.
  By syntactic weakening, we have $D'_3 :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash X : *$.
  Hence by rule on D'_1, D'_2 and D'_3, we have $D' :: \Gamma, \Gamma_1, \Gamma_2 \vdash B\ e : [e/x]\kappa_2$.
  Assume $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$.
  By definition, $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D_1]\!]\ (\gamma_0, \gamma_2)\ (\gamma_0, \gamma_2)(e)$.
  Hence $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D'_1]\!]\ (\gamma_0, \gamma_1, \gamma_2)\ (\gamma_0, \gamma_2)(e)$.
  Since $\mathrm{FV}(e) \cap \mathrm{dom}(\Gamma_1) = \emptyset$, we know $(\gamma_0, \gamma_2)(e) = (\gamma_0, \gamma_1, \gamma_2)(e)$.
  Hence $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D'_1]\!]\ (\gamma_0, \gamma_1, \gamma_2)\ (\gamma_0, \gamma_1, \gamma_2)(e)$.
  From the definition, $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

- Case $\Gamma_0, \Gamma_2 \vdash \lambda\alpha : \kappa_1.\, A : \Pi\alpha : \kappa_1.\, \kappa_2$:
  By inversion, we know that $D_1 :: \Gamma_0, \Gamma_2 \vdash \kappa_1 : \mathsf{kind}$ and $D_2 :: \Gamma_0, \Gamma_2, \alpha : \kappa_1 \vdash A : \kappa_2$.

  By induction, we have $D_1' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash \kappa_1 : \mathsf{kind}$ such that
  for all $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$,
  $[\![D_1]\!]\ (\gamma_0, \gamma_2) = [\![D_1']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

  By induction, we have $D_2' :: \Gamma_0, \Gamma_1, \Gamma_2, \alpha : \kappa_1 \vdash A : \kappa_2$ such that
  for all $\gamma_0, \gamma_1, \gamma_2'$ such that $(\gamma_0, \gamma_2') \in \|\Gamma_0, (\Gamma_2, \alpha : \kappa_1)\|$ and $(\gamma_0, \gamma_1, \gamma_2') \in \|\Gamma_0, \Gamma_1, \Gamma_2, \alpha : \kappa_1\|$,
  $[\![D_1]\!]\ (\gamma_0, \gamma_2) = [\![D_1']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

  Now by rule on D_1' and D_2', we construct $D' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash \lambda\alpha : \kappa_1.\, A : \Pi\alpha : \kappa_1.\, \kappa_2$.
  Assume $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$.
  By inspection of the semantics, $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.
- Case $\Gamma_0, \Gamma_2 \vdash \lambda x : X.\, A : \Pi x : X.\, \kappa_2$:
  By inversion, we know that $D_1 :: \Gamma_0, \Gamma_2 \vdash X : *$ and $D_2 :: \Gamma_0, \Gamma_2, x : X \vdash A : \kappa_2$.

  By induction, we have $D_1' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash X : *$ such that
  for all $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$,
  $[\![D_1]\!]\ (\gamma_0, \gamma_2) = [\![D_1']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

  By induction, we have $D_2' :: \Gamma_0, \Gamma_1, \Gamma_2, x : X \vdash A : \kappa_2$ such that
  for all $\gamma_0, \gamma_1, \gamma_2'$ such that $(\gamma_0, \gamma_2') \in \|\Gamma_0, (\Gamma_2, x : X)\|$ and $(\gamma_0, \gamma_1, \gamma_2') \in \|\Gamma_0, \Gamma_1, \Gamma_2, x : X\|$,
  $[\![D_1]\!]\ (\gamma_0, \gamma_2) = [\![D_1']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

  Now by rule on D_1' and D_2', we construct $D' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash \lambda x : X.\, A : \Pi x : X.\, \kappa_2$.
  Assume $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$.
  By inspection of the semantics, $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.
- Case $\Gamma_0, \Gamma_2 \vdash \Pi x : X.\, Y : *$:
  By inversion, we know that $D_1 :: \Gamma_0, \Gamma_2 \vdash X : *$ and $D_2 :: \Gamma_0, \Gamma_2, x : X \vdash Y : *$.

  By induction, we have $D_1' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash X : *$ such that
  for all $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$,
  $[\![D_1]\!]\ (\gamma_0, \gamma_2) = [\![D_1']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

  By induction, we have $D_2' :: \Gamma_0, \Gamma_1, \Gamma_2, x : X \vdash Y : *$ such that
  for all $\gamma_0, \gamma_1, \gamma_2'$ such that $(\gamma_0, \gamma_2') \in \|\Gamma_0, (\Gamma_2, x : X)\|$ and $(\gamma_0, \gamma_1, \gamma_2') \in \|\Gamma_0, \Gamma_1, \Gamma_2, x : X\|$,
  $[\![D_1]\!]\ (\gamma_0, \gamma_2) = [\![D_1']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

  Now by rule on D_1' and D_2', we construct $D' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash \Pi x : X.\, Y : *$.
  Assume $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$.
  By inspection of the semantics, $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.
- Case $\Gamma_0, \Gamma_2 \vdash \Pi\alpha : \kappa_1.\, Y : *$:
  By inversion, we know that $D_1 :: \Gamma_0, \Gamma_2 \vdash \kappa_1 : \mathsf{kind}$ and $D_2 :: \Gamma_0, \Gamma_2, \alpha : \kappa_1 \vdash Y : *$.

By induction, we have $D_1' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash \kappa_1 : \mathsf{kind}$ such that
for all $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$,
$[\![D_1]\!]\ (\gamma_0, \gamma_2) = [\![D_1']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

By induction, we have $D_2' :: \Gamma_0, \Gamma_1, \Gamma_2, \alpha : \kappa_1 \vdash Y : *$ such that
for all $\gamma_0, \gamma_1, \gamma_2'$ such that $(\gamma_0, \gamma_2') \in \|\Gamma_0, (\Gamma_2, \alpha : \kappa_1)\|$ and $(\gamma_0, \gamma_1, \gamma_2') \in \|\Gamma_0, \Gamma_1, \Gamma_2, \alpha : \kappa_1\|$,
$[\![D_1]\!]\ (\gamma_0, \gamma_2) = [\![D_1']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

Now by rule on D_1' and D_2', we construct $D' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash \Pi\alpha : \kappa_1.\, Y : *$.
Assume $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$.
By inspection of the semantics, $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

- Case $\Gamma_0, \Gamma_2 \vdash e =_X e' : *$:
  By inversion, we know $D_1 :: \Gamma_0, \Gamma_2 \vdash X : *$ and $D_2 :: \Gamma_0, \Gamma_2 \vdash e : X$ and $D_3 :: \Gamma_0, \Gamma_2 \vdash e' : X$.

  By induction, we have $D_1' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash X : *$ such that
  for all $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$,
  $[\![D_1]\!]\ (\gamma_0, \gamma_2) = [\![D_1']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

  By syntactic weakening, $D_2' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash e : X$ and $D_3' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash e' : X$.
  Furthermore, we know $\mathrm{FV}(e) \cap \mathrm{dom}(\Gamma_1) = \mathrm{FV}(e') \cap \mathrm{dom}(\Gamma_1) = \emptyset$.
  Hence $(\gamma_0, \gamma_1, \gamma_2)(e) = (\gamma_0, \gamma_2)(e)$.
  Hence $(\gamma_0, \gamma_1, \gamma_2)(e') = (\gamma_0, \gamma_2)(e')$.
  By rule, we have $D' :: \Gamma_0, \Gamma_1, \Gamma_2 \vdash e =_X e' : *$.
  Assume $\gamma_0, \gamma_1, \gamma_2$ such that $(\gamma_0, \gamma_2) \in \|\Gamma_0, \Gamma_2\|$ and $(\gamma_0, \gamma_1, \gamma_2) \in \|\Gamma_0, \Gamma_1, \Gamma_2\|$.

  Now, assume $(e_0, e_1) \in [\![D]\!]\ (\gamma_0, \gamma_2)$.
  Then $e_0 \mapsto^* \mathsf{refl}$ and $e_1 \mapsto^* \mathsf{refl}$ and $(\gamma_0, \gamma_2)(e, e') \in [\![D_1]\!]\ (\gamma_0, \gamma_2)$.
  Hence we know that $(\gamma_0, \gamma_2)(e, e') \in [\![D_1']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.
  Hence we know that $(\gamma_0, \gamma_1, \gamma_2)(e, e') \in [\![D_1']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.
  Hence we know that $[\![D]\!]\ (\gamma_0, \gamma_2) \subseteq [\![D']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

  Now, assume $(e_0, e_1) \in [\![D']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.
  Then $e_0 \mapsto^* \mathsf{refl}$ and $e_1 \mapsto^* \mathsf{refl}$ and $(\gamma_0, \gamma_1, \gamma_2)(e, e') \in [\![D_1']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.
  Hence we know that $(\gamma_0, \gamma_1, \gamma_2)(e, e') \in [\![D_1]\!]\ (\gamma_0, \gamma_2)$.
  Hence we know that $(\gamma_0, \gamma_2)(e, e') \in [\![D_1']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.
  Hence we know that $[\![D']\!]\ (\gamma_0, \gamma_2) \subseteq [\![D]\!]\ (\gamma_0, \gamma_2)$.

  Hence $[\![D]\!]\ (\gamma_0, \gamma_2) = [\![D']\!]\ (\gamma_0, \gamma_1, \gamma_2)$.

□

Theorem 5 (Substitution for Pre-Contexts)**.** *We have that:*

1. *If* $\Gamma \vdash e : X$*, and* $(\gamma, \gamma(e)/x, \gamma') \in \|\Gamma, x : X, \Gamma'\|$*, then* $(\gamma, \gamma') \in \|\Gamma, [e/x]\Gamma'\|$*.*

2. *If* $\Gamma \vdash A : \kappa$*, and* $(\gamma, (\gamma(A), R)/\alpha, \gamma') \in \|\Gamma, \alpha : \kappa, \Gamma'\|$*, then* $(\gamma, \gamma') \in \|\Gamma, [A/\alpha]\Gamma'\|$*.*

Proof. In each case, we proceed by induction on Γ'.

1. • $\Gamma' = \cdot$:
   Hence $\gamma' = \cdot$.
   Since $[e/x](\cdot) = \cdot$, it is immediately the case that $(\gamma, \cdot) \in \|\Gamma, [e/x]\cdot\|$.
   • $\Gamma' = \Gamma'', y : Y$:
   Hence $\gamma' = (\gamma'', \overline{e'}/y)$.
   By induction, $(\gamma, \gamma'') \in \|\Gamma, [e/x]\Gamma''\|$.
   By definition $(\gamma, \gamma'', \overline{e'}/y) \in \|\Gamma, [e/x]\Gamma'', y : [e/x]Y\|$.
   • $\Gamma' = \Gamma'', \beta : \kappa_0$:
   Hence $\gamma' = (\gamma'', (\overline{B}, R)/\beta)$.
   By induction, $(\gamma, \gamma'') \in \|\Gamma, [e/x]\Gamma''\|$.
   Since kind pre-interpretations ignore term substitutions, $R \in \|[e/x]\kappa_0\|$.
   Hence by definition $(\gamma, \gamma'', (\overline{B}, R)/\beta) \in \|\Gamma, [e/x]\Gamma'', \beta : [e/x]\kappa_0\|$.

2. • $\Gamma' = \cdot$:
   Hence $\gamma' = \cdot$.
   Since $[e/x](\cdot) = \cdot$, it is immediately the case that $(\gamma, \cdot) \in \|\Gamma, [e/x]\cdot\|$.
   • $\Gamma' = \Gamma'', y : Y$:
   Hence $\gamma' = (\gamma'', \overline{e'}/y)$.
   By induction, $(\gamma, \gamma'') \in \|\Gamma, [e/x]\Gamma''\|$.
   By definition $(\gamma, \gamma'', \overline{e'}/y) \in \|\Gamma, [A/\alpha]\Gamma'', y : [A/\alpha]Y\|$.
   • $\Gamma' = \Gamma'', \beta : \kappa_0$:
   Hence $\gamma' = (\gamma'', (\overline{B}, R)/\beta)$.
   By induction, $(\gamma, \gamma'') \in \|\Gamma, [e/x]\Gamma''\|$.
   Since kind pre-interpretations ignore type substitutions, $R \in \|[A/\alpha]\kappa_0\|$.
   Hence by definition $(\gamma, \gamma'', (\overline{B}, R)/\beta) \in \|\Gamma, [A/\alpha]\Gamma'', \beta : [A/\alpha]\kappa_0\|$.

□

Theorem 6 (Substitution of Terms)**.** *Suppose that* $\Gamma \vdash e : X$ *and* $(\gamma, \gamma(e)/x, \gamma') \in \|\Gamma, x : X, \Gamma'\|$*. Then:*

1. *For all* $D :: \Gamma, x : X, \Gamma' \vdash \kappa_0 : \mathsf{kind}$*, there exists* $D' :: \Gamma, [e/x]\Gamma' \vdash [e/x]\kappa_0 : \mathsf{kind}$ *such that* $[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D']\!]\ (\gamma, \gamma')$*.*

2. *For all* $D :: \Gamma, x : X, \Gamma' \vdash C : \kappa_0$*, there exists* $D' :: \Gamma, [e/x]\Gamma' \vdash [e/x]C : [e/x]\kappa_0$ *such that* $[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D']\!]\ (\gamma, \gamma')$*.*

Proof. Assume $\Gamma \vdash e : X$.
We proceed by mutual induction on the kinding and typing derivations.

1. We proceed by case analysis of the derivation $D :: \Gamma, x : X, \Gamma' \vdash \kappa_0 : \mathsf{kind}$.

- Case $D :: \Gamma, x : X, \Gamma' \vdash * : \mathsf{kind}$:
  Assume $(\gamma, \gamma(e)/x, \gamma') \in \|\Gamma, x : X, \Gamma'\|$.
  By rule, we have $D' :: \Gamma, [e/x]\Gamma' \vdash * : \mathsf{kind}$.
  Note that $[e/x]* = *$, and so $D' :: \Gamma, [e/x]\Gamma' \vdash [e/x]* : \mathsf{kind}$.
  By definition $[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D']\!]\ (\gamma, \gamma') = \textsc{Cand}$.
- Case $D :: \Gamma, x : X, \Gamma' \vdash \Pi y : Y.\ \kappa_2 : \mathsf{kind}$:
  By inversion, we have $D_0 :: \Gamma, x : X, \Gamma' \vdash Y : *$ and $D_1 :: \Gamma, x : X, \Gamma', y : Y \vdash \kappa_2 : \mathsf{kind}$.

  By induction, we know there is a $D_0' :: \Gamma, [e/x]\Gamma' \vdash [e/x]Y : [e/x]*$
  such that for all $(\gamma, \gamma(e)/x, \gamma', \overline{e'}/y) \in \|\Gamma, x : X, \Gamma', y : Y\|$,
  we know $[\![D_0]\!]\ (\gamma, \gamma(e)/x, \gamma', \overline{e'}/y) = [\![D_0']\!]\ (\gamma, \gamma', \overline{e'}/x)$.

  By induction, we know there is a $D_1' :: \Gamma, [e/x]\Gamma', y : [e/x]Y \vdash [e/x]\kappa_2 : \mathsf{kind}$
  such that for all $(\gamma, \gamma(e)/x, \gamma', \overline{e'}/y) \in \|\Gamma, x : X, \Gamma', y : Y\|$,
  we know $[\![D_1]\!]\ (\gamma, \gamma(e)/x, \gamma', \overline{e'}/y) = [\![D_1']\!]\ (\gamma, \gamma', \overline{e'}/x)$.

  By rule on D_0' and D_1', we get $D' :: \Gamma, [e/x]\Gamma' \vdash \Pi y : [e/x]Y.\ [e/x]\kappa_2 : \mathsf{kind}$.
  By definition of substitution, $D' :: \Gamma, [e/x]\Gamma' \vdash [e/x](\Pi y : Y.\ \kappa_2) : \mathsf{kind}$.

  Assume $(\gamma, \gamma(e)/x, \gamma') \in \|\Gamma, x : X, \Gamma'\|$.
  By inspection of the definition of $[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma')$, we see
  it equals $[\![D']\!]\ (\gamma, \gamma')$.
- Case $D :: \Gamma, x : X, \Gamma' \vdash \Pi\beta : \kappa_1.\ \kappa_2 : \mathsf{kind}$:
  By inversion, we have $D_0 :: \Gamma, x : X, \Gamma' \vdash \kappa_1 : \mathsf{kind}$ and $D_1 :: \Gamma, x : X, \Gamma', \beta : \kappa_1 \vdash \kappa_2 : \mathsf{kind}$.

  By induction, we know there is a $D_0' :: \Gamma, [e/x]\Gamma' \vdash [e/x]\kappa_1 : [e/x]\mathsf{kind}$
  such that for all $(\gamma, \gamma(e)/x, \gamma', (\overline{B}, R)/\beta) \in \|\Gamma, x : X, \Gamma', \beta : \kappa_1\|$,
  we know $[\![D_0]\!]\ (\gamma, \gamma(e)/x, \gamma', (\overline{B}, R)/\beta) = [\![D_0']\!]\ (\gamma, \gamma', (\overline{B}, R)/\beta)$.

  By induction, we know there is a $D_1' :: \Gamma, [e/x]\Gamma', \beta : [e/x]\kappa_1 \vdash [e/x]\kappa_2 : \mathsf{kind}$
  such that for all $(\gamma, \gamma(e)/x, \gamma', (\overline{B}, R)/\beta) \in \|\Gamma, x : X, \Gamma', \beta : \kappa_1\|$,
  we know $[\![D_1]\!]\ (\gamma, \gamma(e)/x, \gamma', (\overline{B}, R)/\beta) = [\![D_1']\!]\ (\gamma, \gamma', (\overline{B}, R)/\beta)$.

  By rule on D_0' and D_1', we get $D' :: \Gamma, [e/x]\Gamma' \vdash \Pi\beta : [e/x]\kappa_1.\ [e/x]\kappa_2 : \mathsf{kind}$.
  By definition of substitution, $D' :: \Gamma, [e/x]\Gamma' \vdash [e/x](\Pi\beta : \kappa_1.\ \kappa_2) : \mathsf{kind}$.

  Assume $(\gamma, \gamma(e)/x, \gamma') \in \|\Gamma, x : X, \Gamma'\|$.
  By inspection of the definition of $[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma')$, we see
  it equals $[\![D']\!]\ (\gamma, \gamma')$.

2. We proceed by case analysis of the derivation $D :: \Gamma, x : X, \Gamma' \vdash C : \kappa_0$.

   - Case $\Gamma, x : X, \Gamma' \vdash \beta : \kappa_0$:
     By inversion, we know that $\beta : \kappa_0 \in \Gamma, \alpha : \kappa, \Gamma'$.

Since x is a term variable, we know either $\beta \in \Gamma$ or $\beta \in \Gamma'$.
Therefore, it follows that $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')(\beta) = (\gamma, \gamma')(\beta)$.

If $\beta \in \Gamma$, then by rule we have $D' :: \Gamma, [e/x]]\Gamma' \vdash \beta : \kappa_0$.
Since the $x \notin \mathrm{FV}(\kappa_0)$, we know $D' :: \Gamma, [e/x]]\Gamma' \vdash \beta : [e/x]]\kappa_0$.
If $\beta \in \Gamma'$, then by rule we have $D' :: \Gamma, [e/x]]\Gamma' \vdash \beta : [e/x]]\kappa_0$.

Assume $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') \in \|\Gamma, \alpha : \kappa, \Gamma'\|$.
In either case, $[\![D']\!]\ (\gamma, \gamma') = (\gamma, \gamma')(\beta)$.
Hence $[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.

- Case $D :: \Gamma, x : X, \Gamma' \vdash \lambda y : Y.\, C : \Pi y : Y.\, \kappa_2$:
  By inversion, $D_0 :: \Gamma, x : X, \Gamma' \vdash Y : *$ and $D_1 :: \Gamma, x : X, \Gamma', y : Y \vdash C : \kappa_2$.

  By induction, we know there is a $D_0' :: \Gamma, [e/x]\Gamma' \vdash [e/x]Y : *$
  such that for all $(\gamma, \gamma(e)/x, \gamma', \overline{e'}/y) \in \|\Gamma, x : X, \Gamma', y : Y\|$,
  we know $[\![D_0]\!]\ (\gamma, \gamma(e)/x, \gamma', \overline{e'}/y) = [\![D_0']\!]\ (\gamma, \gamma', \overline{e'}/y)$.

  By induction, we know there is a $D_1' :: \Gamma, [e/x]\Gamma', y : [e/x]Y \vdash [e/x]C : [e/x]\kappa_2$
  such that for all $(\gamma, \gamma(e)/x, \gamma', \overline{e'}/y) \in \|\Gamma, x : X, \Gamma', y : Y\|$,
  we know $[\![D_1]\!]\ (\gamma, \gamma(e)/x, \gamma', \overline{e'}/y) = [\![D_1']\!]\ (\gamma, \gamma', \overline{e'}/y)$.

  Assume $(\gamma, \gamma(e)/x, \gamma') \in \|\Gamma, x : X, \Gamma'\|$.
  By inspection of the definition of $[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma')$, we see
  it equals $[\![D']\!]\ (\gamma, \gamma')$.
- Case $D :: \Gamma, x : X, \Gamma' \vdash \lambda\beta : \kappa_1.\, C : \Pi\beta : \kappa_1.\, \kappa_2$:
  By inversion, $D_0 :: \Gamma, x : X, \Gamma' \vdash \kappa_1 : \mathsf{kind}$ and $D_1 :: \Gamma, x : X, \Gamma', \beta : \kappa_1 \vdash C : \kappa_2$.

  By induction, we know there is a $D_0' :: \Gamma, [e/x]\Gamma' \vdash [e/x]\kappa_1 : \mathsf{kind}$
  such that for all $(\gamma, \gamma(e)/x, \gamma', (\overline{B}, R)/\beta) \in \|\Gamma, x : X, \Gamma', \beta : \kappa_1\|$,
  we know $[\![D_0]\!]\ (\gamma, \gamma(e)/x, \gamma', (\overline{B}, R)/\beta) = [\![D_0']\!]\ (\gamma, \gamma', (\overline{B}, R)/\beta)$.

  By induction, we know there is a $D_1' :: \Gamma, [e/x]\Gamma', \beta : [e/x]\kappa_1 \vdash [e/x]C : [e/x]\kappa_2$
  such that for all $(\gamma, \gamma(e)/x, \gamma', (\overline{B}, R)/\beta) \in \|\Gamma, x : X, \Gamma', \beta : \kappa_1\|$,
  we know $[\![D_1]\!]\ (\gamma, \gamma(e)/x, \gamma', (\overline{B}, R)/\beta) = [\![D_1']\!]\ (\gamma, \gamma', (\overline{B}, R)/\beta)$.

  Assume $(\gamma, \gamma(e)/x, \gamma') \in \|\Gamma, x : X, \Gamma'\|$.
  By inspection of the definition of $[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma')$, we see
  it equals $[\![D']\!]\ (\gamma, \gamma')$.
- Case $D :: \Gamma, x : X, \Gamma' \vdash C\ B : [B/\beta]\kappa_2$:
  By inversion, $D_0 :: \Gamma, x : X, \Gamma' \vdash C : \Pi\beta : \kappa_1.\, \kappa_2$.
  By inversion, $D_1 :: \Gamma, x : X, \Gamma' \vdash B : \kappa_1$.

  By induction, we know there is a $D_0' :: \Gamma, [e/x]\Gamma' \vdash [e/x]C : \Pi\beta : [e/x]\kappa_1.\, \kappa_2$
  s.t. for all $(\gamma, \gamma(e)/x, \gamma') \in \|\Gamma, x : X, \Gamma'\|$, $[\![D_0]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D_0']\!]\ (\gamma, \gamma')$.

By induction, we know there is a $D'_1 :: \Gamma, [e/x]\Gamma' \vdash [e/x]B : [e/x]\kappa_1$
s.t. for all $(\gamma, \gamma(e)/x, \gamma') \in \|\Gamma, x : X, \Gamma'\|$, $[\![D_0]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D'_0]\!]\ (\gamma, \gamma')$.

Via application rule on D'_0 and D'_1, get $D' :: \Gamma, [e/x]\Gamma' \vdash [e/x]C\ [e/x]B : [[e/x]B/\beta]\kappa_2$.
By properties of equality, this is $D' :: \Gamma, [e/x]\Gamma' \vdash [e/x](C\ B) : [e/x]([B/\beta]\kappa_2)$.

Assume $(\gamma, \gamma(e)/x, \gamma') \in \|\Gamma, x : X, \Gamma'\|$.
Note $(\gamma, \gamma') \in \|\Gamma, [e/x]\Gamma'\|$.
We know:

$$[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D_0]\!]\ \ (\gamma, \gamma(e)/x, \gamma')\ \ (\gamma, \gamma(e)/x, \gamma')(e)(B)\ ([\![D_1]\!]\ \ (\gamma, \gamma(e)/x, \gamma'))$$

By induction hypotheses, we know

$$[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D'_0]\!]\ \ (\gamma, \gamma')\ (\gamma, \gamma(e)/x, \gamma')(B)\ ([\![D'_1]\!]\ \ (\gamma, \gamma'))$$

By definition of substitution, we know $(\gamma, \gamma(e)/x, \gamma')(B) = (\gamma, \gamma')([e/x]B)$.
Hence

$$[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D'_0]\!]\ \ (\gamma, \gamma')\ (\gamma, \gamma')(B)\ ([\![D'_1]\!]\ \ (\gamma, \gamma'))$$

Hence $[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.

- Case $D :: \Gamma, x : X, \Gamma' \vdash C\ e' : [e'/y]\kappa_2$:
By inversion, $D_0 :: \Gamma, x : X, \Gamma' \vdash C : \Pi y : Y.\ \kappa_2$.
By inversion, $D_1 :: \Gamma, x : X, \Gamma' \vdash e' : Y$.
By inversion, $D_2 :: \Gamma, x : X, \Gamma' \vdash Y : *$.

By induction, we know there is a $D'_0 :: \Gamma, [e/x]\Gamma' \vdash [e/x]C : \Pi y : [e/x]Y.\ \kappa_2$
s.t. for all $(\gamma, \gamma(e)/x, \gamma') \in \|\Gamma, x : X, \Gamma'\|$, $[\![D_0]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D'_0]\!]\ (\gamma, \gamma')$.

By syntactic substitution, we have $D'_1 :: \Gamma, [e/x]\Gamma' \vdash [e/x]e' : [e/x]Y$.
By syntactic substitution, we have $D'_2 :: \Gamma, [e/x]\Gamma' \vdash [e/x]Y : *$.

Via application rule on D'_0, D'_1 and D'_2, get $D' :: \Gamma, [e/x]\Gamma' \vdash [e/x]C\ [e/x]e' : [[e/x]e'/y]\kappa_2$.
By substitution properties, this is $D' :: \Gamma, [e/x]\Gamma' \vdash [e/x](C\ e') : [e/x]([e'/y]\kappa_2)$.

Assume $(\gamma, \gamma(e)/x, \gamma') \in \|\Gamma, x : X, \Gamma'\|$.
Note $(\gamma, \gamma') \in \|\Gamma, [e/x]\Gamma'\|$.
We know:

$$[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D_0]\!]\ \ (\gamma, \gamma(e)/x, \gamma')\ (\gamma, \gamma(e)/x, \gamma')(e')$$

By induction hypothesis, we know

$$[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D'_0]\!]\ \ (\gamma, \gamma')\ (\gamma, \gamma(e)/x, \gamma')(e')$$

By properties of substitution, $(\gamma, \gamma(e)/x, \gamma')(e) = (\gamma, \gamma')([e/x]e')$.
Hence

$$[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D'_0]\!]\ (\gamma, \gamma')\ \ (\gamma, \gamma')([e/x]e')$$

Hence $[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.

- Case $D :: \Gamma, x : X, \Gamma' \vdash \Pi y : Y.\ Z : *$:
  By inversion, $D_0 :: \Gamma, x : X, \Gamma' \vdash Y : *$ and $D_1 :: \Gamma, x : X, \Gamma', y : Y \vdash Z : *$.

  By induction, there is $D'_0 :: \Gamma, [e/x]\Gamma' \vdash [e/x]Y : *$
  s.t. for all $(\gamma, \gamma(e)/x, \gamma') \in \|\Gamma, x : X, \Gamma'\|$, $[\![D_0]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D'_0]\!]\ (\gamma, \gamma')$.
  By induction, there is $D'_1 :: \Gamma, [e/x]\Gamma', y : [e/x]Y \vdash [e/x]Z : *$
  s.t. for all $(\gamma, \gamma(e)/x, \gamma', \overline{e'}/y) \in \|\Gamma, x : X, \Gamma', y : Y\|$,
  $[\![D_1]\!]\ (\gamma, \gamma(e)/x, \gamma', \overline{e'}/y) = [\![D'_1]\!]\ (\gamma, \gamma', \overline{e'}/y)$.

  By pi-rule on D'_0 and D'_1, we have $D' :: \Gamma, [e/x] \vdash \Pi y : [e/x]Y.\ [e/x]Z : *$.
  By properties of substitution, we have $D' :: \Gamma, [e/x] \vdash [e/x](\Pi y : Y.\ Z) : *$.

  Assume $(\gamma, \gamma(e)/x, \gamma') \in \|\Gamma, x : X, \Gamma'\|$.
  We want to show $[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.

  Assume $(f, f') \in [\![D]\!]\ (\gamma, \gamma(e)/x, \gamma')$.
  Then $f \downarrow$ and $f' \downarrow$ and
  for all $(t, t') \in [\![D_0]\!]\ (\gamma, \gamma(e)/x, \gamma')$, we know $(f\ t, f'\ t') \in [\![D_1]\!]\ (\gamma, \gamma(e)/x, \gamma', (t, t')/y)$.
  Assume $(t, t') \in [\![D'_0]\!]\ (\gamma, \gamma')$.
  Then we know by induction that $(t, t') \in [\![D_0]\!]\ (\gamma, \gamma(e)/x, \gamma')$.
  Hence we know that $(f\ t, f'\ t') \in [\![D_1]\!]\ (\gamma, \gamma(e)/x, \gamma', (t, t')/y)$.
  By induction hypothesis, we know $(f\ t, f'\ t') \in [\![D'_1]\!]\ (\gamma, \gamma', (t, t')/y)$.
  Hence $(f, f') \in [\![D']\!]\ (\gamma, \gamma')$.

  Assume $(f, f') \in [\![D']\!]\ (\gamma, \gamma')$.
  Then $f \downarrow$ and $f' \downarrow$ and
  for all $(t, t') \in [\![D'_0]\!]\ (\gamma, \gamma')$, we know $(f\ t, f'\ t') \in [\![D'_1]\!]\ (\gamma, \gamma', (t, t')/y)$.
  Assume $(t, t') \in [\![D_0]\!]\ (\gamma, \gamma')$.
  Then we know by induction that $(t, t') \in [\![D'_0]\!]\ (\gamma, \gamma')$.
  Hence we know that $(f\ t, f'\ t') \in [\![D'_1]\!]\ (\gamma, \gamma', (t, t')/y)$.
  By induction hypothesis, we know $(f\ t, f'\ t') \in [\![D_1]\!]\ (\gamma, \gamma(e)/x, \gamma', (t, t')/y)$.
  Hence $(f, f') \in [\![D]\!]\ (\gamma, \gamma(e)/x, \gamma')$.

  Hence $(f, f') \in [\![D]\!]\ (\gamma, \gamma(e)/x, \gamma')$ iff $(f, f') \in [\![D']\!]\ (\gamma, \gamma')$.
  Hence $[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.
- Case $D :: \Gamma, x : X, \Gamma' \vdash \Pi\beta : \kappa_1.\ Z : *$:
  By inversion, $D_0 :: \Gamma, x : X, \Gamma' \vdash \kappa_1 : \mathsf{kind}$ and $D_1 :: \Gamma, x : X, \Gamma', \beta : \kappa_1 \vdash Z : *$.

  By induction, there is $D'_0 :: \Gamma, [e/x]\Gamma' \vdash [e/x]\kappa_1 : *$
  s.t. for all $(\gamma, \gamma(e)/x, \gamma') \in \|\Gamma, x : X, \Gamma'\|$, $[\![D_0]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D'_0]\!]\ (\gamma, \gamma')$.
  By induction, there is $D'_1 :: \Gamma, x : X, \Gamma', \beta : \kappa_1 \vdash Z : *$
  s.t. for all $(\gamma, \gamma(e)/x, \gamma', (\overline{B}, R)/\beta) \in \|\Gamma, x : X, \Gamma', y : Y\|$,
  $[\![D_1]\!]\ (\gamma, \gamma(e)/x, \gamma', (\overline{B}, R)/\beta) = [\![D'_1]\!]\ (\gamma, \gamma', (\overline{B}, R)/\beta)$.

By all-rule on D_0' and D_1', we have $D' :: \Gamma, [e/x] \vdash \Pi y : [e/x]Y.\, [e/x]Z : *$.
By properties of substitution, we have $D' :: \Gamma, [e/x] \vdash [e/x](\Pi y : Y.\, Z) : *$.

Assume $(\gamma, \gamma(e)/x, \gamma') \in \|\Gamma, x : X, \Gamma'\|$.
We want to show $[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.

Assume $(f, f') \in [\![D]\!]\ (\gamma, \gamma(e)/x, \gamma')$.
Then $f \downarrow$ and $f' \downarrow$ and
for all $(B, B') \in \text{Type}^2, R \in [\![D_0]\!]\ (\gamma, \gamma(e)/x, \gamma')$,
we know $(f\ A, f'\ A') \in [\![D_1]\!]\ (\gamma, \gamma(e)/x, \gamma', (\overline{B}, R)/\beta)$.
Assume $(B, B') \in \text{Type}^2$ and $R \in [\![D_0']\!]\ (\gamma, \gamma')$.
Then we know by induction that $R \in [\![D_0]\!]\ (\gamma, \gamma(e)/x, \gamma')$.
Hence we know that $(f\ B, f'\ B') \in [\![D_1]\!]\ (\gamma, \gamma(e)/x, \gamma', (\overline{B}, R)/\beta)$.
By induction hypothesis, we know $(f\ B, f'\ B') \in [\![D_1']\!]\ (\gamma, \gamma', (\overline{B}, R)/\beta)$.
Hence $(f, f') \in [\![D']\!]\ (\gamma, \gamma')$.

Assume $(f, f') \in [\![D']\!]\ (\gamma, \gamma')$.
Then $f \downarrow$ and $f' \downarrow$ and
for all $(t, t') \in [\![D_0']\!]\ (\gamma, \gamma')$, we know $(f\ t, f'\ t') \in [\![D_1']\!]\ (\gamma, \gamma', (t, t')/y)$.
Assume $\overline{B} \in \text{Type}^2$ and $R \in [\![D_0]\!]\ (\gamma, \gamma')$.
Then we know by induction that $R \in [\![D_0']\!]\ (\gamma, \gamma')$.
Hence we know that $(f\ B, f'\ B') \in [\![D_1']\!]\ (\gamma, \gamma', (\overline{B}, R)/\beta)$.
By induction hypothesis, we know $(f\ B, f'\ B') \in [\![D_1]\!]\ (\gamma, \gamma(e)/x, \gamma', (\overline{B}, R)/\beta)$.
Hence $(f, f') \in [\![D]\!]\ (\gamma, \gamma(e)/x, \gamma')$.

Hence $(f, f') \in [\![D]\!]\ (\gamma, \gamma(e)/x, \gamma')$ iff $(f, f') \in [\![D']\!]\ (\gamma, \gamma')$.
Hence $[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.

- Case $D :: \Gamma, x : X, \Gamma' \vdash e_1 =_Y e_2 : *$:
By inversion, we have $D_0 :: \Gamma, x : X, \Gamma' \vdash Y : *$
and $D_1 :: \Gamma, x : X, \Gamma' \vdash e_1 : Y$ and $D_2 :: \Gamma, x : X, \Gamma' \vdash e_2 : Y$.
By induction, there is $D_0' :: \Gamma, [e/x]\Gamma' \vdash [e/x]Y : *$
s.t. for all $(\gamma, \gamma(e)/x, \gamma') \in \|\Gamma, x : X, \Gamma'\|$, $[\![D_0]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D_0']\!]\ (\gamma, \gamma')$.
By syntactic substitution, we have $D_1' :: \Gamma, [e/x]\Gamma' \vdash [e/x]e_1 : [e/x]Y$.
By syntactic substitution, we have $D_2' :: \Gamma, [e/x]\Gamma' \vdash [e/x]e_2 : [e/x]Y$.
By rule, we have $D' :: \Gamma, [e/x]\Gamma' \vdash [e/x]e_1 =_{[e/x]Y} [e/x]e_2 : *$.

  Assume $(\gamma, \gamma(e)/x, \gamma') \in \|\Gamma, x : X, \Gamma'\|$.
  We want to show $[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.

  Assume $(p, p') \in [\![D]\!]\ (\gamma, \gamma(e)/x, \gamma')$.
  Then $p \mapsto^* \mathsf{refl}$ and $p' \mapsto^* \mathsf{refl}$ and $(\gamma, \gamma(e)/x, \gamma')(e_1, e_2) \in [\![D_0]\!]\ (\gamma, \gamma(e)/x, \gamma')$.
  By properties of substitution, $(\gamma, \gamma')([e/x]e_1, [e/x]e_2) \in [\![D_0]\!]\ (\gamma, \gamma(e)/x, \gamma')$.
  By induction, $(\gamma, \gamma')([e/x]e_1, [e/x]e_2) \in [\![D_0']\!]\ (\gamma, \gamma')$.
  Hence $(p, p') \in [\![D']\!]\ (\gamma, \gamma')$.

Assume $(p, p') \in [\![D']\!]\ (\gamma, \gamma')$.
Then $p \mapsto^* \mathsf{refl}$ and $p' \mapsto^* \mathsf{refl}$ and $(\gamma, \gamma')([e/x]e_1, [e/x]e_2) \in [\![D'_0]\!]\ (\gamma, \gamma')$.
By properties of substitution, $(\gamma, \gamma(e)/x, \gamma')(e_1, e_2) \in [\![D'_0]\!]\ (\gamma, \gamma')$.
By induction, $(\gamma, \gamma(e)/x, \gamma')(e_1, e_2) \in [\![D_0]\!]\ (\gamma, \gamma(e)/x, \gamma')$.
Hence $(p, p') \in [\![D]\!]\ (\gamma, \gamma(e)/x, \gamma')$.

Hence $(p, p') \in [\![D]\!]\ (\gamma, \gamma(e)/x, \gamma')$ iff $(p, p') \in [\![D']\!]\ (\gamma, \gamma')$.
Hence $[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.

- Case $D :: \Gamma, x : X, \Gamma' \vdash Y : \kappa\mathsf{kind}$.
  By inversion, $D_0 :: \Gamma, x : X, \Gamma' \vdash Y : \kappa_0$ and $D_1 :: \Gamma, x : X, \Gamma' \vdash \kappa \equiv \kappa_0 : \mathsf{kind}$.

  By induction, there is $D'_0 :: \Gamma, [e/x]\Gamma' \vdash [e/x]Y : \kappa_0$
  s.t. for all $(\gamma, \gamma(e)/x, \gamma') \in \|\Gamma, x : X, \Gamma'\|$, $[\![D_0]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D'_0]\!]\ (\gamma, \gamma')$.
  By syntactic substitution, $D'_1 :: \Gamma, [e/x]\Gamma' \vdash [e/x]\kappa \equiv [e/x]\kappa_0 : \mathsf{kind}$.
  By rule, we have $D' :: \Gamma, [e/x]\Gamma' \vdash [e/x]\kappa : \mathsf{kind}$.

  We want to show $[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.
  By definition, $[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D_0]\!]\ (\gamma, \gamma(e)/x, \gamma')$.
  By induction, $[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D'_0]\!]\ (\gamma, \gamma')$.
  By definition, $[\![D]\!]\ (\gamma, \gamma(e)/x, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.

□

Theorem 7 (Substitution of Types). *Suppose that* $\Gamma \vdash A : \kappa$ *and* $(\gamma, (\gamma(A), [\![D_1]\!]\ \gamma)/\alpha, \gamma') \in \|\Gamma, \alpha : \kappa, \Gamma'\|$. *Then:*

1. *For all* $D :: \Gamma, \alpha : \kappa, \Gamma' \vdash \kappa_0 : \mathsf{kind}$*, there exists* $D' :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]\kappa_0 : \mathsf{kind}$ *such that* $[\![D]\!]\ (\gamma, (\gamma(A), [\![D_1]\!]\ \gamma)/\alpha, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.
2. *For all* $D :: \Gamma, \alpha : \kappa, \Gamma' \vdash C : \kappa_0$*, there exists* $D' :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]C : [A/\alpha]\kappa_0$ *such that* $[\![D]\!]\ (\gamma, (\gamma(A), [\![D_1]\!]\ \gamma)/\alpha, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.

Proof. Assume $T :: \Gamma \vdash A : \kappa$.
We proceed by mutual induction on the kinding and typing derivations.

1. We proceed by case analysis of the derivation $D :: \Gamma, \alpha : \kappa, \Gamma' \vdash \kappa_0 : \mathsf{kind}$.

   - Case $D :: \Gamma, \alpha : \kappa, \Gamma' \vdash * : \mathsf{kind}$:
     Assume $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') \in \|\Gamma, \alpha : \kappa, \Gamma'\|$.
     By rule, we have $D' :: \Gamma, [A/\alpha]\Gamma' \vdash * : \mathsf{kind}$.
     Note that $[A/\alpha]* = *$, and so $D' :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]* : \mathsf{kind}$.
     By definition $[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D']\!]\ (\gamma, \gamma') = \textsc{Cand}$.
   - Case $D :: \Gamma, \alpha : \kappa, \Gamma' \vdash \Pi y : Y.\ \kappa_2 : \mathsf{kind}$:
     By inversion, we have $D_0 :: \Gamma, \alpha : \kappa, \Gamma' \vdash Y : *$ and $D_1 :: \Gamma, \alpha : \kappa, \Gamma', y : Y \vdash \kappa_2 : \mathsf{kind}$.

By induction, we know there is a $D'_0 :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]Y : [A/\alpha]*$
such that for all $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', \overline{e'}/y) \in \|\Gamma, \alpha : \kappa, \Gamma', y : Y\|$,
we know $[\![D_0]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', \overline{e'}/y) = [\![D'_0]\!]\ (\gamma, \gamma', \overline{e'}/y)$.

By induction, we know there is a $D'_1 :: \Gamma, [A/\alpha]\Gamma', y : [A/\alpha]Y \vdash [A/\alpha]\kappa_2 : \mathsf{kind}$
such that for all $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', \overline{e'}/y) \in \|\Gamma, \alpha : \kappa, \Gamma', y : Y\|$,
we know $[\![D_1]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', \overline{e'}/y) = [\![D'_1]\!]\ (\gamma, \gamma', \overline{e'}/y)$.

By rule on D'_0 and D'_1, we get $D' :: \Gamma, [A/\alpha]\Gamma' \vdash \Pi y : [A/\alpha]Y.\ [A/\alpha]\kappa_2 : \mathsf{kind}$.
By definition of substitution, $D' :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha](\Pi y : Y.\ \kappa_2) : \mathsf{kind}$.

Assume $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') \in \|\Gamma, \alpha : \kappa, \Gamma'\|$.
By inspection of the definition of $[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')$, we see
it equals $[\![D']\!]\ (\gamma, \gamma')$.

- Case $D :: \Gamma, \alpha : \kappa, \Gamma' \vdash \Pi\beta : \kappa_1.\ \kappa_2 : \mathsf{kind}$:
  By inversion, we have $D_0 :: \Gamma, \alpha : \kappa, \Gamma' \vdash \kappa_1 : \mathsf{kind}$ and $D_1 :: \Gamma, \alpha : \kappa, \Gamma', \beta : \kappa_1 \vdash \kappa_2 : \mathsf{kind}$.

  By induction, we know there is a $D'_0 :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]\kappa_1 : [A/\alpha]\mathsf{kind}$
  such that for all $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', (\overline{B}, R)/\beta) \in \|\Gamma, \alpha : \kappa, \Gamma', \beta : \kappa_1\|$,
  we know $[\![D_0]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', (\overline{B}, R)/\beta) = [\![D'_0]\!]\ (\gamma, \gamma', (\overline{B}, R)/\beta)$.

  By induction, we know there is a $D'_1 :: \Gamma, [A/\alpha]\Gamma', \beta : [A/\alpha]\kappa_1 \vdash [A/\alpha]\kappa_2 : \mathsf{kind}$
  such that for all $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', (\overline{B}, R)/\beta) \in \|\Gamma, \alpha : \kappa, \Gamma', \beta : \kappa_1\|$,
  we know $[\![D_1]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', (\overline{B}, R)/\beta) = [\![D'_1]\!]\ (\gamma, \gamma', (\overline{B}, R)/\beta)$.

  By rule on D'_0 and D'_1, we get $D' :: \Gamma, [A/\alpha]\Gamma' \vdash \Pi\beta : [A/\alpha]\kappa_1.\ [A/\alpha]\kappa_2 : \mathsf{kind}$.
  By definition of substitution, $D' :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha](\Pi\beta : \kappa_1.\ \kappa_2) : \mathsf{kind}$.

  Assume $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') \in \|\Gamma, \alpha : \kappa, \Gamma'\|$.
  By inspection of the definition of $[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')$, we see
  it equals $[\![D']\!]\ (\gamma, \gamma')$.

2. We proceed by case analysis of the derivation $D :: \Gamma, \alpha : \kappa, \Gamma' \vdash C : \kappa_0$.

   - Case $\Gamma, \alpha : \kappa, \Gamma' \vdash \beta : \kappa_0$:
     By inversion, we know that $\beta : \kappa_0 \in \Gamma, \alpha : \kappa, \Gamma'$.
     Since α is a type variable, we know either $\beta \in \Gamma$ or $\alpha = \beta$ or $\beta \in \Gamma'$.
     - If $\alpha = \beta$:
       By weakening on T we get a $D' :: \Gamma, [A/\alpha]\Gamma' \vdash A : \kappa_0$
       such that $[\![D']\!]\ (\gamma, \gamma') = [\![T]\!]\ \gamma$.
       Note $[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha) = [\![T]\!]\ \gamma$.
       Hence $[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha) = [\![D']\!]\ (\gamma, \gamma')$.
     - Otherwise:
       * If $\beta \in \Gamma$, then by rule we have $D' :: \Gamma, [A/\alpha]\Gamma' \vdash \beta : \kappa_0$.
         Since the $\alpha \notin \mathrm{FV}(\kappa_0)$, we know $D' :: \Gamma, [A/\alpha]\Gamma' \vdash \beta : [A/\alpha]\kappa_0$.

- $*$ If $\beta \in \Gamma'$, then by rule we have $D' :: \Gamma, [A/\alpha]\Gamma' \vdash \beta : [A/\alpha]\kappa_0$.

  Assume $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') \in \|\Gamma, \alpha : A, \Gamma'\|$.
  In either case, $[\![D']\!]\ (\gamma, \gamma') = (\gamma, \gamma')(\beta)$.

  Hence $[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.

- Case $D :: \Gamma, \alpha : \kappa, \Gamma' \vdash \lambda y : Y.\ C : \Pi y : Y.\ \kappa_2$:
  By inversion, $D_0 :: \Gamma, \alpha : \kappa, \Gamma' \vdash Y : *$ and $D_1 :: \Gamma, \alpha : \kappa, \Gamma', y : Y \vdash C : \kappa_2$.

  By induction, we know there is a $D_0' :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]Y : *$
  such that for all $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', \overline{e'}/y) \in \|\Gamma, \alpha : \kappa, \Gamma', y : Y\|$,
  we know $[\![D_0]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', \overline{e'}/y) = [\![D_0']\!]\ (\gamma, \gamma', \overline{e'}/y)$.

  By induction, we know there is a $D_1' :: \Gamma, [A/\alpha]\Gamma', y : [A/\alpha]Y \vdash [A/\alpha]C : [A/\alpha]\kappa_2$
  such that for all $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', \overline{e'}/y) \in \|\Gamma, \alpha : \kappa, \Gamma', y : Y\|$,
  we know $[\![D_1]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', \overline{e'}/y) = [\![D_1']\!]\ (\gamma, \gamma', \overline{e'}/y)$.

  Assume $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') \in \|\Gamma, \alpha : \kappa, \Gamma'\|$.
  By inspection of the definition of $[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')$, we see
  it equals $[\![D']\!]\ (\gamma, \gamma')$.

- Case $D :: \Gamma, \alpha : \kappa, \Gamma' \vdash \lambda\beta : \kappa_1.\ C : \Pi\beta : \kappa_1.\ \kappa_2$:
  By inversion, $D_0 :: \Gamma, \alpha : \kappa, \Gamma' \vdash \kappa_1 : \mathsf{kind}$ and $D_1 :: \Gamma, \alpha : \kappa, \Gamma', \beta : \kappa_1 \vdash C : \kappa_2$.

  By induction, we know there is a $D_0' :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]\kappa_1 : \mathsf{kind}$
  such that for all $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', (\overline{B}, R)/\beta) \in \|\Gamma, \alpha : \kappa, \Gamma', \beta : \kappa_1\|$,
  we know $[\![D_0]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', (\overline{B}, R)/\beta) = [\![D_0']\!]\ (\gamma, \gamma', (\overline{B}, R)/\beta)$.

  By induction, we know there is a $D_1' :: \Gamma, [A/\alpha]\Gamma', \beta : [A/\alpha]\kappa_1 \vdash [A/\alpha]C : [A/\alpha]\kappa_2$
  such that for all $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', (\overline{B}, R)/\beta) \in \|\Gamma, \alpha : \kappa, \Gamma', \beta : \kappa_1\|$,
  we know $[\![D_1]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', (\overline{B}, R)/\beta) = [\![D_1']\!]\ (\gamma, \gamma', (\overline{B}, R)/\beta)$.

  Assume $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') \in \|\Gamma, \alpha : \kappa, \Gamma'\|$.
  By inspection of the definition of $[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')$, we see
  it equals $[\![D']\!]\ (\gamma, \gamma')$.

- Case $D :: \Gamma, \alpha : \kappa, \Gamma' \vdash C\ B : [B/\beta]\kappa_2$:
  By inversion, $D_0 :: \Gamma, \alpha : \kappa, \Gamma' \vdash C : \Pi\beta : \kappa_1.\ \kappa_2$.
  By inversion, $D_1 :: \Gamma, \alpha : \kappa, \Gamma' \vdash B : \kappa_1$.

  By induction, we know there is a $D_0' :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]C : \Pi\beta : [A/\alpha]\kappa_1.\ \kappa_2$
  s.t. for all $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') \in \|\Gamma, \alpha : \kappa, \Gamma'\|$, $[\![D_0]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D_0']\!]\ (\gamma, \gamma')$.

  By induction, we know there is a $D_1' :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]B : [A/\alpha]\kappa_1$
  s.t. for all $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') \in \|\Gamma, \alpha : \kappa, \Gamma'\|$, $[\![D_0]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') =$

$[\![D'_0]\!]\ (\gamma, \gamma')$.

Via application rule on D'_0 and D'_1, get $D' :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]C\ [A/\alpha]B : [[A/\alpha]B/\beta]\kappa_2$.
By properties of equality, this is $D' :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha](C\ B) : [A/\alpha]([B/\beta]\kappa_2)$.

Assume $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') \in \|\Gamma, \alpha : \kappa, \Gamma'\|$.
Note $(\gamma, \gamma') \in \|\Gamma, [A/\alpha]\Gamma'\|$.
We know:

$[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D_0]\!]\ \ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')\ \ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')(e)(B)\ ([\![D_1]\!]\ \ (\gamma,$

By induction hypotheses, we know

$[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D'_0]\!]\ \ (\gamma, \gamma')\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')(B)\ ([\![D'_1]\!]\ \ (\gamma, \gamma'))$

By definition of substitution, we know $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')(B) = (\gamma, \gamma')([A/\alpha]B)$.
Hence

$$[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D'_0]\!]\ \ (\gamma, \gamma')\ (\gamma, \gamma')(B)\ ([\![D'_1]\!]\ \ (\gamma, \gamma'))$$

Hence $[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.

- Case $D :: \Gamma, \alpha : \kappa, \Gamma' \vdash C\ e' : [e'/y]\kappa_2$:
  By inversion, $D_0 :: \Gamma, \alpha : \kappa, \Gamma' \vdash C : \Pi y : Y.\ \kappa_2$.
  By inversion, $D_1 :: \Gamma, \alpha : \kappa, \Gamma' \vdash e' : Y$.
  By inversion, $D_2 :: \Gamma, \alpha : \kappa, \Gamma' \vdash Y : *$.

  By induction, we know there is a $D'_0 :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]C : \Pi y : [A/\alpha]Y.\ \kappa_2$
  s.t. for all $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') \in \|\Gamma, \alpha : \kappa, \Gamma'\|$, $[\![D_0]\!]\ \ (\gamma, (\gamma(A), [\![T]\!]\ \ \gamma)/\alpha, \gamma') =$
  $[\![D'_0]\!]\ (\gamma, \gamma')$.

  By syntactic substitution, we have $D'_1 :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]e' : [A/\alpha]Y$.
  By syntactic substitution, we have $D'_2 :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]Y : *$.

  Via application rule on D'_0, D'_1 and D'_2, get $D' :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]C\ [A/\alpha]e' : [[A/\alpha]e'/y]\kappa_2$.
  By properties of equality, this is $D' :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha](C\ e') : [A/\alpha]([e'/y]\kappa_2)$.

  Assume $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') \in \|\Gamma, \alpha : \kappa, \Gamma'\|$.
  Note $(\gamma, \gamma') \in \|\Gamma, [A/\alpha]\Gamma'\|$.
  We know:

  $[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D_0]\!]\ \ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')(e')$

  By induction hypothesis, we know

  $$[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D'_0]\!]\ \ (\gamma, \gamma')\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')(e')$$

  By properties of substitution, $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')(e) = (\gamma, \gamma')([A/\alpha]e')$.
  Hence

  $$[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D'_0]\!]\ \ (\gamma, \gamma')\ \ (\gamma, \gamma')([A/\alpha]e')$$

  Hence $[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.

- Case $D :: \Gamma, \alpha : \kappa, \Gamma' \vdash \Pi y : Y.\ Z : *$:
  By inversion, $D_0 :: \Gamma, \alpha : \kappa, \Gamma' \vdash Y : *$ and $D_1 :: \Gamma, \alpha : \kappa, \Gamma', y : Y \vdash Z : *$.

  By induction, there is $D_0' :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]Y : *$
  s.t. for all $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') \in \|\Gamma, \alpha : \kappa, \Gamma'\|$, $[\![D_0]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D_0']\!]\ (\gamma, \gamma')$.
  By induction, there is $D_1' :: \Gamma, [A/\alpha]\Gamma', y : [A/\alpha]Y \vdash [A/\alpha]Z : *$
  s.t. for all $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', \overline{e'}/y) \in \|\Gamma, \alpha : \kappa, \Gamma', y : Y\|$,
  $[\![D_1]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', \overline{e'}/y) = [\![D_1']\!]\ (\gamma, \gamma', \overline{e'}/y)$.

  By pi-rule on D_0' and D_1', we have $D' :: \Gamma, [A/\alpha] \vdash \Pi y : [A/\alpha]Y.\ [A/\alpha]Z : *$.
  By properties of substitution, we have $D' :: \Gamma, [A/\alpha] \vdash [A/\alpha](\Pi y : Y.\ Z) : *$.

  Assume $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') \in \|\Gamma, \alpha : \kappa, \Gamma'\|$.
  We want to show $[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.

  Assume $(f, f') \in [\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')$.
  Then $f \downarrow$ and $f' \downarrow$ and
  for all $(t, t') \in [\![D_0]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')$, we know $(f\ t, f'\ t') \in [\![D_1]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', (t, t')/y)$
  Assume $(t, t') \in [\![D_0']\!]\ (\gamma, \gamma')$.
  Then we know by induction that $(t, t') \in [\![D_0]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')$.
  Hence we know that $(f\ t, f'\ t') \in [\![D_1]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', (t, t')/y)$.
  By induction hypothesis, we know $(f\ t, f'\ t') \in [\![D_1']\!]\ (\gamma, \gamma', (t, t')/y)$.
  Hence $(f, f') \in [\![D']\!]\ (\gamma, \gamma')$.

  Assume $(f, f') \in [\![D']\!]\ (\gamma, \gamma')$.
  Then $f \downarrow$ and $f' \downarrow$ and
  for all $(t, t') \in [\![D_0']\!]\ (\gamma, \gamma')$, we know $(f\ t, f'\ t') \in [\![D_1']\!]\ (\gamma, \gamma', (t, t')/y)$.
  Assume $(t, t') \in [\![D_0]\!]\ (\gamma, \gamma')$.
  Then we know by induction that $(t, t') \in [\![D_0']\!]\ (\gamma, \gamma')$.
  Hence we know that $(f\ t, f'\ t') \in [\![D_1']\!]\ (\gamma, \gamma', (t, t')/y)$.
  By induction hypothesis, we know $(f\ t, f'\ t') \in [\![D_1]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', (t, t')/y)$.
  Hence $(f, f') \in [\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')$.

  Hence $(f, f') \in [\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')$ iff $(f, f') \in [\![D']\!]\ (\gamma, \gamma')$.
  Hence $[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.
- Case $D :: \Gamma, \alpha : \kappa, \Gamma' \vdash \Pi \beta : \kappa_1.\ Z : *$:
  By inversion, $D_0 :: \Gamma, \alpha : \kappa, \Gamma' \vdash \kappa_1 : \mathsf{kind}$ and $D_1 :: \Gamma, \alpha : \kappa, \Gamma', \beta : \kappa_1 \vdash Z : *$.

  By induction, there is $D_0' :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]\kappa_1 : *$
  s.t. for all $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') \in \|\Gamma, \alpha : \kappa, \Gamma'\|$, $[\![D_0]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D_0']\!]\ (\gamma, \gamma')$.
  By induction, there is $D_1' :: \Gamma, \alpha : \kappa, \Gamma', \beta : \kappa_1 \vdash Z : *$
  s.t. for all $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', (\overline{B}, R)/\beta) \in \|\Gamma, \alpha : \kappa, \Gamma', y : Y\|$,

$[\![D_1]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', (\overline{B}, R)/\beta) = [\![D_1']\!]\ (\gamma, \gamma', (\overline{B}, R)/\beta)$.

By all-rule on D_0' and D_1', we have $D' :: \Gamma, [A/\alpha] \vdash \Pi y : [A/\alpha]Y.\ [A/\alpha]Z : *$.
By properties of substitution, we have $D' :: \Gamma, [A/\alpha] \vdash [A/\alpha](\Pi y : Y.\ Z) : *$.

Assume $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') \in \|\Gamma, \alpha : \kappa, \Gamma'\|$.
We want to show $[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.

Assume $(f, f') \in [\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')$.
Then $f \downarrow$ and $f' \downarrow$ and
for all $(B, B') \in \text{Type}^2, R \in [\![D_0]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')$,
we know $(f\ A, f'\ A') \in [\![D_1]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', (\overline{B}, R)/\beta)$.
Assume $(B, B') \in \text{Type}^2$ and $R \in [\![D_0']\!]\ (\gamma, \gamma')$.
Then we know by induction that $R \in [\![D_0]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')$.
Hence we know that $(f\ B, f'\ B') \in [\![D_1]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', (\overline{B}, R)/\beta)$.
By induction hypothesis, we know $(f\ B, f'\ B') \in [\![D_1']\!]\ (\gamma, \gamma', (\overline{B}, R)/\beta)$.
Hence $(f, f') \in [\![D']\!]\ (\gamma, \gamma')$.

Assume $(f, f') \in [\![D']\!]\ (\gamma, \gamma')$.
Then $f \downarrow$ and $f' \downarrow$ and
for all $(t, t') \in [\![D_0']\!]\ (\gamma, \gamma')$, we know $(f\ t, f'\ t') \in [\![D_1']\!]\ (\gamma, \gamma', (t, t')/y)$.
Assume $\overline{B} \in \text{Type}^2$ and $R \in [\![D_0]\!]\ (\gamma, \gamma')$.
Then we know by induction that $R \in [\![D_0']\!]\ (\gamma, \gamma')$.
Hence we know that $(f\ B, f'\ B') \in [\![D_1']\!]\ (\gamma, \gamma', (\overline{B}, R)/\beta)$.
By induction hypothesis, we know $(f\ B, f'\ B') \in [\![D_1]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma', (\overline{B}, R)/\beta)$.
Hence $(f, f') \in [\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')$.

Hence $(f, f') \in [\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')$ iff $(f, f') \in [\![D']\!]\ (\gamma, \gamma')$.
Hence $[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.

- Case $D :: \Gamma, \alpha : \kappa, \Gamma' \vdash e_1 =_Y e_2 : *$:
  By inversion, we have $D_0 :: \Gamma, \alpha : \kappa, \Gamma' \vdash Y : *$
  and $D_1 :: \Gamma, \alpha : \kappa, \Gamma' \vdash e_1 : Y$ and $D_2 :: \Gamma, \alpha : \kappa, \Gamma' \vdash e_2 : Y$.
  By induction, there is $D_0' :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]Y : *$
  s.t. for all $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') \in \|\Gamma, \alpha : \kappa, \Gamma'\|$, $[\![D_0]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D_0']\!]\ (\gamma, \gamma')$.
  By syntactic substitution, we have $D_1' :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]e_1 : [A/\alpha]Y$.
  By syntactic substitution, we have $D_2' :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]e_2 : [A/\alpha]Y$.
  By rule, we have $D' :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]e_1 =_{[A/\alpha]Y} [A/\alpha]e_2 : *$.

  Assume $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') \in \|\Gamma, \alpha : \kappa, \Gamma'\|$.
  We want to show $[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.

  Assume $(p, p') \in [\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')$.
  Then $p \mapsto^* \mathsf{refl}$ and $p' \mapsto^* \mathsf{refl}$ and $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')(e_1, e_2) \in [\![D_0]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')$.

By properties of substitution, $(\gamma, \gamma')([A/\alpha]e_1, [A/\alpha]e_2) \in [\![D_0]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')$.
By induction, $(\gamma, \gamma')([A/\alpha]e_1, [A/\alpha]e_2) \in [\![D'_0]\!]\ (\gamma, \gamma')$.
Hence $(p, p') \in [\![D']\!]\ (\gamma, \gamma')$.
Assume $(p, p') \in [\![D']\!]\ (\gamma, \gamma')$.
Then $p \mapsto^* \mathsf{refl}$ and $p' \mapsto^* \mathsf{refl}$ and $(\gamma, \gamma')([A/\alpha]e_1, [A/\alpha]e_2) \in [\![D'_0]\!]\ (\gamma, \gamma')$.
By properties of substitution, $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')(e_1, e_2) \in [\![D'_0]\!]\ (\gamma, \gamma')$.
By induction, $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')(e_1, e_2) \in [\![D_0]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')$.
Hence $(p, p') \in [\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')$.

Hence $(p, p') \in [\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')$ iff $(p, p') \in [\![D']\!]\ (\gamma, \gamma')$.
Hence $[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.

- Case $D :: \Gamma, \alpha : \kappa, \Gamma' \vdash B : \kappa_1$.
  By inversion, $D_0 :: \Gamma, \alpha : \kappa, \Gamma' \vdash B : \kappa_0$ and $D_1 :: \Gamma, \alpha : \kappa, \Gamma' \vdash \kappa_0 \equiv \kappa_1 : \mathsf{kind}$.

  By induction, there is $D'_0 :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]B : \kappa_0$
  s.t. for all $(\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') \in \|\Gamma, \alpha : \kappa, \Gamma'\|$, $[\![D_0]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D'_0]\!]\ (\gamma, \gamma')$.
  By syntactic substitution, $D'_1 :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]\kappa \equiv [A/\alpha]\kappa_0 : \mathsf{kind}$.
  By rule, we have $D' :: \Gamma, [A/\alpha]\Gamma' \vdash [A/\alpha]\kappa_1 : \mathsf{kind}$.

  We want to show $[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.
  By definition, $[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D_0]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma')$.
  By induction, $[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D'_0]\!]\ (\gamma, \gamma')$.
  By definition, $[\![D]\!]\ (\gamma, (\gamma(A), [\![T]\!]\ \gamma)/\alpha, \gamma') = [\![D']\!]\ (\gamma, \gamma')$.

□

Theorem 8 (Fundamental Property)**.** *We have that:*

1. *If* $D :: \Gamma \vdash \kappa : \mathsf{kind}$*, then for all* $\gamma, \gamma' \in [\![D_0 :: \Gamma\ \mathsf{ok}]\!]$ *such that* $\gamma \sim \gamma'$*,* $[\![D]\!]\ \gamma = [\![D]\!]\ \gamma'$*.*
2. *If* $D :: \Gamma \vdash A : \kappa$*, then for all* $\gamma, \gamma' \in [\![D_0 :: \Gamma\ \mathsf{ok}]\!]$ *such that* $\gamma \sim \gamma'$*,* $[\![D]\!]\ \gamma = [\![D]\!]\ \gamma'$*.*
3. *If* $D :: \Gamma \vdash e : X$ *then for all* $D_1 :: \Gamma \vdash X : *$ *and* $\gamma, \gamma' \in [\![D_0 :: \Gamma\ \mathsf{ok}]\!]$ *such that* $\gamma \sim \gamma'$*,* $\gamma(e) \sim_{[\![D_1]\!]\ \gamma} \gamma'(e)$*.*
4. *If* $D :: \Gamma \vdash A : \kappa$*, then for all* $\gamma \in [\![D_0 :: \Gamma\ \mathsf{ok}]\!]$*,* $[\![D]\!]\ \gamma \in [\![D_1 :: \Gamma \vdash \kappa : \mathsf{kind}]\!]\ \gamma$*.*
5. *If* $D :: \Gamma \vdash \kappa \equiv \kappa' : \mathsf{kind}$*, then for all* $\gamma \in [\![D_0 :: \Gamma\ \mathsf{ok}]\!]$*,* $D_1 :: \Gamma \vdash \kappa : \mathsf{kind}$ *and* $D_2 :: \Gamma \vdash \kappa : \mathsf{kind}$*,* $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$*.*
6. *If* $D :: \Gamma \vdash A \equiv A' : \kappa$*, then for all* $\gamma \in [\![D_0 :: \Gamma\ \mathsf{ok}]\!]$*,* $D_1 :: \Gamma \vdash A : \kappa$ *and* $D_2 :: \Gamma \vdash A' : \kappa$*,* $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$*.*
7. *If* $D :: \Gamma \vdash e_1 \equiv e_2 : X$*, then for all* $\gamma \in [\![D_0 :: \Gamma\ \mathsf{ok}]\!]$*,* $D_1 :: \Gamma \vdash X : *$*,* $\gamma(e_1) \sim_{[\![D_1]\!]\ \gamma} \gamma(e_2)$*.*

Proof. 1. Assume $D :: \Gamma \vdash \kappa : \mathsf{kind}$. We proceed by induction on D.

- Case $D :: \Gamma \vdash * : \mathsf{kind}$.
  Assume $\gamma, \gamma' \in [\![D_0 :: \Gamma\ \mathsf{ok}]\!]$ such that $\gamma \sim \gamma'$.
  By definition, $[\![D :: \Gamma \vdash * : \mathsf{kind}]\!]\ \gamma = [\![D :: \Gamma \vdash * : \mathsf{kind}]\!]\ \gamma' = \textsc{Cand}$.

- Case $D :: \Gamma \vdash \Pi\alpha : \kappa'.\ \kappa'' : \mathsf{kind}$.
  By inversion, we have $D' :: \Gamma \vdash \kappa' : \mathsf{kind}$ and $D'' :: \Gamma, \alpha : \kappa' \vdash \kappa' : \mathsf{kind}$.
  (a) By induction, for all $\gamma, \gamma' \in [\![D_0 :: \Gamma\ \mathsf{ok}]\!]$ such that $\gamma \sim \gamma'$, $[\![D']\!]\ \gamma = [\![D']\!]\ \gamma'$.
  (b) By induction, for all $\gamma, \gamma' \in [\![D_0 :: \Gamma, \alpha : \kappa'\ \mathsf{ok}]\!]$ such that $\gamma \sim \gamma'$, $[\![D'']\!]\ \gamma = [\![D'']\!]\ \gamma'$.
  Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $\gamma, \gamma' \in [\![D_0]\!]$ such that $\gamma \sim \gamma'$.
  We want to show $[\![D]\!]\ \gamma = [\![D]\!]\ \gamma'$.

  $\Longrightarrow$: Assume $T \in [\![D]\!]\ \gamma$. We will show $T \in [\![D]\!]\ \gamma'$.
  By assumption, we know that:
  1. $\forall \overline{A}, \overline{B}, R \in \|\kappa'\|, T(\overline{A}, R) = T(\overline{B}, R)$.
  2. $\forall \overline{A}, R \in [\![D' :: \Gamma \vdash \kappa' : \mathsf{kind}]\!]\ \gamma,\ T(\overline{A}, R) \in [\![D'' :: \Gamma \vdash \kappa'' : \mathsf{kind}]\!]\ (\gamma, (\overline{A}, R)/\alpha)$.
  3. $\forall \overline{A}, R \notin [\![D' :: \Gamma \vdash \kappa' : \mathsf{kind}]\!]\ \gamma,\ T(\overline{A}, R) = !_{\kappa'}$.
  We want to show these three properties with γ' for γ.

  By hypothesis (a), we know $[\![D']\!]\ \gamma = [\![D']\!]\ \gamma'$, so 1. and 3. follow immediately.

  To show 2, assume $\overline{A}, R \in [\![D' :: \Gamma \vdash \kappa' : \mathsf{kind}]\!]\ \gamma'$.
  Since we have D_0 and D', we have by rule a $D'_0 :: \Gamma, \alpha : \kappa'\ \mathsf{ok}$.
  By hypothesis (a), we know that $R \in [\![D' :: \Gamma \vdash \kappa' : \mathsf{kind}]\!]\ \gamma$.
  Hence by 2., $T(\overline{A}, R) \in [\![D'']\!]\ (\gamma, (\overline{A}, R)/\alpha) \in [\![D'']\!]\ (\gamma, (\overline{A}, R)/\alpha)$.
  By definition, $(\gamma, (\overline{A}, R)/\alpha) \in [\![D'_0]\!]$.
  By definition, $(\gamma, (\overline{A}, R)/\alpha) \sim (\gamma', (\overline{A}, R)/\alpha)$.
  By hypothesis (b), we know that $[\![D'']\!]\ (\gamma, (\overline{A}, R)/\alpha) = [\![D'']\!]\ (\gamma', (\overline{A}, R)/\alpha)$.
  So $\forall \overline{A}, R \in [\![D' :: \Gamma \vdash \kappa' : \mathsf{kind}]\!]\ \gamma',\ T(\overline{A}, R) \in [\![D'' :: \Gamma, \alpha : \kappa' \vdash \kappa'' : \mathsf{kind}]\!]\ (\gamma', (\overline{A}, R)/\alpha)$.

  $\Longleftarrow$: Assume $T \in [\![D]\!]\ \gamma'$. We will show $T \in [\![D]\!]\ \gamma$.
  By assumption, we know that:
  1. $\forall \overline{A}, \overline{B}, R \in \|\kappa'\|, T(\overline{A}, R) = T(\overline{B}, R)$.
  2. $\forall \overline{A}, R \in [\![D' :: \Gamma \vdash \kappa' : \mathsf{kind}]\!]\ \gamma',\ T(\overline{A}, R) \in [\![D'' :: \Gamma \vdash \kappa'' : \mathsf{kind}]\!]\ (\gamma', (\overline{A}, R)/\alpha)$.
  3. $\forall \overline{A}, R \notin [\![D' :: \Gamma \vdash \kappa' : \mathsf{kind}]\!]\ \gamma',\ T(\overline{A}, R) = !_{\kappa'}$.
  We want to show these three properties with γ for γ'.

  By hypothesis (a), we know $[\![D']\!]\ \gamma' = [\![D']\!]\ \gamma$, so 1. and 3. follow immediately.

  To show 2, assume $\overline{A}, R \in [\![D' :: \Gamma \vdash \kappa' : \mathsf{kind}]\!]\ \gamma$.
  Since we have D_0 and D', we have by rule a $D'_0 :: \Gamma, \alpha : \kappa'\ \mathsf{ok}$.
  By hypothesis (a), we know that $R \in [\![D' :: \Gamma \vdash \kappa' : \mathsf{kind}]\!]\ \gamma'$.
  Hence by 2., $T(\overline{A}, R) \in [\![D'']\!]\ (\gamma', (\overline{A}, R)/\alpha) \in [\![D'']\!]\ (\gamma', (\overline{A}, R)/\alpha)$.
  By definition, $(\gamma', (\overline{A}, R)/\alpha) \in [\![D'_0]\!]$.
  By definition, $(\gamma', (\overline{A}, R)/\alpha) \sim (\gamma, (\overline{A}, R)/\alpha)$.

By hypothesis (b), we know that $[\![D'']\!]\ (\gamma', (\overline{A}, R)/\alpha) = [\![D'']\!]\ (\gamma, (\overline{A}, R)/\alpha)$.
So $\forall \overline{A}, R \in [\![D' :: \Gamma \vdash \kappa' : \mathsf{kind}]\!]\ \gamma$, $T(\overline{A}, R) \in [\![D'' :: \Gamma, \alpha : \kappa' \vdash \kappa'' : \mathsf{kind}]\!]\ (\gamma, (\overline{A}, R)/\alpha)$.

- Case $D :: \Gamma \vdash \Pi x : X.\ \kappa'' : \mathsf{kind}$.
  By inversion, $D' :: \Gamma \vdash X : *$ and $D'' :: \Gamma, x : X \vdash \kappa'' : \mathsf{kind}$.
  (a) By mutual induction, for all $\gamma, \gamma' \in [\![\Gamma\ \mathsf{ok}]\!]$ such that $\gamma \sim \gamma'$, $[\![D']\!]\ \gamma = [\![D']\!]\ \gamma'$.
  (b) By induction, for all $\gamma, \gamma' \in [\![\Gamma, x : X\ \mathsf{ok}]\!]$ such that $\gamma \sim \gamma'$, $[\![D'']\!]\ \gamma = [\![D'']\!]\ \gamma'$.
  Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $\gamma, \gamma' \in [\![D_0]\!]$ such that $\gamma \sim \gamma'$.
  We want to show $[\![D]\!]\ \gamma = [\![D]\!]\ \gamma'$.

  $\Longrightarrow$: Assume $R \in [\![D]\!]\ \gamma$. We will show $R \in [\![D]\!]\ \gamma'$.
  Since $R \in [\![D]\!]\ \gamma$, we know:
  1. $\forall \bar{e}, \overline{e'} \in [\![D']\!]\ \gamma$, if $\bar{e} \sim \overline{e'}$ then $R\ \bar{e} = R\ \overline{e'}$.
  2. $\forall \bar{e} \in [\![D']\!]\ \gamma$, $R\ \bar{e} \in [\![D'']\!]\ (\gamma, \bar{e}/x)$.
  3. $\forall \bar{e} \notin [\![D']\!]\ \gamma$. $R\ \bar{e} = !_{\kappa'}$.
  We need to prove 1,2, and 3 with γ' for γ.
  By (a), 1. and 3. follow immediately. To show 2, assume $\bar{e} \in [\![D']\!]\ \gamma'$.
  Then, by (a) we know that $\bar{e} \in [\![D']\!]\ \gamma$.
  Hence $R\ \bar{e} \in [\![D'']\!]\ (\gamma, \bar{e}/x)$.
  Using D_0 and D', by rule we have $D'_0 :: \Gamma, x : X\ \mathsf{ok}$.
  So $(\gamma, \bar{e}/x) \in [\![D'_0]\!]$ and $(\gamma', \bar{e}/x) \in [\![D'_0]\!]$.
  Since $\gamma \sim \gamma$ and $\bar{e} \sim \bar{e}$, we know that $(\gamma, \bar{e}/x) \sim (\gamma', \bar{e}/x)$.
  Hence by (b), $[\![D'']\!]\ (\gamma, \bar{e}/x) = [\![D'']\!]\ (\gamma', \bar{e}/x)$.
  Hence $R\ \bar{e} \in [\![D'']\!]\ (\gamma', \bar{e}/x)$.
  So $\forall \bar{e} \in [\![D']\!]\ \gamma'$, $R\ \bar{e} \in [\![D'']\!]\ (\gamma', \bar{e}/x)$.

  $\Longleftarrow$: Assume $R \in [\![D]\!]\ \gamma'$. We will show $R \in [\![D]\!]\ \gamma$.
  Since $R \in [\![D]\!]\ \gamma'$, we know:
  1. $\forall \bar{e}, \overline{e'} \in [\![D']\!]\ \gamma'$, if $\bar{e} \sim \overline{e'}$ then $R\ \bar{e} = R\ \overline{e'}$.
  2. $\forall \bar{e} \in [\![D']\!]\ \gamma'$, $R\ \bar{e} \in [\![D'']\!]\ (\gamma', \bar{e}/x)$.
  3. $\forall \bar{e} \notin [\![D']\!]\ \gamma'$. $R\ \bar{e} = !_{\kappa'}$.
  We need to prove 1,2, and 3 with γ for γ'.
  By (a), 1. and 3. follow immediately. To show 2, assume $\bar{e} \in [\![D']\!]\ \gamma$.
  Then, by (a) we know that $\bar{e} \in [\![D']\!]\ \gamma'$.
  Hence $R\ \bar{e} \in [\![D'']\!]\ (\gamma', \bar{e}/x)$.
  Using D_0 and D', by rule we have $D'_0 :: \Gamma, x : X\ \mathsf{ok}$.
  So $(\gamma', \bar{e}/x) \in [\![D'_0]\!]$ and $(\gamma, \bar{e}/x) \in [\![D'_0]\!]$.
  Since $\gamma' \sim \gamma'$ and $\bar{e} \sim \bar{e}$, we know that $(\gamma', \bar{e}/x) \sim (\gamma, \bar{e}/x)$.
  Hence by (b), $[\![D'']\!]\ (\gamma', \bar{e}/x) = [\![D'']\!]\ (\gamma, \bar{e}/x)$.
  Hence $R\ \bar{e} \in [\![D'']\!]\ (\gamma, \bar{e}/x)$.
  So $\forall \bar{e} \in [\![D']\!]\ \gamma$, $R\ \bar{e} \in [\![D'']\!]\ (\gamma, \bar{e}/x)$.

2. Assume $D :: \Gamma \vdash A : \kappa$.

   - Case $D :: \Gamma \vdash \alpha : \kappa$:

Assume $D_0 :: \Gamma$ ok and $\gamma, \gamma' \in [\![D_0]\!]$ such that $\gamma \sim \gamma'$.
Then $\gamma(\alpha) = \gamma'(\alpha)$, so $[\![D]\!]\ \gamma = [\![D]\!]\ \gamma'$.

- Case $D :: \Gamma \vdash C\ A : [A/\alpha]\kappa''$.
  By inversion, $D' :: \Gamma \vdash C : \Pi\alpha : \kappa'.\ \kappa''$ and $D' :: \Gamma \vdash A : \kappa'$.
  By induction, for all $\gamma, \gamma' \in [\![D_0 :: \Gamma\ \mathsf{ok}]\!]$ such that $\gamma \sim \gamma'$, $[\![D']\!]\ \gamma = [\![D']\!]\ \gamma'$.
  By induction, for all $\gamma, \gamma' \in [\![D_0 :: \Gamma\ \mathsf{ok}]\!]$ such that $\gamma \sim \gamma'$, $[\![D'']\!]\ \gamma = [\![D'']\!]\ \gamma'$.
  Assume $\gamma, \gamma' \in [\![D_0 :: \Gamma\ \mathsf{ok}]\!]$ such that $\gamma \sim \gamma'$.
  Note that $[\![D]\!]\ \gamma = [\![D']\!]\ \gamma\ (\gamma(A), [\![D'']\!]\ \gamma)$.
  By validity, we know that $D'_1 :: \Gamma \vdash \Pi\alpha : \kappa'.\ \kappa'' : \mathsf{kind}$.
  Hence by mutual induction, we know $[\![D]\!]\ \gamma \in [\![D'_1 :: \Gamma \vdash \Pi\alpha : \kappa'.\ \kappa'' : \mathsf{kind}]\!]\ \gamma$.
  Hence $[\![D']\!]\ \gamma\ (\gamma(A), [\![D'']\!]\ \gamma) = [\![D']\!]\ \gamma\ (\gamma'(A), [\![D'']\!]\ \gamma)$.
  By induction, Hence, $[\![D]\!]\ \gamma = [\![D']\!]\ \gamma'\ (\gamma'(A), [\![D'']\!]\ \gamma')$.
  Hence $[\![D]\!]\ \gamma = [\![D]\!]\ \gamma'$.

- Case $D :: \Gamma \vdash \lambda\alpha : \kappa'.\ B : \Pi\alpha : \kappa'.\ \kappa''$.
  By inversion, $D' :: \Gamma \vdash \kappa' : \mathsf{kind}$ and $D'' :: \Gamma, \alpha : \kappa' \vdash B : \kappa''$.
  (a) By induction, for all $D_0 :: \Gamma$ ok and $\gamma, \gamma' \in [\![D_0]\!]$ s.t. $\gamma \sim \gamma'$, $[\![D']\!]\ \gamma = [\![D']\!]\ \gamma$.
  (b) By induction, for all $D_0 :: \Gamma, \alpha : \kappa'$ ok and $\gamma, \gamma' \in [\![D_0]\!]$ s.t. $\gamma \sim \gamma'$, $[\![D'']\!]\ \gamma = [\![D'']\!]\ \gamma$.
  Assume $D_0 :: \Gamma$ ok and $\gamma, \gamma' \in [\![D_0]\!]$ s.t. $\gamma \sim \gamma'$.
  Note that from D' and D_0, we have $D'_0 :: \Gamma, \alpha : \kappa'$ ok.
  Assume we have $(\overline{A}, R) \in \mathrm{Type}^2 \times \|\kappa'\|$.
  Now we will compare $[\![D]\!]\ \gamma\ (\overline{A}, R)$ with $[\![D]\!]\ \gamma'\ (\overline{A}, R)$.

  \- Suppose $R \in [\![D']\!]\ \gamma$:
  Then $[\![D]\!]\ \gamma\ (\bar{a}, R) = [\![D'']\!]\ (\gamma, (\overline{A}, R)/\alpha)$.
  Then by (a), we know that $R \in [\![D']\!]\ \gamma'$.
  Hence $[\![D]\!]\ \gamma'\ (\bar{a}, R) = [\![D'']\!]\ (\gamma', (\overline{A}, R)/\alpha)$.
  Hence $(\gamma, (\overline{A}, R)/\alpha) \in [\![D'_0]\!]$ and $(\gamma', (\overline{A}, R)/\alpha) \in [\![D'_0]\!]$.
  Note that $(\gamma, (\overline{A}, R)/\alpha) \sim (\gamma', (\overline{A}, R)/\alpha)$.
  Hence by (b), we know $[\![D'']\!]\ (\gamma, (\overline{A}, R)/\alpha) = [\![D'']\!]\ (\gamma', (\overline{A}, R)/\alpha)$.
  So $[\![D]\!]\ \gamma\ (\bar{a}, R) = [\![D]\!]\ \gamma'\ (\bar{a}, R)$.

  \- Suppose $R \notin [\![D']\!]\ \gamma$:
  Then $[\![D]\!]\ \gamma\ (\overline{A}, R) = !_{\kappa''}$.
  By (a), we know $R \notin [\![D']\!]\ \gamma$.
  Then $[\![D]\!]\ \gamma'\ (\overline{A}, R) = !_{\kappa''}$.
  So $[\![D]\!]\ \gamma\ (\overline{A}, R) = [\![D]\!]\ \gamma'\ (\overline{A}, R)$.

  Therefore $[\![D]\!]\ \gamma\ (\overline{A}, R) = [\![D]\!]\ \gamma'\ (\overline{A}, R)$.
  Therefore $[\![D]\!]\ \gamma = [\![D]\!]\ \gamma'$.

- Case $D :: \Gamma \vdash \lambda x : X.\ B : \Pi x : X.\ \kappa''$.
  By inversion, $D' :: \Gamma \vdash X : *$ and $D'' :: \Gamma, x : X \vdash B : \kappa''$.
  (a) By induction, for all $D_0 :: \Gamma$ ok and $\gamma, \gamma' \in [\![D_0]\!]$ s.t. $\gamma \sim \gamma'$, $[\![D']\!]\ \gamma = [\![D']\!]\ \gamma$.
  (b) By induction, for all $D_0 :: \Gamma, x : X$ ok and $\gamma, \gamma' \in [\![D_0]\!]$ s.t. $\gamma \sim \gamma'$, $[\![D'']\!]\ \gamma = [\![D'']\!]\ \gamma$.
  Assume $D_0 :: \Gamma$ ok and $\gamma, \gamma' \in [\![D_0]\!]$ s.t. $\gamma \sim \gamma'$.

Note that from D' and D_0, we have $D_0' :: \Gamma, \alpha : \kappa'\ \mathsf{ok}$.
Assume we have $\bar{e} \in \mathrm{Exp}^2$.
Now we will compare $[\![D]\!]\ \gamma\ \bar{e}$ with $[\![D]\!]\ \gamma'\ \bar{e}$.

- Suppose $\bar{e} \in [\![D']\!]\ \gamma$:
Then we know $[\![D]\!]\ \gamma\ \bar{e} = [\![D'']\!]\ (\gamma, \bar{e}/x)$.
By (a), $\bar{e} \in [\![D']\!]\ \gamma'$.
Then we know $[\![D]\!]\ \gamma = [\![D'']\!]\ (\gamma', \bar{e}/x)$.
We know that $\bar{e} \sim \bar{e}$, and so $(\gamma, \bar{e}/x) \sim (\gamma', \bar{e}/x)$.
Hence by (b), $[\![D'']\!]\ (\gamma, \bar{e}/x) = [\![D'']\!]\ (\gamma', \bar{e}/x)$.
Hence $[\![D]\!]\ \gamma\ \bar{e} = [\![D]\!]\ \gamma'\ \bar{e}$.

- Suppose $\bar{e} \notin [\![D']\!]\ \gamma$:
Then $[\![D]\!]\ \gamma\ \bar{e} = !_{\kappa''}$.
By (a), $\bar{e} \notin [\![D']\!]\ \gamma'$.
Then $[\![D]\!]\ \gamma'\ \bar{e} = !_{\kappa''}$.
Hence $[\![D]\!]\ \gamma\ \bar{e} = [\![D]\!]\ \gamma'\ \bar{e}$.

So for all $\bar{e} \in \mathrm{Exp}^2$, $[\![D]\!]\ \gamma\ \bar{e} = [\![D]\!]\ \gamma'\ \bar{e}$.
So $[\![D]\!]\ \gamma = [\![D]\!]\ \gamma'$.

- Case $D :: \Gamma \vdash A\ e : [e/x]B$:
By inversion, $D' :: \Gamma \vdash e : X$ and $D'' :: \Gamma \vdash A : \Pi x : X.\ \kappa''$ and $D''' :: \Gamma \vdash X : *$.
(a) By mutual induction, for all $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash X : *$ and $\gamma, \gamma' \in [\![D_0]\!]$ such that $\gamma \sim \gamma'$, $\gamma(e) \sim_{[\![D_1]\!]\ \gamma} \gamma'(e)$.
(b) By induction, for all $D_0 :: \Gamma\ \mathsf{ok}$ and $\gamma, \gamma' \in [\![D_0]\!]$ such that $\gamma \sim \gamma'$, $[\![D'']\!]\ \gamma = [\![D'']\!]\ \gamma'$.
(c) By mutual induction, for all $D_0 :: \Gamma\ \mathsf{ok}$ and $D_2 :: \Gamma \vdash \Pi x : X.\ \kappa'' : \mathsf{kind}$ and $\gamma \in [\![D_0]\!]$, $[\![D'']\!]\ \gamma \in [\![D_2]\!]\ \gamma$.
Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $\gamma, \gamma' \in [\![D_0]\!]$ such that $\gamma \sim \gamma'$.
By validity, we know $D_1 :: \Gamma \vdash X : *$.
By validity, we know $D_2 :: \Gamma \vdash \Pi x : X.\ \kappa'' : \mathsf{kind}$.
(a') Hence $\gamma(e) \sim_{[\![D_1]\!]\ \gamma} \gamma'(e)$.
(b') Hence $[\![D'']\!]\ \gamma = [\![D'']\!]\ \gamma'$.
(c') Hence $[\![D'']\!]\ \gamma \in [\![D_2]\!]\ \gamma$.
By (c') and the definition of $[\![D_2]\!]\ \gamma$, we have a $D_1' :: \Gamma \vdash X : *$ such that for all $\bar{e}, \overline{e'}$ s.t. $\bar{e} \sim_{[\![D_1']\!]\ \gamma} \overline{e'}$, $[\![D']\!]\ \gamma\ \bar{e} = [\![D']\!]\ \gamma\ \overline{e'}$.
By coherence, we know that $[\![D_1]\!]\ \gamma = [\![D_1']\!]\ \gamma$.
Hence (a') implies $\gamma(e) \sim_{[\![D_1']\!]\ \gamma} \gamma'(e)$.
So $[\![D']\!]\ \gamma\ \gamma(e) = [\![D']\!]\ \gamma\ \gamma'(e)$.
By (b'), we know $[\![D']\!]\ \gamma\ \gamma(e) = [\![D']\!]\ \gamma'\ \gamma'(e)$.
Hence $[\![D]\!]\ \gamma = [\![D]\!]\ \gamma'$.

- Case $D :: \Gamma \vdash A : \kappa$:
By inversion, we know $D' :: \Gamma \vdash A : \kappa'$ and $D'' :: \Gamma \vdash \kappa \equiv \kappa' : \mathsf{kind}$.
(a) By induction, for all $D_0 :: \Gamma\ \mathsf{ok}$ and $\gamma, \gamma' \in [\![\Gamma\ \mathsf{ok}]\!]$ s.t. $\gamma \sim \gamma'$, $[\![D']\!]\ \gamma = [\![D']\!]\ \gamma'$.
Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $\gamma, \gamma' \in [\![\Gamma\ \mathsf{ok}]\!]$ s.t. $\gamma \sim \gamma'$.

By (a), we know $[\![D']\!]\ \gamma = [\![D']\!]\ \gamma'$.
By definition, $[\![D]\!] = [\![D']\!]$, so $[\![D]\!]\ \gamma = [\![D]\!]\ \gamma'$.

- Case $D :: \Gamma \vdash \Pi x : X.\, Y : *$.
  By inversion, we know $D' :: \Gamma \vdash X : *$ and $D'' :: \Gamma, x : X \vdash Y : *$.
  (a) By induction, for all $D_0 :: \Gamma\ \mathsf{ok}$ and $\gamma, \gamma' \in [\![D_0]\!]$ s.t. $\gamma \sim \gamma'$, $[\![D']\!]\ \gamma = [\![D']\!]\ \gamma'$.
  (b) By induction, for all $D_0' :: \Gamma, x : X\ \mathsf{ok}$ and $\gamma, \gamma' \in [\![D_0']\!]$ s.t. $\gamma \sim \gamma'$, $[\![D'']\!]\ \gamma = [\![D'']\!]\ \gamma'$.

  Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $\gamma, \gamma' \in [\![\Gamma\ \mathsf{ok}]\!]$ s.t. $\gamma \sim \gamma'$.
  Note that from D' and D_0 we have $D_0' :: \Gamma, x : X\ \mathsf{ok}$.
  We want to show that $[\![D]\!]\ \gamma = [\![D]\!]\ \gamma'$.

  $\Longrightarrow$: Assume $(e_1, e_1') \in [\![D]\!]\ \gamma$. We will show $(e_1, e_1') \in [\![D]\!]\ \gamma'$.
  From the hypothesis, we know:
  1. $e_1 \downarrow$ and $e_1' \downarrow$.
  2. For all $(e_2, e_2') \in [\![D']\!]\ \gamma$. $(e_1\ e_2, e_1'\ e_2') \in [\![D'']\!]\ (\gamma, (e_1, e_1')/x)$.
  We need to show 1 and 2 with γ' for γ.
  1'. is immediate. To show 2'., assume $(e_2, e_2') \in [\![D']\!]\ \gamma'$.
  By (a), we know that $[\![D']\!]\ \gamma = [\![D']\!]\ \gamma'$, so $(e_2, e_2') \in [\![D']\!]\ \gamma$.
  Hence by 2., we know $(e_1\ e_2, e_1'\ e_2') \in [\![D'']\!]\ (\gamma, (e_1, e_1')/x)$.
  Note that $(\gamma, (e_1, e_1')/x) \sim (\gamma', (e_1, e_1')/x)$.
  Hence by (b), $[\![D'']\!]\ (\gamma, (e_1, e_1')/x) = [\![D'']\!]\ (\gamma', (e_1, e_1')/x)$.
  Hence $(e_1\ e_2, e_1'\ e_2') \in [\![D'']\!]\ (\gamma', (e_1, e_1')/x)$.
  Hence for all $(e_2, e_2') \in [\![D']\!]\ \gamma'$. $(e_1\ e_2, e_1'\ e_2') \in [\![D'']\!]\ (\gamma', (e_1, e_1')/x)$.
  Hence $(e_1, e_1') \in [\![D]\!]\ \gamma'$.

  $\Longleftarrow$: Assume $(e_1, e_1') \in [\![D]\!]\ \gamma'$. We will show $(e_1, e_1') \in [\![D]\!]\ \gamma$.
  From the hypothesis, we know:
  1. $e_1 \downarrow$ and $e_1' \downarrow$.
  2. For all $(e_2, e_2') \in [\![D']\!]\ \gamma'$. $(e_1\ e_2, e_1'\ e_2') \in [\![D'']\!]\ (\gamma', (e_1, e_1')/x)$.
  We need to show 1 and 2 with γ for γ'.
  1'. is immediate. To show 2'., assume $(e_2, e_2') \in [\![D']\!]\ \gamma$.
  By (a), we know that $[\![D']\!]\ \gamma' = [\![D']\!]\ \gamma$, so $(e_2, e_2') \in [\![D']\!]\ \gamma'$.
  Hence by 2., we know $(e_1\ e_2, e_1'\ e_2') \in [\![D'']\!]\ (\gamma', (e_1, e_1')/x)$.
  Note that $(\gamma', (e_1, e_1')/x) \sim (\gamma, (e_1, e_1')/x)$.
  Hence by (b), $[\![D'']\!]\ (\gamma', (e_1, e_1')/x) = [\![D'']\!]\ (\gamma, (e_1, e_1')/x)$.
  Hence $(e_1\ e_2, e_1'\ e_2') \in [\![D'']\!]\ (\gamma, (e_1, e_1')/x)$.
  Hence for all $(e_2, e_2') \in [\![D']\!]\ \gamma$. $(e_1\ e_2, e_1'\ e_2') \in [\![D'']\!]\ (\gamma, (e_1, e_1')/x)$.
  Hence $(e_1, e_1') \in [\![D]\!]\ \gamma$.

- Case $D :: \Gamma \vdash \Pi\alpha : \kappa.\, Y : *$.
  By inversion, we have $D' :: \Gamma \vdash \kappa : \mathsf{kind}$ and $D'' :: \Gamma, \alpha : \kappa \vdash Y : *$.
  (a) By mutual induction, for all $D_0 :: \Gamma\ \mathsf{ok}$ and $\gamma, \gamma' \in [\![D_0]\!]$ s.t. $\gamma \sim \gamma'$, $[\![D']\!]\ \gamma = [\![D']\!]\ \gamma'$.
  (b) By induction, for all $D_0' :: \Gamma, \alpha : \kappa\ \mathsf{ok}$ and $\gamma, \gamma' \in [\![D_0']\!]$ s.t. $\gamma \sim \gamma'$, $[\![D'']\!]\ \gamma = [\![D'']\!]\ \gamma'$.

  Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $\gamma, \gamma' \in [\![\Gamma\ \mathsf{ok}]\!]$ s.t. $\gamma \sim \gamma'$.

Note that from D' and D_0 we have $D_0' :: \Gamma, \alpha : \kappa$ ok.
We want to show that $[\![D]\!]\ \gamma = [\![D]\!]\ \gamma'$.

$\Longrightarrow$: Assume $(e, e') \in [\![D]\!]\ \gamma$. We will show $(e, e') \in [\![D]\!]\ \gamma'$.
From the hypothesis, we know:
1. $e \downarrow$ and $e' \downarrow$.
2. $\forall A, A' \in \text{Type}, R \in [\![D']\!]\ \gamma, (e\ A, e\ A') \in [\![D']\!]\ (\gamma, ((A, A'), R)/\alpha)$. We want to show 1 and 2 with γ' for γ.
1'. is immediate. To show 2', assume $A, A' \in \text{Type}, R \in [\![D']\!]\ \gamma'$.
By (a), we know that $[\![D']\!]\ \gamma = [\![D']\!]\ \gamma'$.
Hence $R \in [\![D']\!]\ \gamma$, and by 2., we know $(e\ A, e\ A') \in [\![D']\!]\ (\gamma, ((A, A'), R)/\alpha)$.
Note that $(\gamma, ((A, A'), R)/\alpha) \sim_{D_0'} (\gamma', ((A, A'), R)/\alpha)$.
Hence by (b), $[\![D'']\!]\ (\gamma, ((A, A'), R)/\alpha) = [\![D'']\!]\ (\gamma', ((A, A'), R)/\alpha)$.
Hence $(e\ A, e\ A') \in [\![D']\!]\ (\gamma', ((A, A'), R)/\alpha)$.
Hence 2', $\forall A, A' \in \text{Type}, R \in [\![D']\!]\ \gamma', (e\ A, e\ A') \in [\![D']\!]\ (\gamma', ((A, A'), R)/\alpha)$.
So $(e, e') \in [\![D]\!]\ \gamma'$.

$\Longleftarrow$: Assume $(e, e') \in [\![D]\!]\ \gamma'$. We will show $(e, e') \in [\![D]\!]\ \gamma$.
From the hypothesis, we know:
1. $e \downarrow$ and $e' \downarrow$.
2. $\forall A, A' \in \text{Type}, R \in [\![D']\!]\ \gamma', (e\ A, e\ A') \in [\![D']\!]\ (\gamma', ((A, A'), R)/\alpha)$. We want to show 1 and 2 with γ for γ'.
1'. is immediate. To show 2', assume $A, A' \in \text{Type}, R \in [\![D']\!]\ \gamma$.
By (a), we know that $[\![D']\!]\ \gamma' = [\![D']\!]\ \gamma$.
Hence $R \in [\![D']\!]\ \gamma'$, and by 2., we know $(e\ A, e\ A') \in [\![D']\!]\ (\gamma', ((A, A'), R)/\alpha)$.
Note that $(\gamma', ((A, A'), R)/\alpha) \sim_{D_0'} (\gamma, ((A, A'), R)/\alpha)$.
Hence by (b), $[\![D'']\!]\ (\gamma', ((A, A'), R)/\alpha) = [\![D'']\!]\ (\gamma, ((A, A'), R)/\alpha)$.
Hence $(e\ A, e\ A') \in [\![D']\!]\ (\gamma, ((A, A'), R)/\alpha)$.
Hence 2', $\forall A, A' \in \text{Type}, R \in [\![D']\!]\ \gamma, (e\ A, e\ A') \in [\![D']\!]\ (\gamma, ((A, A'), R)/\alpha)$.
So $(e, e') \in [\![D]\!]\ \gamma$.

- Case $D :: \Gamma \vdash e_1 =_X e_2 : *$.
  By inversion, we get $D_1 :: \Gamma \vdash e_1 : X$, $D_2 :: \Gamma \vdash e_2 : X$, $D' :: \Gamma \vdash X : *$.
  (a) By mutual induction, for all $D_0 :: \Gamma$ ok, and $D' :: \Gamma \vdash X : *$, and $\gamma, \gamma' \in [\![D_0]\!]$ s.t. $\gamma \sim \gamma'$, $\gamma(e_1) \sim_{[\![D']\!]\ \gamma} \gamma'(e_1)$.
  (b) By mutual induction, for all $D_0 :: \Gamma$ ok, and $D' :: \Gamma \vdash X : *$, and $\gamma, \gamma' \in [\![D_0]\!]$ s.t. $\gamma \sim \gamma'$, $\gamma(e_2) \sim_{[\![D']\!]\ \gamma} \gamma'(e_2)$.
  (c) By induction, for all $D_0 :: \Gamma$ ok and $\gamma, \gamma' \in [\![D_0]\!]$ s.t. $\gamma \sim \gamma'$, $[\![D']\!]\ \gamma = [\![D']\!]\ \gamma'$.

  Assume $D_0 :: \Gamma$ ok and $\gamma, \gamma' \in [\![\Gamma\ \text{ok}]\!]$ s.t. $\gamma \sim \gamma'$.
  Note that from D' and D_0 we have $D_0' :: \Gamma, \alpha : \kappa$ ok.
  We want to show that $[\![D]\!]\ \gamma = [\![D]\!]\ \gamma'$.

  $\Longrightarrow$: Assume $(e, e') \in [\![D]\!]\ \gamma$. We will show $(e, e') \in [\![D]\!]\ \gamma'$.
  By hypothesis, we know that:

1. $e \mapsto^* \mathsf{refl}$ and $e' \mapsto^* \mathsf{refl}$.
2. $(\gamma_1(e_1), \gamma_2(e_2)) \in [\![D']\!]\ \gamma$.
We want to show 1 and 2 with γ for γ'.
1'. is immediate.
By (a), we know $(\gamma_1'(e_1), \gamma_2'(e_1)) \in [\![D']\!]\ \gamma$ and $(\gamma_1(e_1), \gamma_2'(e_1)) \in [\![D']\!]\ \gamma$.
From 2., we know $(\gamma_1(e_1), \gamma_2(e_2)) \in [\![D']\!]\ \gamma$, so $(\gamma_1'(e_1), \gamma_2(e_2)) \in [\![D']\!]\ \gamma$.
By (b), we know $(\gamma_1'(e_2), \gamma_2(e_2)) \in [\![D']\!]\ \gamma$ and $(\gamma_1'(e_2), \gamma_2'(e_2)) \in [\![D']\!]\ \gamma$.
Hence $(\gamma_1'(e_1), \gamma_2'(e_2)) \in [\![D']\!]\ \gamma$.
By (c), $(\gamma_1'(e_1), \gamma_2'(e_2)) \in [\![D']\!]\ \gamma'$.

$\Longleftarrow$: Assume $(e, e') \in [\![D]\!]\ \gamma'$. We will show $(e, e') \in [\![D]\!]\ \gamma$.
By hypothesis, we know that:
1. $e \mapsto^* \mathsf{refl}$ and $e' \mapsto^* \mathsf{refl}$.
2. $(\gamma_1'(e_1), \gamma_2'(e_2)) \in [\![D']\!]\ \gamma'$.
We want to show 1 and 2 with γ' for γ.
1'. is immediate.
By (a), we know $(\gamma_1(e_1), \gamma_2(e_1)) \in [\![D']\!]\ \gamma'$ and $(\gamma_1'(e_1), \gamma_2(e_1)) \in [\![D']\!]\ \gamma'$.
From 2., we know $(\gamma_1'(e_1), \gamma_2'(e_2)) \in [\![D']\!]\ \gamma'$, so $(\gamma_1(e_1), \gamma_2'(e_2)) \in [\![D']\!]\ \gamma'$.
By (b), we know $(\gamma_1(e_2), \gamma_2'(e_2)) \in [\![D']\!]\ \gamma'$ and $(\gamma_1(e_2), \gamma_2(e_2)) \in [\![D']\!]\ \gamma'$.
Hence $(\gamma_1(e_1), \gamma_2(e_2)) \in [\![D']\!]\ \gamma'$.
By (c), $(\gamma_1(e_1), \gamma_2(e_2)) \in [\![D']\!]\ \gamma$.

3. Assume $D :: \Gamma \vdash e : X$.

- Case $\Gamma \vdash x : X$.
  Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash X : *$ and $\gamma, \gamma' \in [\![\Gamma\ \mathsf{ok}]\!]$ s.t. $\gamma \sim_{D_0} \gamma'$.
  By the structure of $\gamma \sim_{D_0} \gamma'$, we have Γ_0 and γ_0, γ_0' such that:
  1. there is a Γ_1 such that $\Gamma = \Gamma_0, \Gamma_1$
  2. there are γ_1, γ_1' such that $\gamma = \gamma_0, \gamma_1$ and $\gamma' = \gamma_0', \gamma_1'$
  3. there is a $D_0' :: \Gamma_0\ \mathsf{ok}$.
  4. there is a $D_1' :: \Gamma_0 \vdash X : *$
  5. $\gamma_0, \gamma_0' \in [\![D_0']\!]$ and $\gamma_0 \sim_{D_0'} \gamma_0'$.
  Hence we know that $\gamma(x) \sim_{[\![D_1']\!]\ \gamma_0} \gamma'(x)$.
  By weakening, we have a $D_1'' :: \Gamma \vdash X : *$ such that $[\![D_1']\!]\ \gamma_0 = [\![D_1'']\!]\ [\![D_1'']\!]\ \gamma$.
  By coherence, $[\![D_1'']\!]\ \gamma = [\![D_1]\!]\ \gamma$.
  Hence $\gamma(x) \sim_{[\![D_1]\!]\ \gamma} \gamma'(x)$.
- Case $D :: \Gamma \vdash \lambda y : Y.\ e : \Pi y : Y.\ Z$.
  By inversion, $D' :: \Gamma \vdash Y : *$ and $D'' :: \Gamma, y : Y \vdash e : Z$.
  (a) by mutual induction, for all $D_0 :: \Gamma\ \mathsf{ok}$ and $\gamma, \gamma' \in [\![D_0]\!]$ such that $\gamma \sim \gamma'$, $[\![D']\!]\ \gamma = [\![D']\!]\ \gamma'$.
  (b) by induction, for all $D_0' :: \Gamma, y : Y\ \mathsf{ok}$ and $D_1 :: \Gamma, y : Y \vdash Z : *$ and $\gamma, \gamma' \in [\![D_0']\!]$ such that $\gamma \sim \gamma'$, $\gamma(e) \sim_{[\![D_1]\!]\ \gamma} \gamma'(e)$.
  Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash \Pi y : Y.\ Z : *$ and $\gamma, \gamma' \in [\![\Gamma\ \mathsf{ok}]\!]$ s.t. $\gamma \sim_{D_0} \gamma'$.
  By inversion on D_1 we have $D_2 :: \Gamma \vdash Y : *$ and $D_3 :: \Gamma, y : Y \vdash Z : *$.

By D' and D_0, we know $D'_0 :: \Gamma, y : Y$ ok.
We want to show $\gamma(\lambda y : Y.\ e) \sim_{[\![D_1]\!]\ \gamma} \gamma'(\lambda y : Y.\ e)$.
So we need to show that:
1. $(\lambda y.\ \gamma_1(e), \lambda y.\ \gamma_2(e)) \in [\![D_1]\!]\ \gamma$
2. $(\lambda y.\ \gamma'_1(e), \lambda y.\ \gamma'_2(e)) \in [\![D_1]\!]\ \gamma$
3. $(\lambda y.\ \gamma_1(e), \lambda y.\ \gamma'_2(e)) \in [\![D_1]\!]\ \gamma$
Assume we have $\bar{t} = (t_1, t_2) \in [\![D_2]\!]\ \gamma$.
By coherence and (a), $[\![D_2]\!]\ \gamma = [\![D']\!]\ \gamma = [\![D']\!]\ \gamma' = [\![D_2]\!]\ \gamma'$.
Hence $\bar{t} \in [\![D']\!]\ \gamma$.
Hence $(\gamma, \bar{t}/y) \sim_{D'_0} (\gamma', \bar{t}/y)$.
By (b), we know that $(\gamma, \bar{t}/y)(e) \sim_{[\![D_3]\!]\ (\gamma, \bar{t}/y)} (\gamma', \bar{t}/y)(e)$.
Hence:
1. $((\gamma_1, t_1/y)(e), (\gamma_2, t_2/y)(e)) \in [\![D_3]\!]\ (\gamma, \bar{t}/y)$
2. $((\gamma'_1, t_1/y)(e), (\gamma'_2, t_2/y)(e)) \in [\![D_3]\!]\ (\gamma, \bar{t}/y)$
3. $((\gamma_1, t_1/y)(e), (\gamma'_2, t_2/y)(e)) \in [\![D_3]\!]\ (\gamma, \bar{t}/y)$
Note that $(\lambda y.\ \gamma_i(e))\ t_i \mapsto (\gamma_i, t_i/y)(e)$ and $(\lambda y.\ \gamma'_i(e))\ t_i \mapsto (\gamma'_i, t_i/y)(e)$.
Hence:
1. $((\lambda y.\ \gamma_1(e))\ t_1, (\lambda y.\ \gamma_2(e))\ t_2) \in [\![D_3]\!]\ (\gamma, \bar{t}/y)$
2. $((\lambda y.\ \gamma'_1(e))\ t_1, (\lambda y.\ \gamma'_2(e))\ t_2) \in [\![D_3]\!]\ (\gamma, \bar{t}/y)$
3. $((\lambda y.\ \gamma_1(e))\ t_1, (\lambda y.\ \gamma'_2(e))\ t_2) \in [\![D_3]\!]\ (\gamma, \bar{t}/y)$

- Case $D :: \Gamma \vdash e\ t : [t/y]Z$.
  By inversion, $D' :: \Gamma \vdash e : \Pi y : Y.\ Z$ and $D'' :: \Gamma \vdash t : Y$.
  (a) by induction, for all $D_0 :: \Gamma$ ok and $D_1 :: \Gamma \vdash \Pi y : Y.\ Z : *$ and $\gamma, \gamma' \in [\![D_0]\!]$ such that $\gamma \sim \gamma'$, $\gamma(e) \sim_{[\![D_1]\!]\ \gamma} \gamma'(e)$.
  (b) by induction, for all $D_0 :: \Gamma$ ok and $D_2 :: \Gamma \vdash Y : *$ and $\gamma, \gamma' \in [\![D_0]\!]$ such that $\gamma \sim \gamma'$, $\gamma(t) \sim_{[\![D_2]\!]\ \gamma} \gamma'(t)$.
  Assume $D_0 :: \Gamma$ ok and $D_4 :: \Gamma \vdash [t/y]Z : *$ and $\gamma, \gamma' \in [\![\Gamma \text{ ok}]\!]$ s.t. $\gamma \sim_{D_0} \gamma'$.
  We want to show that $\gamma(e\ t) \sim_{[\![D_4]\!]\ \gamma} \gamma'(e\ t)$.
  By validity, we have $D_1 :: \Gamma \vdash \Pi y : Y.\ Z : *$.
  By inversion on D_1, we get $D_2 :: \Gamma \vdash Y : *$ and $D_3 :: \Gamma, y : Y \vdash Z : *$.
  By (a), we have $\gamma(e) \sim_{[\![D_1]\!]\ \gamma} \gamma'(e)$.
  By (b), we have $\gamma(t) \sim_{[\![D_2]\!]\ \gamma} \gamma'(t)$.
  By induction on D_3 it follows that $\gamma(e\ t) \sim_{[\![D_3]\!]\ (\gamma, \gamma(t)/y)} \gamma'(e\ t)$.
  By substitution of terms, we have $D'_4 :: \Gamma \vdash [t/y]Z : *$ such that $[\![D'_4]\!]\ \gamma = [\![D_3]\!]\ (\gamma, \gamma(t)/y)$.
  By coherence $[\![D'_4]\!]\ \gamma = [\![D_4]\!]\ \gamma$.
  Hence $\gamma(e\ t) \sim_{[\![D_4]\!]\ \gamma} \gamma'(e\ t)$.

- Case $D :: \Gamma \vdash \lambda\alpha : \kappa.\ e : \Pi\alpha : \kappa.\ Y$.
  By inversion, we get $D' :: \Gamma \vdash \kappa :$ kind and $D'' :: \Gamma, \alpha : \kappa \vdash e : Y$.
  (a) By mutual induction, for all $D_0 :: \Gamma$ ok and $\gamma, \gamma' \in [\![D_0]\!]$ such that $\gamma \sim \gamma'$, we have $[\![D']\!]\ \gamma = [\![D']\!]\ \gamma'$.
  (b) By induction, for all $D'_0 :: \Gamma, \alpha : \kappa$ ok and $D_3 :: \Gamma, \alpha : \kappa \vdash Y : *$, and $\gamma, \gamma' \in [\![D'_0]\!]$ such that $\gamma \sim \gamma'$, $\gamma(e) \sim_{[\![D_3]\!]\ \gamma} \gamma'(e)$.
  Assume $D_0 :: \Gamma$ ok and $D_1 :: \Gamma \vdash \Pi\alpha : \kappa.\ Y : *$ and $\gamma, \gamma' \in [\![\Gamma \text{ ok}]\!]$ s.t. $\gamma \sim_{D_0} \gamma'$.

By inversion on D_1, we get $D_2 :: \Gamma \vdash \kappa : \mathsf{kind}$ and $D_3 :: \Gamma, \alpha : \kappa \vdash Y : *$.
With D_0 and D_2, we get $D_0' :: \Gamma, \alpha : \kappa\ \mathsf{ok}$.
We want to show $\gamma(\lambda\alpha : \kappa.\ e) \sim_{[\![D_1]\!]\ \gamma} \gamma'(\lambda\alpha : \kappa.\ e)$.
That is, we need to show:
1. $(\lambda\alpha : \gamma_1(\kappa).\ \gamma_1(e), \lambda\alpha : \gamma_2(\kappa).\ \gamma_2(e)) \in [\![D_1]\!]\ \gamma$
2. $(\lambda\alpha : \gamma_1'(\kappa).\ \gamma_1'(e), \lambda\alpha : \gamma_2'(\kappa).\ \gamma_2'(e)) \in [\![D_1]\!]\ \gamma$
3. $(\lambda\alpha : \gamma_1(\kappa).\ \gamma_1(e), \lambda\alpha : \gamma_2'(\kappa).\ \gamma_2'(e)) \in [\![D_1]\!]\ \gamma$
4. $(\lambda\alpha : \gamma_1'(\kappa).\ \gamma_1'(e), \lambda\alpha : \gamma_2(\kappa).\ \gamma_2(e)) \in [\![D_1]\!]\ \gamma$
To show these, note that the termination condition holds trivially.
Assume that we have $A_1, A_2 \in \text{Type}, R \in [\![D_2]\!]\ \gamma$.
By coherence, we know that $[\![D_2]\!]\ \gamma = [\![D']\!]\ \gamma$, and so $R \in [\![D']\!]\ \gamma$.
Hence $(\gamma, ((A_1, A_2), R)/\alpha) \sim_{D_0'} (\gamma', ((A_1, A_2), R)/\alpha)$.
By (b), $(\gamma, ((A_1, A_2), R)/\alpha)(e) \sim_{[\![D_3]\!]\ (\gamma, ((A_1, A_2), R)/\alpha)} (\gamma', ((A_1, A_2), R)/\alpha)(e)$.
Note that:
1. $(\lambda\alpha : \gamma_i(\kappa).\ \gamma_i(e))\ A_i \mapsto (\gamma_i, A_i/\alpha)(e)$
2. $(\lambda\alpha : \gamma_i'(\kappa).\ \gamma_i'(e))\ A_i \mapsto (\gamma_i', A_i/\alpha)(e)$
Therefore:
1. $((\lambda\alpha : \gamma_1(\kappa).\ \gamma_1(e))\ A_1, (\lambda\alpha : \gamma_2(\kappa).\ \gamma_2(e))\ A_2) \in [\![D_3]\!]\ (\gamma, ((A_1, A_2), R)/\alpha)$
2. $((\lambda\alpha : \gamma_1'(\kappa).\ \gamma_1'(e))\ A_1, (\lambda\alpha : \gamma_2'(\kappa).\ \gamma_2'(e))\ A_2) \in [\![D_3]\!]\ (\gamma, ((A_1, A_2), R)/\alpha)$
3. $((\lambda\alpha : \gamma_1(\kappa).\ \gamma_1(e))\ A_1, (\lambda\alpha : \gamma_2'(\kappa).\ \gamma_2'(e))\ A_2) \in [\![D_3]\!]\ (\gamma, ((A_1, A_2), R)/\alpha)$
4. $((\lambda\alpha : \gamma_1'(\kappa).\ \gamma_1'(e))\ A_1, (\lambda\alpha : \gamma_2(\kappa).\ \gamma_2(e))\ A_2) \in [\![D_3]\!]\ (\gamma, ((A_1, A_2), R)/\alpha)$
Hence:
1. $(\lambda\alpha : \gamma_1(\kappa).\ \gamma_1(e), \lambda\alpha : \gamma_2(\kappa).\ \gamma_2(e)) \in [\![D_1]\!]\ \gamma$
2. $(\lambda\alpha : \gamma_1'(\kappa).\ \gamma_1'(e), \lambda\alpha : \gamma_2'(\kappa).\ \gamma_2'(e)) \in [\![D_1]\!]\ \gamma$
3. $(\lambda\alpha : \gamma_1(\kappa).\ \gamma_1(e), \lambda\alpha : \gamma_2'(\kappa).\ \gamma_2'(e)) \in [\![D_1]\!]\ \gamma$
4. $(\lambda\alpha : \gamma_1'(\kappa).\ \gamma_1'(e), \lambda\alpha : \gamma_2(\kappa).\ \gamma_2(e)) \in [\![D_1]\!]\ \gamma$

- Case $D :: \Gamma \vdash e\ B : [B/\beta]Y$.
  By inversion, $D' :: \Gamma \vdash e : \Pi\beta : \kappa.\ Y$ and $D'' :: \Gamma \vdash B : \kappa$.
  (a) By induction, for all $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash \Pi\alpha : \kappa.\ Y : *$, and $\gamma, \gamma' \in [\![D_0]\!]$ such that $\gamma \sim \gamma'$, $\gamma(e) \sim_{[\![D_1]\!]\ \gamma} \gamma'(e)$.
  (b) By mutual induction, for all $D_0 :: \Gamma\ \mathsf{ok}$ and $\gamma, \gamma' \in [\![D_0]\!]$ such that $\gamma \sim \gamma'$, we have $[\![D'']\!]\ \gamma = [\![D'']\!]\ \gamma'$.
  (c) By mutual induction, $D_0 :: \Gamma\ \mathsf{ok}$ and $D_2 :: \Gamma \vdash \kappa : \mathsf{kind}$, and $\gamma \in [\![D_0]\!]$, $[\![D'']\!]\ \gamma \in [\![D_2]\!]\ \gamma$.

  Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $D_4 :: \Gamma \vdash [B/\beta]Y : *$ and $\gamma, \gamma' \in [\![\Gamma\ \mathsf{ok}]\!]$ s.t. $\gamma \sim_{D_0} \gamma'$.
  By validity, we have $D_1 :: \Gamma \vdash \Pi\beta : \kappa.\ Y : *$.
  Hence by (a), $\gamma(e) \sim_{[\![D_1]\!]\ \gamma} \gamma'(e)$.
  By inversion on D_1, we have $D_2 :: \Gamma \vdash \kappa : \mathsf{kind}$ and $D_3 :: \Gamma, \alpha : \kappa \vdash Y : *$.
  Hence by (c), $[\![D'']\!]\ \gamma \in [\![D_2]\!]\ \gamma$.
  Furthermore, by (b) we know $[\![D'']\!]\ \gamma = [\![D'']\!]\ \gamma'$.
  Hence $(\gamma, (\gamma(B), [\![D'']\!]\ \gamma)/\beta) \sim_{D_0'} (\gamma, (\gamma'(B), [\![D'']\!]\ \gamma')/\beta)$.
  By definition, if $(e_1, e_2) \in [\![D_1]\!]\ \gamma$, then:
  1'. $e_1 \downarrow$ and $e_2 \downarrow$.

2'. For all $(A_1, A_2), R \in [\![D_2]\!]\ \gamma$, $(e_1\ A_1, e_2\ A_2) \in [\![D_3]\!]\ (\gamma, ((A_1, A_2), R)/\alpha)$.
Note that we know:
1. $(\gamma_1(e), \gamma_2(e)) \in [\![D_1]\!]\ \gamma$.
2. $(\gamma_1'(e), \gamma_2'(e)) \in [\![D_1]\!]\ \gamma$.
3. $(\gamma_1(e), \gamma_2'(e)) \in [\![D_1]\!]\ \gamma$.
4. $(\gamma_1'(e), \gamma_2(e)) \in [\![D_1]\!]\ \gamma$.
Hence using 2', we can conclude:
1. $(\gamma_1(e)\ \gamma_1(B), \gamma_2(e)\ \gamma_2(B)) \in [\![D_3]\!]\ (\gamma, (\gamma_1(B), \gamma_2(B), [\![D'']\!]\ \gamma)/\beta)$.
2. $(\gamma_1'(e)\ \gamma_1'(B), \gamma_2'(e)\ \gamma_2'(B)) \in [\![D_3]\!]\ (\gamma, (\gamma_1'(B), \gamma_2'(B), [\![D'']\!]\ \gamma)/\beta)$.
3. $(\gamma_1(e)\ \gamma_1(B), \gamma_2'(e)\ \gamma_2'(B)) \in [\![D_3]\!]\ (\gamma, (\gamma_1(B), \gamma_2'(B), [\![D'']\!]\ \gamma)/\beta)$.
4. $(\gamma_1'(e)\ \gamma_1'(B), \gamma_2(e)\ \gamma_2(B)) \in [\![D_3]\!]\ (\gamma, (\gamma_1'(B), \gamma_2(B), [\![D'']\!]\ \gamma)/\beta)$.
By induction on D_3, all the $[\![D_3]\!] \ldots$ in 1–4 here are equal.
By substitution, we know that $[\![D_3]\!]\ (\gamma, (\gamma(B), [\![D'']\!]\ \gamma)/\beta) = [\![D_4]\!]\ \gamma$.
Hence using properties of substitution, we can conclude:
1. $(\gamma_1(e\ B), \gamma_2(e\ B)) \in [\![D_4]\!]\ \gamma$.
2. $(\gamma_1'(e\ B), \gamma_2'(e\ B)) \in [\![D_4]\!]\ \gamma$.
3. $(\gamma_1(e\ B), \gamma_2'(e\ B)) \in [\![D_4]\!]\ \gamma$.
4. $(\gamma_1'(e\ B), \gamma_2(e\ B)) \in [\![D_4]\!]\ \gamma$.
Hence $\gamma(e\ B) \sim_{[\![D_4]\!]\ \gamma} \gamma'(e\ B)$.

- Case $D :: \Gamma \vdash \mathsf{refl} : e_1 =_X e_2$.
  By inversion, we have $D' :: \Gamma \vdash e_1 \equiv e_2 : X$.
  (a) by mutual induction, for all $D_0 :: \Gamma\ \mathsf{ok}$ and $D_3 :: \Gamma \vdash X : *$ and $\gamma \in [\![\Gamma\ \mathsf{ok}]\!]$, $(\gamma_1(e_1), \gamma_2(e_2)) \in [\![D_3]\!]\ \gamma$.

  Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $E :: \Gamma \vdash e_1 =_X e_2 : *$ and $\gamma, \gamma' \in [\![\Gamma\ \mathsf{ok}]\!]$ s.t. $\gamma \sim_{D_0} \gamma'$.
  By inversion on E, we get $D_3 :: \Gamma \vdash X : *$ and $D_1 :: \Gamma \vdash e_1 : X$ and $D_2 :: \Gamma \vdash e_2 : X$.
  Now, note that $\gamma(\mathsf{refl}) = \gamma'(\mathsf{refl}) = (\mathsf{refl}, \mathsf{refl})$.
  Hence, to show $\gamma(e) \sim_{[\![E]\!]\ \gamma} \gamma'(e)$, it suffices to show $(\mathsf{refl}, \mathsf{refl}) \in [\![E]\!]\ \gamma$.
  By definition, $(\mathsf{refl}, \mathsf{refl}) \in [\![E]\!]\ \gamma$ when $(\gamma_1(e_1), \gamma_2(e_2)) \in [\![D_3]\!]\ \gamma$.
  This follows from (a).
- Case $D :: \Gamma \vdash e : X$
  By inversion, $D' :: \Gamma \vdash e : Y$ and $D'' :: \Gamma \vdash X \equiv Y : *$.
  (a) By induction, for all $D_0 :: \Gamma\ \mathsf{ok}$ and $D_2 :: \Gamma \vdash Y : *$, and $\gamma, \gamma' \in [\![D_0]\!]$ such that $\gamma \sim \gamma'$, $\gamma(e) \sim_{[\![D_2]\!]\ \gamma} \gamma'(e)$.
  (b) By mutual induction, for all $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash X : *$ and $D_2 :: \Gamma \vdash Y : *$ and $\gamma \in [\![D_0]\!]$, $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$.
  Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash X : *$, and $\gamma, \gamma' \in [\![D_0]\!]$ such that $\gamma \sim \gamma'$.
  By validity, we know $D_2 :: \Gamma \vdash Y : *$.
  Hence we know by (b) that $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$.
  By (a), we know $\gamma(e) \sim_{[\![D_2]\!]\ \gamma} \gamma'(e)$.
  Hence $\gamma(e) \sim_{[\![D_1]\!]\ \gamma} \gamma'(e)$.

4. Assume $D :: \Gamma \vdash A : \kappa$.

- Case $D :: \Gamma \vdash \alpha : \kappa$.
  Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash \kappa : \mathsf{kind}$ and $\gamma \in [\![\Gamma\ \mathsf{ok}]\!]$.
  By the structure of $\gamma \in [\![D_0]\!]$, we have Γ_0 and γ_0 such that:
  1. there is a Γ_1 such that $\Gamma = \Gamma_0, \Gamma_1$
  2. there are γ_1 such that $\gamma = \gamma_0, \gamma_1$
  3. there is a $D'_0 :: \Gamma_0\ \mathsf{ok}$.
  4. there is a $D'_1 :: \Gamma_0 \vdash \kappa : \mathsf{kind}$
  5. $\gamma_0 \in [\![D'_0]\!]$. Hence we know that $\gamma(\alpha) \in [\![D'_1]\!]\ \gamma_0$.
  By weakening, we have a $D''_1 :: \Gamma \vdash \kappa : \mathsf{kind}$ such that $[\![D'_1]\!]\ \gamma_0 = [\![D''_1]\!]\ \gamma$.
  By coherence, $[\![D''_1]\!]\ \gamma = [\![D]\!]\ \gamma$.
  Hence $\gamma(\alpha) \in [\![D]\!]\ \gamma$.
  Hence $[\![D]\!]\ \gamma \in [\![D]\!]\ \gamma$.

- Case $D :: \Gamma \vdash A\ B : [B/\beta]\kappa''$.
  By inversion, we have $D' :: \Gamma \vdash A : \Pi\beta : \kappa'.\ \kappa''$ and $D'' :: \Gamma \vdash B : \kappa'$.
  (a) By induction, for all $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash \Pi\beta : \kappa'.\ \kappa'' : \mathsf{kind}$ and $\gamma \in [\![D_0]\!]$, $[\![D']\!]\ \gamma \in [\![D_1]\!]$.
  (b) By induction, for all $D_0 :: \Gamma\ \mathsf{ok}$ and $D_2 :: \Gamma \vdash \kappa' : \mathsf{kind}$ and $\gamma \in [\![D_0]\!]$, $[\![D'']\!]\ \gamma \in [\![D_2]\!]$.
  Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $D_3 :: \Gamma \vdash [B/\beta]\kappa'' : \mathsf{kind}$ and $\gamma \in [\![\Gamma\ \mathsf{ok}]\!]$.
  Then we know that $[\![D']\!]\ \gamma \in [\![D_1]\!]\ \gamma$ and $[\![D'']\!]\ \gamma \in [\![D_2]\!]\ \gamma$.
  By inversion on D_1, we have $D'_2 :: \Gamma \vdash \kappa' : \mathsf{kind}$ and $D'_3 :: \Gamma, \beta : \kappa' \vdash \kappa'' : \mathsf{kind}$.
  By definition, for all $\overline{C}, R \in [\![D'_2]\!]\ \gamma$, we know $[\![D']\!]\ \gamma\ (\overline{C}, R) \in [\![D'_3]\!]\ (\gamma, (\overline{C}, R)/\beta)$.
  By coherence, we know that $[\![D_2]\!]\ \gamma = [\![D'_2]\!]\ \gamma'$.
  Hence $[\![D']\!]\ \gamma\ (\gamma(B), [\![D'']\!]\ \gamma) \in [\![D'_3]\!]\ (\gamma, (\gamma(B), [\![D'']\!]\ \gamma)/\beta)$.
  By substitution, there is a $D'_1 :: \Gamma \vdash [B/\beta]\kappa'' : \mathsf{kind}$ such that $[\![D'_1]\!]\ \gamma = [\![D'_3]\!]\ (\gamma, (\gamma(B), [\![D'']\!]\ \gamma)/\beta)$.
  By coherence, $[\![D_1]\!]\ \gamma = [\![D'_1]\!]\ \gamma$.
  Hence $[\![D']\!]\ \gamma\ (\gamma(B), [\![D'']\!]\ \gamma) \in [\![D_1]\!]\ \gamma$.
  Hence $[\![D]\!]\ \gamma \in [\![D_1]\!]\ \gamma$.

- Case $D :: \Gamma \vdash \lambda\alpha : \kappa'.\ B : \Pi\alpha : \kappa'.\ \kappa''$.
  By inversion, we get $D' :: \Gamma, \alpha : \kappa' \vdash B : \kappa''$ and $D'' :: \Gamma \vdash \kappa' : \mathsf{kind}$.
  By induction, for all $D'_0 :: \Gamma, \alpha : \kappa'\ \mathsf{ok}$ and $D_2 :: \Gamma, \alpha : \kappa' \vdash \kappa'' : \mathsf{kind}$ and $\gamma' \in [\![D'_0]\!]$, we know $[\![D']\!]\ \gamma' \in [\![D_1]\!]\ \gamma'$.
  Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $D_3 :: \Gamma \vdash \Pi\alpha : \kappa'.\ \kappa'' : \mathsf{kind}$ and $\gamma \in [\![\Gamma\ \mathsf{ok}]\!]$.
  By inversion on D_3, we get $D_1 :: \Gamma \vdash \kappa' : \mathsf{kind}$ and $D_2 :: \Gamma, \alpha : \kappa' \vdash \kappa'' : \mathsf{kind}$.
  Now, we want to show $[\![D]\!]\ \gamma \in [\![D_3]\!]\ \gamma$.
  To show this, assume we have $\overline{A}, R \in [\![D_1]\!]\ \gamma$.
  Note that from D_0 and D_1, we have $D'_0 :: \Gamma, \alpha : \kappa'\ \mathsf{ok}$.
  Furthermore $(\gamma, (\overline{A}, R)/\alpha) \in [\![D'_0]\!]$.
  Hence $[\![D']\!]\ (\gamma, (\overline{A}, R)/\alpha) \in [\![D_2]\!]\ (\gamma, (\overline{A}, R)/\alpha)$.
  Hence $[\![D]\!]\ \gamma \in [\![D_3]\!]\ \gamma$.

- Case $D :: \Gamma \vdash A\ e : [e/x]\kappa'$.
  By inversion, we have $D' :: \Gamma \vdash A : \Pi x : X.\ \kappa'$ and $D'' :: \Gamma \vdash e : X$.
  (a) By induction, we know for all $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash \Pi x : X.\ \kappa' : \mathsf{kind}$ and $\gamma \in [\![D_0]\!]$,

$[\![D']\!]\ \gamma \in [\![D_1]\!]$.
(b) By mutual induction, we know for all $D_0 :: \Gamma\ \mathsf{ok}$ and $D_2 :: \Gamma \vdash X : *$ and $\gamma, \gamma' \in [\![D_0]\!]$ such that $\gamma \sim_{D_0} \gamma'$, $\gamma(e) \sim_{[\![D_2]\!]\ \gamma} \gamma'(e)$.

Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $D_3 :: \Gamma \vdash [e/x]\kappa' : \mathsf{kind}$ and $\gamma \in [\![\Gamma\ \mathsf{ok}]\!]$.
By validity we get $D_1 :: \Gamma \vdash \Pi x : X.\ \kappa' : \mathsf{kind}$.
By inversion on D_1, we get $D_2 :: \Gamma \vdash X : *$ and $D_3' :: \Gamma, x : X \vdash \kappa' : \mathsf{kind}$.
Hence we know that $[\![D']\!]\ \gamma \in [\![D_1]\!]\ \gamma$.
Hence we know that $\gamma(e) \sim_{[\![D_2]\!]\ \gamma} \gamma(e)$, and so $\gamma(e) \in [\![D_2]\!]\ \gamma$.
So, by definition we know that $[\![D']\!]\ \gamma\ \gamma(e) \in [\![D_3']\!]\ (\gamma, \gamma(e)/x)$.
By subsitution we get $D_3'' :: \Gamma \vdash [e/x]\kappa' : \mathsf{kind}$ such that $[\![D_3'']\!]\ \gamma = [\![D_3']\!]\ (\gamma, \gamma(e)/x)$.
By coherence, $[\![D_2'']\!]\ \gamma = [\![D_3]\!]\ \gamma$.
Hence $[\![D']\!]\ \gamma\ \gamma(e) \in [\![D_3]\!]\ \gamma$.
Hence $[\![D]\!]\ \gamma \in [\![D_3]\!]\ \gamma$.

- Case $D :: \Gamma \vdash \lambda x : X.\ B : \Pi x : X.\ \kappa$.
By inversion, we have $D' :: \Gamma \vdash X : *$ and $D'' :: \Gamma, x : X \vdash B : \kappa$.
(a) By induction, for all $D_0' :: \Gamma, x : X\ \mathsf{ok}$ and $D_2 :: \Gamma, x : X \vdash \kappa : \mathsf{kind}$ and $\gamma' \in [\![D_0']\!]$, $[\![D'']\!]\ \gamma' \in [\![D_2]\!]\ \gamma'$.

Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash \Pi x : X.\ \kappa : \mathsf{kind}$ and $\gamma \in [\![\Gamma\ \mathsf{ok}]\!]$.
By inversion on D_1, we get $D_2 :: \Gamma, x : X \vdash \kappa : \mathsf{kind}$ and $D_3' :: \Gamma \vdash X : *$.
We want to show $[\![D]\!]\ \gamma \in [\![D_1]\!]\ \gamma$.
It suffices to show for all $\bar{e} \in [\![D_3']\!]\ \gamma$, $[\![D]\!]\ \gamma\ \bar{e} \in [\![D_2]\!]\ (\gamma, \bar{e}/x)$.
From D_0 and D_1', we get $D_0' :: \Gamma, x : X\ \mathsf{ok}$.
Hence $(\gamma, \bar{e}/x) \in [\![D_0']\!]$.
By (a), we know that $[\![D'']\!]\ (\gamma, \bar{e}/x) \in [\![D_2]\!]\ (\gamma, \bar{e}/x)$.
However, note that $[\![D]\!]\ \gamma\ \bar{e} = [\![D'']\!]\ (\gamma, \bar{e}/x)$.
Hence $[\![D]\!]\ \gamma \in [\![D_2]\!]\ (\gamma, \bar{e}/x)$.
Hence for all $\bar{e} \in [\![D_3']\!]\ \gamma$, $[\![D]\!]\ \gamma\ \bar{e} \in [\![D_2]\!]\ (\gamma, \bar{e}/x)$.
Hence $[\![D]\!]\ \gamma \in [\![D_1]\!]\ \gamma$.

- Case $D :: \Gamma \vdash \Pi\alpha : \kappa.\ X : *$.
By inversion, we get $D' :: \Gamma \vdash \kappa : \mathsf{kind}$ and $D'' :: \Gamma, \alpha : \kappa \vdash X : *$.
By induction, for all $D_0' :: \Gamma, \alpha : \kappa\ \mathsf{ok}$ and $D_2 :: \Gamma, \alpha : \kappa \vdash * : \mathsf{kind}$ and $\gamma \in [\![\Gamma, \alpha : \kappa\ \mathsf{ok}]\!]$, we have $[\![D'']\!] \in [\![D_2]\!]$.
Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash * : \mathsf{kind}$ and $\gamma \in [\![\Gamma\ \mathsf{ok}]\!]$.
Then we have $D_0' :: \Gamma, \alpha : \kappa\ \mathsf{ok}$.
Now we want to show that $[\![D]\!]\ \gamma \in [\![D_1]\!]\ \gamma = \textsc{Cand}$.
So we need to show that:
1. $[\![D]\!]\ \gamma$ is a QPER.
2. $\forall (e_1, e_2) \in [\![D]\!]\ \gamma$, $e_1 \downarrow$ and $e_2 \downarrow$.
3. $[\![D]\!]\ \gamma$ is closed under expansions and reduction.

1. Assume $(e_1, e_2) \in [\![D]\!]\ \gamma$ and $(e_1', e_2') \in [\![D]\!]\ \gamma$ and $(e_1, e_2') \in [\![D]\!]\ \gamma$.

We want $(e'_1, e_2) \in [\![D]\!]\ \gamma$.
It suffices to show that for all $A_1, A_2, R \in [\![D']\!]\ \gamma$, $(e'_1\ A_1, e_2\ A_2) \in [\![D'']\!]\ (\gamma, ((A_1, A_2), R)/\alpha)$.
Assume $A_1, A_2, R \in [\![D']\!]\ \gamma$.
Then we know that $(e_1\ A_1, e_2\ A_2) \in [\![D'']\!]\ (\gamma, ((A_1, A_2), R)/\alpha)$ and $(e'_1\ A_1, e'_2\ A_2) \in [\![D'']\!]\ (\gamma, ((A_1, A_2), R)/\alpha)$ and $(e_1\ A_1, e'_2\ A_2) \in [\![D'']\!]\ (\gamma, ((A_1, A_2), R)/\alpha)$.
By induction, we know that $[\![D'']\!]\ (\gamma, ((A_1, A_2), R)/\alpha)$ is a QPER.
Hence $(e'_1\ A_1, e_2\ A_2) \in [\![D'']\!]\ (\gamma, ((A_1, A_2), R)/\alpha)$.
Hence for all $A_1, A_2, R \in [\![D']\!]\ \gamma$, $(e'_1\ A_1, e_2\ A_2) \in [\![D'']\!]\ (\gamma, ((A_1, A_2), R)/\alpha)$.
Hence $[\![D]\!]\ \gamma$ is a QPER.

2. follows by the definition of $[\![D]\!]\ \gamma$.

3. Assume $(e_1, e_2) \in [\![D]\!]\ \gamma$ and that $e_1 \leftrightarrow^* e'_1$ and $e_2 \leftrightarrow^* e'_2$.
We want to show $(e'_1, e'_2) \in [\![D]\!]\ \gamma$.
It suffices to show that for all $A_1, A_2, R \in [\![D']\!]\ \gamma$, $(e'_1\ A_1, e'_2\ A_2) \in [\![D'']\!]\ (\gamma, ((A_1, A_2), R)/\alpha)$.
Assume $A_1, A_2, R \in [\![D']\!]\ \gamma$.
By assumption $(e_1\ A_1, e_2\ A_2) \in [\![D'']\!]\ (\gamma, ((A_1, A_2), R)/\alpha)$.
Since $e_1 \leftrightarrow^* e'_1$, it follows that $e_1\ A_1 \leftrightarrow^* e'_1\ A_1$.
Since $e_2 \leftrightarrow^* e'_2$, it follows that $e_2\ A_2 \leftrightarrow^* e'_2\ A_2$.
By induction, $[\![D'']\!]\ (\gamma, ((A_1, A_2), R)/\alpha)$ is closed under expansions and reductions.
Hence $(e'_1\ A_1, e'_2\ A_2) \in [\![D'']\!]\ (\gamma, ((A_1, A_2), R)/\alpha)$.
Hence for all $A_1, A_2, R \in [\![D']\!]\ \gamma$, $(e'_1\ A_1, e'_2\ A_2) \in [\![D'']\!]\ (\gamma, ((A_1, A_2), R)/\alpha)$.
Hence $(e'_1, e'_2) \in [\![D]\!]\ \gamma$.

- Case $D :: \Gamma \vdash \Pi y : Y.\ X : *$.
By inversion, we get $D' :: \Gamma \vdash Y : *$ and $D'' :: \Gamma, y : Y \vdash X : *$.
By induction, for all $D'_0 :: \Gamma, y : Y\ \mathsf{ok}$ and $D_2 :: \Gamma, y : Y \vdash * : \mathsf{kind}$ and $\gamma \in [\![\Gamma, y : Y\ \mathsf{ok}]\!]$, we have $[\![D'']\!] \in [\![D_2]\!]$.
Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash * : \mathsf{kind}$ and $\gamma \in [\![\Gamma\ \mathsf{ok}]\!]$.
Then we have $D'_0 :: \Gamma, y : Y\ \mathsf{ok}$.
Now we want to show that $[\![D]\!]\ \gamma \in [\![D_1]\!]\ \gamma = \textsc{Cand}$.
So we need to show that:
1. $[\![D]\!]\ \gamma$ is a QPER.
2. $\forall (e_1, e_2) \in [\![D]\!]\ \gamma$, $e_1 \downarrow$ and $e_2 \downarrow$.
3. $[\![D]\!]\ \gamma$ is closed under expansions and reduction.

1. Assume $(e_1, e_2) \in [\![D]\!]\ \gamma$ and $(e'_1, e'_2) \in [\![D]\!]\ \gamma$ and $(e_1, e'_2) \in [\![D]\!]\ \gamma$.
We want $(e'_1, e_2) \in [\![D]\!]\ \gamma$.
It suffices to show that for all $\bar{t} \in [\![D']\!]\ \gamma$, $(e'_1\ t_1, e_2\ t_2) \in [\![D'']\!]\ (\gamma, \bar{t}/y)$.
Assume $\bar{t} \in [\![D']\!]\ \gamma$.
Then we know that $(\gamma, \bar{t}/y) \in [\![D'_0]\!]$.
Then we know that $(e_1\ t_1, e_2\ t_2) \in [\![D'']\!]\ (\gamma, \bar{t}/y)$ and $(e'_1\ t_1, e'_2\ t_2) \in [\![D'']\!]\ (\gamma, \bar{t}/y)$ and $(e_1\ t_1, e'_2\ t_2) \in [\![D'']\!]\ (\gamma, \bar{t}/y)$.
By induction, we know that $[\![D'']\!]\ (\gamma, \bar{t}/y)$ is a QPER.
Hence $(e'_1\ t_1, e_2\ t_2) \in [\![D'']\!]\ (\gamma, \bar{t}/y)$.

Hence for all $\bar{t} \in [\![D']\!]\ \gamma$, $(e_1'\ t_1, e_2\ t_2) \in [\![D'']\!]\ (\gamma, \bar{t}/y)$.
Hence $[\![D]\!]\ \gamma$ is a QPER.

2. follows by the definition of $[\![D]\!]\ \gamma$.

3. Assume $(e_1, e_2) \in [\![D]\!]\ \gamma$ and that $e_1 \leftrightarrow^* e_1'$ and $e_2 \leftrightarrow^* e_2'$.
We want to show $(e_1', e_2') \in [\![D]\!]\ \gamma$.
It suffices to show that for all $\bar{t} \in [\![D']\!]\ \gamma$, $(e_1'\ t_1, e_2'\ t_2) \in [\![D'']\!]\ (\gamma, \bar{t}/y)$.
Assume $\bar{t} \in [\![D']\!]\ \gamma$.
Then we know that $(\gamma, \bar{t}/y) \in [\![D_0']\!]$.
By assumption $(e_1\ t_1, e_2\ t_2) \in [\![D'']\!]\ (\gamma, \bar{t}/y)$.
Since $e_1 \leftrightarrow^* e_1'$, it follows that $e_1\ t_1 \leftrightarrow^* e_1'\ t_1$.
Since $e_2 \leftrightarrow^* e_2'$, it follows that $e_2\ t_2 \leftrightarrow^* e_2'\ t_2$.
By induction, $[\![D'']\!]\ (\gamma, \bar{t}/y)$ is closed under expansions and reductions.
Hence $(e_1'\ t_1, e_2'\ t_2) \in [\![D'']\!]\ (\gamma, \bar{t}/y)$.
Hence for all $\bar{t} \in [\![D']\!]\ \gamma$, $(e_1'\ t_1, e_2'\ t_2) \in [\![D'']\!]\ (\gamma, \bar{t}/y)$.
Hence $(e_1', e_2') \in [\![D]\!]\ \gamma$.

- Case $D :: \Gamma \vdash e_1 =_X e_2 : *$.
Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash * : \mathsf{kind}$ and $\gamma \in [\![\Gamma\ \mathsf{ok}]\!]$.

We want to show that $[\![D]\!]\ \gamma \in [\![D_1]\!]\ \gamma = \textsc{Cand}$.
So we need to show that:
1. $[\![D]\!]\ \gamma$ is a QPER.
2. $\forall (e_1, e_2) \in [\![D]\!]\ \gamma$, $e_1 \downarrow$ and $e_2 \downarrow$.
3. $[\![D]\!]\ \gamma$ is closed under expansions and reduction.

1. Assume that $(e_1, e_2) \in [\![D]\!]\ \gamma$ and $(e_1', e_2') \in [\![D]\!]\ \gamma$ and $(e_1, e_2') \in [\![D]\!]\ \gamma$.
We want to show $(e_1', e_2) \in [\![D]\!]\ \gamma$.
It suffices to show that $e_1' \mapsto^* \mathsf{refl}$ and $e_2 \mapsto^* \mathsf{refl}$.
Note that $e_1 \mapsto^* \mathsf{refl}$ and $e_2 \mapsto^* \mathsf{refl}$ and $e_1' \mapsto^* \mathsf{refl}$ and $e_2' \mapsto^* \mathsf{refl}$.
So $(e_1', e_2) \in [\![D]\!]\ \gamma$.
So $[\![D]\!]\ \gamma$ is a QPER.

2. follows immediately from the definition.

3. Assume that $(e_1, e_2) \in [\![D]\!]\ \gamma$, and that $e_1 \leftrightarrow^* e_1'$ and $e_2 \leftrightarrow^* e_2'$.
We know that $e_1 \mapsto^* \mathsf{refl}$ and $e_2 \mapsto^* \mathsf{refl}$.
Therefore it follows that $e_1' \mapsto^* \mathsf{refl}$ and $e_2' \mapsto^* \mathsf{refl}$.
Hence $(e_1', e_2') \in [\![D]\!]\ \gamma$.

- Case $D :: \Gamma \vdash A : \kappa'$.
By inversion, we have $D' :: [\![\Gamma \vdash A : \kappa]\!]$ and $D'' :: \Gamma \vdash \kappa \equiv \kappa' : \mathsf{kind}$.
(a) By induction, we know for all $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: [\![\Gamma \vdash \mathsf{kind} : \kappa]\!]$ and $\gamma \in [\![D_0]\!]$, $[\![D']\!]\ \gamma \in [\![D_1]\!]\ \gamma$.

(b) By mutual induction, for all $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: [\![\Gamma \vdash \mathsf{kind} : \kappa]\!]$ and $D_2 :: [\![\Gamma \vdash \kappa' : \mathsf{kind}]\!]$, $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$. Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $D_2 :: \Gamma \vdash \kappa : \mathsf{kind}$ and $\gamma \in [\![\Gamma\ \mathsf{ok}]\!]$.
By validity on D'', we know $D_1 :: \Gamma \vdash \kappa' : \mathsf{kind}$.
Therefore by (b), $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$.
From (a), we know $[\![D']\!]\ \gamma \in [\![D_2]\!]\ \gamma$.
Hence $[\![D']\!]\ \gamma \in [\![D_1]\!]\ \gamma$.

5. Assume $\Gamma \vdash \kappa \equiv \kappa' : \mathsf{kind}$.

- Case $D :: \Gamma \vdash \kappa \equiv \kappa : \mathsf{kind}$:
  Immediate.

- Case $D :: \Gamma \vdash \kappa \equiv \kappa' : \mathsf{kind}$:
  By inversion, $D' :: \Gamma \vdash \kappa' \equiv \kappa : \mathsf{kind}$.
  (a) By induction, for all $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash \kappa : \mathsf{kind}$ and $D_2 :: \Gamma \vdash \kappa' : \mathsf{kind}$ and $\gamma \in [\![D_0]\!]$, $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$.
  Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash \kappa : \mathsf{kind}$ and $D_2 :: \Gamma \vdash \kappa' : \mathsf{kind}$ and $\gamma \in [\![D_0]\!]$.
  By (a), $[\![D_2]\!]\ \gamma = [\![D_1]\!]\ \gamma$.
  Hence $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$.

- Case $D :: \Gamma \vdash \kappa \equiv \kappa'' : \mathsf{kind}$:
  By inversion, $D' :: \Gamma \vdash \kappa \equiv \kappa' : \mathsf{kind}$ and $D'' :: \Gamma \vdash \kappa' \equiv \kappa'' : \mathsf{kind}$.
  (a) By induction, for all $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash \kappa : \mathsf{kind}$ and $D_2 :: \Gamma \vdash \kappa' : \mathsf{kind}$ and $\gamma \in [\![D_0]\!]$, $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$.
  (b) By induction, for all $D_0 :: \Gamma\ \mathsf{ok}$ and $D_2 :: \Gamma \vdash \kappa' : \mathsf{kind}$ and $D_3 :: \Gamma \vdash \kappa'' : \mathsf{kind}$ and $\gamma \in [\![D_0]\!]$, $[\![D_2]\!]\ \gamma = [\![D_3]\!]\ \gamma$.
  Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash \kappa : \mathsf{kind}$ and $D_3 :: \Gamma \vdash \kappa'' : \mathsf{kind}$ and $\gamma \in [\![D_0]\!]$.
  By validity on D', we get $D_2 :: \Gamma \vdash \kappa' : \mathsf{kind}$.
  By (a), $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$.
  By (b), $[\![D_2]\!]\ \gamma = [\![D_3]\!]\ \gamma$.
  Hence $[\![D_1]\!]\ \gamma = [\![D_3]\!]\ \gamma$.

- Case $D :: \Gamma \vdash [e_1/x]\kappa \equiv [e_2/x]\kappa : \mathsf{kind}$:
  By inversion, $D' :: \Gamma, x : X \vdash \kappa : \mathsf{kind}$ and $D'' :: \Gamma \vdash e_1 \equiv e_2 : X$.
  (a) By mutual induction, for all $D_0 :: \Gamma\ \mathsf{ok}$ and $D_5 :: \Gamma \vdash X : *$ and $\gamma \in [\![D_0]\!]$, $\gamma(e_1) \sim_{[\![D_5]\!]\ \gamma} \gamma(e_2)$.
  (b) By mutual induction, for all $D_0' :: \Gamma, x : X\ \mathsf{ok}$ and $\gamma, \gamma' \in [\![D_0']\!]$ s.t. $\gamma \sim_{D_0} \gamma'$, we have $[\![D'']\!]\ \gamma = [\![D'']\!]\ \gamma'$.
  Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash [e_1/x]\kappa : \mathsf{kind}$ and $D_2 :: \Gamma \vdash [e_2/x]\kappa : \mathsf{kind}$ and $\gamma \in [\![D_0]\!]$.
  By validity on D'', we have $D_3 :: \Gamma \vdash e_1 : X$ and $D_4 :: \Gamma \vdash e_2 : X$.
  By validity on D_3, we get $D_5 :: \Gamma \vdash X : *$.
  Note that by (a), $\gamma(e_1) \sim_{[\![D_5]\!]\ \gamma} \gamma(e_2)$.
  Note that by rule on D_0 and D_5, we have $D_0' :: \Gamma, x : X\ \mathsf{ok}$.
  Hence $(\gamma, \gamma(e_1)/x) \sim_{D_0'} (\gamma, \gamma(e_2)/x)$.

By (b), $[\![D'']\!]\ (\gamma, \gamma(e_1)/x) = [\![D'']\!]\ (\gamma, \gamma(e_2)/x)$.
By substitution and then coherence, $[\![D'']\!]\ (\gamma, \gamma(e_1)/x) = [\![D_1]\!]\ \gamma$.
By substitution and then coherence, $[\![D'']\!]\ (\gamma, \gamma(e_2)/x) = [\![D_2]\!]\ \gamma$.
Hence $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$.

- Case $D :: \Gamma \vdash [A_1/\alpha]\kappa \equiv [A_2/\alpha]\kappa : \mathsf{kind}$:
  By inversion, $D' :: \Gamma, \alpha : \kappa' \vdash \kappa : \mathsf{kind}$ and $D'' :: \Gamma \vdash A_1 \equiv A_2 : \kappa'$.
  (a) By mutual induction, for all $D_0 :: \Gamma\ \mathsf{ok}$ and $D_3 :: \Gamma \vdash A_1 : \kappa'$ and $D_4 :: \Gamma \vdash A_2 : \kappa'$, and $\gamma \in [\![D_0]\!]$, $[\![D_3]\!]\ \gamma = [\![D_4]\!]\ \gamma$.
  (b) By mutual induction, for all $D_0' :: \Gamma, \alpha : \kappa'\ \mathsf{ok}$ and $\gamma, \gamma' \in [\![D_0']\!]$ s.t. $\gamma \sim_{D_0} \gamma'$, we have $[\![D'']\!]\ \gamma = [\![D'']\!]\ \gamma'$.
  Assume $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash [A_1/\alpha]\kappa : \mathsf{kind}$ and $D_2 :: \Gamma \vdash [A_2/\alpha]\kappa : \mathsf{kind}$ and $\gamma \in [\![D_0]\!]$.
  By validity on D'', we have $D_3 :: \Gamma \vdash A_1 : \kappa'$ and $D_4 :: \Gamma \vdash A_2 : \kappa'$.
  By validity on D_3, we get $D_5 :: \Gamma \vdash \kappa' : \mathsf{kind}$.
  Note that by rule on D_0 and D_5, we have $D_0' :: \Gamma, \alpha : \kappa'\ \mathsf{ok}$.
  Note that by (a), $[\![D_3]\!]\ \gamma = [\![D_4]\!]\ \gamma$.
  Hence $(\gamma, (\gamma(A_1), [\![D_3]\!]\ \gamma)/\alpha) \sim_{D_0'} (\gamma, (\gamma(A_2), [\![D_4]\!]\ \gamma)/\alpha)$.
  By (b), $[\![D'']\!]\ (\gamma, (\gamma(A_1), [\![D_3]\!]\ \gamma)/\alpha) = [\![D'']\!]\ (\gamma, (\gamma(A_2), [\![D_4]\!]\ \gamma)/\alpha)$.
  By substitution and then coherence, $[\![D'']\!]\ (\gamma, \gamma(A_1)/\alpha) = [\![D_1]\!]\ \gamma$.
  By substitution and then coherence, $[\![D'']\!]\ (\gamma, \gamma(A_2)/\alpha) = [\![D_2]\!]\ \gamma$.
  Hence $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$.

6. Assume $D :: \Gamma \vdash A \equiv A' : \kappa$.

- Case $D :: \Gamma \vdash [e/x]B \equiv [e'/x]B : [e/x]\kappa$.
  By inversion, we have $D' :: \Gamma \vdash e \equiv e' : X$ and $D'' :: \Gamma, x : X \vdash B : \kappa$.
  Assume $D_0 :: \Gamma\ \mathsf{ok}$, $D_1 :: \Gamma \vdash [e/x]B : [e/x]\kappa$ and $D_2 :: \Gamma \vdash [e'/x]B : [e/x]\kappa$ and $\gamma \in [\![D_0]\!]$.
  By validity, we know $D_3 :: \Gamma \vdash X : *$.
  Note that $\gamma \sim_{D_0} \gamma$, so by mutual induction $\gamma(e) \sim_{[\![D_3]\!]\ \gamma} \gamma(e')$.
  From D_0 and D_3, we have $D_0' :: \Gamma, x : X\ \mathsf{ok}$.
  By validity, we have $D_4 :: \Gamma, x : X \vdash \kappa : \mathsf{kind}$.
  Furthermore, we know $(\gamma, \gamma(e)/x) \sim_{D_0'} (\gamma, \gamma(e')/x)$.
  Hence by mutual induction, we know that $[\![D'']\!]\ (\gamma, \gamma(e)/x) = [\![D'']\!]\ (\gamma, \gamma(e')/x)$.
  By substitution, we have $D_1' :: \Gamma \vdash [e/x]B : [e/x]\kappa$ such that $[\![D_1']\!]\ \gamma = [\![D'']\!]\ (\gamma, \gamma(e)/x)$.
  By substitution, we have $D_2' :: \Gamma \vdash [e'/x]B : [e'/x]\kappa$ such that $[\![D_2']\!]\ \gamma = [\![D'']\!]\ (\gamma, \gamma(e')/x)$.
  By rule with D' and D_4, we have $D_5 :: \Gamma \vdash [e/x]\kappa \equiv [e'/x]\kappa : \mathsf{kind}$.
  By rule with D_2' and D_5, we have $D_2'' :: \Gamma \vdash [e'/x]B : [e/x]\kappa$.
  Note that by definition, $[\![D_2'']\!]\ \gamma = [\![D_2']\!]\ \gamma = [\![D'']\!]\ (\gamma, \gamma(e')/x)$.
  By coherence $[\![D_1]\!]\ \gamma = [\![D_1']\!]\ \gamma$.
  By coherence $[\![D_2]\!]\ \gamma = [\![D_2'']\!]\ \gamma$.
  Hence $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$.

- Case $D :: \Gamma \vdash [A/\alpha]B \equiv [A'/\alpha]B : [A/\alpha]\kappa$.
  By inversion, we have $D' :: \Gamma \vdash A \equiv A' : \kappa'$ and $D'' :: \Gamma, \alpha : \kappa' \vdash B : \kappa$.
  Assume $D_0 :: \Gamma\ \mathsf{ok}$, $D_1 :: \Gamma \vdash [A/\alpha]B : [A/\alpha]\kappa$ and $D_2 :: \Gamma \vdash [A'/\alpha]B : [A/\alpha]\kappa$ and

$\gamma \in [\![D_0]\!]$.
By validity, we know that $D_3 :: \Gamma \vdash A : \kappa'$ and $D_3' :: \Gamma \vdash A' : \kappa'$.
By mutual induction, we know that $[\![D_3]\!]\ \gamma = [\![D_3']\!]\ \gamma$.
By validity, we know that $D_4 :: \Gamma \vdash \kappa' : \mathsf{kind}$.
Hence we have $D_0' :: \Gamma, \alpha : \kappa'\ \mathsf{ok}$.
By validity, we have $D_5 :: \Gamma, \alpha : \kappa' \vdash \kappa : \mathsf{kind}$.
Hence $(\gamma, (\gamma(A), [\![D_3]\!]\ \gamma)/\alpha) \sim_{D_0'} (\gamma, (\gamma(A'), [\![D_3']\!]\ \gamma)/\alpha)$.
By mutual induction, $[\![D'']\!]\ (\gamma, (\gamma(A), [\![D_3]\!]\ \gamma)/\alpha) = [\![D'']\!]\ (\gamma, (\gamma(A'), [\![D_3']\!]\ \gamma)/\alpha)$.
By substitution, we have $D_1' :: \Gamma \vdash [A/\alpha]B : [A/\alpha]\kappa$ s.t. $[\![D_1']\!]\ \gamma = [\![D'']\!]\ (\gamma, (\gamma(A), [\![D_3]\!]\ \gamma)/\alpha)$.
By substitution, we have $D_2' :: \Gamma \vdash [A'/\alpha]B : [A'/\alpha]\kappa$ s.t. $[\![D_2']\!]\ \gamma = [\![D'']\!]\ (\gamma, (\gamma(A), [\![D_3']\!]\ \gamma)/\alpha)$.
By rule on D' and D_5, we have $D_6 :: \Gamma \vdash [A/\alpha]\kappa \equiv [A'/\alpha]\kappa : \mathsf{kind}$.
By rule on D_2' and D_6, we have $D_2'' :: \Gamma \vdash [A'/\alpha]B : [A/\alpha]\kappa$.
Note that $[\![D_2'']\!]\ \gamma = [\![D_2']\!]\ \gamma$.
By coherence, $[\![D_1]\!]\ \gamma = [\![D_1']\!]\ \gamma$.
By coherence, $[\![D_2]\!]\ \gamma = [\![D_2'']\!]\ \gamma$.
Hence $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$.

- Case $D :: \Gamma \vdash A \equiv A' : \kappa$.
  By inversion, we have $D' :: \Gamma \vdash A \equiv A' : \kappa'$ and $D'' :: \Gamma \vdash \kappa \equiv \kappa' : \mathsf{kind}$.
  Assume $D_0 :: \Gamma\ \mathsf{ok}$, $D_1 :: \Gamma \vdash A : \kappa$ and $D_2 :: \Gamma \vdash A' : \kappa$ and $\gamma \in [\![D_0]\!]$.
  By validity, we have $D_1' :: \Gamma \vdash A : \kappa'$ and $D_2' :: \Gamma \vdash A' : \kappa'$.
  By induction on D', we know that $[\![D_1']\!]\ \gamma = [\![D_2']\!]\ \gamma$.
  By rule on D_1' and D'', we have $D_1'' :: \Gamma \vdash A : \kappa$.
  By rule on D_2' and D'', we have $D_2'' :: \Gamma \vdash A' : \kappa$.
  Note that by definition, $[\![D_1'']\!]\ \gamma = [\![D_1']\!]\ \gamma$.
  Note that by definition, $[\![D_2'']\!]\ \gamma = [\![D_2']\!]\ \gamma$.
  By coherence, $[\![D_1]\!]\ \gamma = [\![D_1'']\!]\ \gamma$.
  By coherence, $[\![D_2]\!]\ \gamma = [\![D_2'']\!]\ \gamma$.
  Hence $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$.

- Case $D :: \Gamma \vdash A \equiv A : \kappa$. (reflexivity)
  By inversion, we have $D' :: \Gamma \vdash A : \kappa$.
  Assume $D_0 :: \Gamma\ \mathsf{ok}$, $D_1 :: \Gamma \vdash A : \kappa$ and $D_2 :: \Gamma \vdash A : \kappa$ and $\gamma \in [\![D_0]\!]$.
  By coherence, $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$.

- Case $D :: \Gamma \vdash A \equiv A' : \kappa$. (symmetry)
  By inversion, we have $D' :: \Gamma \vdash A' \equiv A : \kappa$.
  Assume $D_0 :: \Gamma\ \mathsf{ok}$, $D_1 :: \Gamma \vdash A : \kappa$ and $D_2 :: \Gamma \vdash A' : \kappa$ and $\gamma \in [\![D_0]\!]$.
  By induction, $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$.

- Case $D :: \Gamma \vdash A \equiv A'' : \kappa$. (transitivity)
  By inversion, we have $D' :: \Gamma \vdash A \equiv A' : \kappa$ and $D'' :: \Gamma \vdash A' \equiv A'' : \kappa$.
  Assume $D_0 :: \Gamma\ \mathsf{ok}$, $D_1 :: \Gamma \vdash A : \kappa$ and $D_2 :: \Gamma \vdash A'' : \kappa$ and $\gamma \in [\![D_0]\!]$.
  By validity, we have $D_3 :: \Gamma \vdash A' : \kappa$.
  By induction, we know that $[\![D_1]\!]\ \gamma = [\![D_3]\!]\ \gamma$.

By induction, we know that $[\![D_3]\!]\ \gamma = [\![D_2]\!]\ \gamma$.
Hence $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$.

- Case $D :: \Gamma \vdash (\lambda x : X.\ B)\ e \equiv [e/x]B : [e/x]\kappa$.
  By inversion, $D' :: \Gamma \vdash \lambda x : X.\ B : \Pi x : X.\ \kappa$ and $D'' :: \Gamma \vdash e : X$.
  Assume $D_0 :: \Gamma\ \mathsf{ok}$, $D_1 :: \Gamma \vdash (\lambda x : X.\ B)\ e : [e/x]\kappa$ and $D_2 :: \Gamma \vdash [e/x]B : [e/x]\kappa$ and $\gamma \in [\![D_0]\!]$.
  By validity, $D_3 :: \Gamma \vdash \Pi x : X.\ \kappa : \mathsf{kind}$.
  By validity, $D_4 :: \Gamma \vdash X : *$.
  By mutual induction, $[\![D']\!]\ \gamma \in [\![D_3]\!]\ \gamma$.
  Since $\gamma \sim_{D_0} \gamma$, we know $\gamma(e) \in [\![D_4]\!]\ \gamma$.
  By inversion on D_1, we have $D'_1 :: \Gamma \vdash \lambda x : X.\ B : \Pi x : X.\ \kappa$ and $D''_1 :: \Gamma \vdash e : X$.
  By coherence, we know that $[\![D'_1]\!]\ \gamma = [\![D_1]\!]\ \gamma$.
  Hence $[\![D_1]\!]\ \gamma = [\![D']\!]\ \gamma\ \gamma(e)$.
  By inversion on D', we have $D_5 :: \Gamma, x : X \vdash B : \kappa$.
  So we know that $[\![D']\!]\ \gamma\ \gamma(e) = [\![D_5]\!]\ (\gamma, \gamma(e)/x)$.
  Note that from D_0 and D_4, we have $D'_0 :: \Gamma, x : X\ \mathsf{ok}$, and $(\gamma, \gamma(e)/x) \in [\![D'_0]\!]$.
  Hence by substitution, we have $D'_2 :: \Gamma \vdash [e/x]B : [e/x]\kappa$ such that $[\![D'_2]\!]\ \gamma = [\![D_5]\!]\ (\gamma, \gamma(e)/x)$.
  By coherence $[\![D_2]\!]\,; \gamma = [\![D'_2]\!]\ \gamma$.
  Hence $[\![D_1]\!]\,; \gamma = [\![D_2]\!]\ \gamma$.

- Case $D :: \Gamma \vdash (\lambda \alpha : \kappa'.\ B)\ A \equiv [A/\alpha]B : [A/\alpha]\kappa$.
  By inversion, $D' :: \Gamma \vdash \lambda x : X.\ B : \Pi \alpha : \kappa'.\ \kappa$ and $D'' :: \Gamma \vdash A : \kappa'$.
  Assume $D_0 :: \Gamma\ \mathsf{ok}$, $D_1 :: \Gamma \vdash (\lambda \alpha : \kappa'.\ B)\ A : [A/\alpha]\kappa$ and $D_2 :: \Gamma \vdash [A/\alpha]B : [A/\alpha]\kappa$ and $\gamma \in [\![D_0]\!]$.
  By validity, $D_3 :: \Gamma \vdash \Pi \alpha : \kappa'.\ \kappa : \mathsf{kind}$.
  By validity, $D_4 :: \Gamma \vdash A : \kappa'$.
  By mutual induction, $[\![D']\!]\ \gamma \in [\![D_3]\!]\ \gamma$.
  By mutual induction, $[\![D'']\!]\ \gamma \in [\![D_4]\!]\ \gamma$.
  By inversion on D_1, we have $D'_1 :: \Gamma \vdash \lambda \alpha : \kappa'.\ B : \Pi \alpha : \kappa'.\ \kappa$ and $D''_1 :: \Gamma \vdash A : \kappa'$.
  By coherence, we know that $[\![D'_1]\!]\ \gamma = [\![D_1]\!]\ \gamma$.
  Hence $[\![D_1]\!]\ \gamma = [\![D']\!]\ \gamma\ (\gamma(A), [\![D'']\!]\ \gamma)$.
  By inversion on D', we have $D_5 :: \Gamma, \alpha : \kappa' \vdash B : \kappa$.
  So we know that $[\![D']\!]\ \gamma\ (\gamma(A), [\![D'']\!]\ \gamma) = [\![D_5]\!]\ (\gamma, (\gamma(A), [\![D'']\!]\ \gamma)/\alpha)$.
  Note that from D_0 and D_4, we have $D'_0 :: \Gamma, \alpha : \kappa'\ \mathsf{ok}$, and $(\gamma, (\gamma(A), [\![D'']\!]\ \gamma)/\alpha) \in [\![D'_0]\!]$.
  Hence by substitution, we have $D'_2 :: \Gamma \vdash [A/\alpha]B : [A/\alpha]\kappa$ such that $[\![D'_2]\!]\ \gamma = [\![D_5]\!]\ (\gamma, (\gamma(A), [\![D'']\!]\ \gamma)/\alpha)$.
  By coherence $[\![D_2]\!]\,; \gamma = [\![D'_2]\!]\ \gamma$.
  Hence $[\![D_1]\!]\,; \gamma = [\![D_2]\!]\ \gamma$.

- Case $D :: \Gamma \vdash B \equiv B' : \Pi x : X.\ \kappa$.
  By inversion, we have $D' :: \Gamma, x : X \vdash B\ x \equiv B'\ x : \kappa$, $D'' :: \Gamma \vdash B : \Pi x : X.\ \kappa$ and $D''' :: \Gamma \vdash B' : \Pi x : X.\ \kappa$.
  Assume $D_0 :: \Gamma\ \mathsf{ok}$, $D_1 :: \Gamma \vdash B : \Pi x : X.\ \kappa$ and $D_2 :: \Gamma \vdash B' : \Pi x : X.\ \kappa$ and $\gamma \in [\![D_0]\!]$.
  By validity, we know $D_3 :: \Gamma \vdash \Pi x : X.\ \kappa : \mathsf{kind}$.

By inversion on D_3, we get $D_4 :: \Gamma \vdash X : *$.
From D_0 and D_4, we get $D'_0 :: \Gamma, x : X\ \mathsf{ok}$.
By mutual induction, $[\![D'']\!]\ \gamma \in [\![D_3]\!]\ \gamma$.
By mutual induction, $[\![D''']\!]\ \gamma \in [\![D_3]\!]\ \gamma$.
By coherence, $[\![D_1]\!]\ \gamma = [\![D'']\!]\ \gamma$ and $[\![D_2]\!]\ \gamma = [\![D''']\!]\ \gamma$.
So $[\![D_1]\!]\ \gamma$ and $[\![D_2]\!]\ \gamma$ are in $\|\Pi x : X.\ \kappa\|$.
Assume $\bar{e} \in \mathrm{Exp} \times \mathrm{Exp}$.
Consider whether $\bar{e} \in [\![D_4]\!]\ \gamma$.

- Case $\bar{e} \in [\![D_4]\!]\ \gamma$:
  Then $(\gamma, \bar{e}/x) \in [\![D'_0]\!]$.
  By weakening and rule on D_1, we get $D'_1 :: \Gamma, x : X \vdash B\ x : \kappa$.
  By weakening and rule on D_2, we get $D'_2 :: \Gamma, x : X \vdash B\ x : \kappa$.
  By induction hypothesis, we know that $[\![D'_1]\!]\ (\gamma, \bar{e}/x) = [\![D'_2]\!]\ (\gamma, \bar{e}/x)$.
  However $[\![D'_1]\!]\ (\gamma, \bar{e}/x) = [\![D_1]\!]\ \gamma\ \bar{e}$.
  However $[\![D'_2]\!]\ (\gamma, \bar{e}/x) = [\![D_2]\!]\ \gamma\ \bar{e}$.
  So $[\![D_1]\!]\ \gamma\ \bar{e} = [\![D_2]\!]\ \gamma\ \bar{e}$.
- Case $\bar{e} \notin [\![D_4]\!]\ \gamma$:
  Then $[\![D_1]\!]\ \gamma\ \bar{e} = !_\kappa = [\![D_2]\!]\ \gamma\ \bar{e}$.

Hence $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$.

- Case $D :: \Gamma \vdash B \equiv B' : \Pi\alpha : \kappa'.\ \kappa$.
  By inversion, we have $D' :: \Gamma, \alpha : \kappa' \vdash B\ \alpha \equiv B'\ \alpha : \kappa$, $D'' :: \Gamma \vdash B : \Pi\alpha : \kappa'.\ \kappa$ and $D''' :: \Gamma \vdash B' : \Pi\alpha : \kappa'.\ \kappa$.
  Assume $D_0 :: \Gamma\ \mathsf{ok}$, $D_1 :: \Gamma \vdash B : \Pi\alpha : \kappa'.\ \kappa$ and $D_2 :: \Gamma \vdash B' : \Pi\alpha : \kappa'.\ \kappa$ and $\gamma \in [\![D_0]\!]$.
  By validity, we know $D_3 :: \Gamma \vdash \Pi\alpha : \kappa'.\ \kappa : \mathsf{kind}$.
  By inversion on D_3, we get $D_4 :: \Gamma \vdash \kappa' : \mathsf{kind}$.
  From D_0 and D_4, we get $D'_0 :: \Gamma, \alpha : \kappa'\ \mathsf{ok}$.
  By mutual induction, $[\![D'']\!]\ \gamma \in [\![D_3]\!]\ \gamma$.
  By mutual induction, $[\![D''']\!]\ \gamma \in [\![D_3]\!]\ \gamma$.
  By coherence, $[\![D_1]\!]\ \gamma = [\![D'']\!]\ \gamma$ and $[\![D_2]\!]\ \gamma = [\![D''']\!]\ \gamma$.
  So $[\![D_1]\!]\ \gamma$ and $[\![D_2]\!]\ \gamma$ are in $\|\Pi\alpha : \kappa'.\ \kappa\|$.
  Assume $(\overline{A}, R) \in (\mathrm{Exp} \times \mathrm{Exp}) \times \|\Pi\alpha : \kappa'.\ \kappa\|$.
  Consider whether $R \in [\![D_4]\!]\ \gamma$.

  - Case $R \in [\![D_4]\!]\ \gamma$:
    Then $(\gamma, (\overline{A}, R)/\alpha) \in [\![D'_0]\!]$.
    By weakening and rule on D_1, we get $D'_1 :: \Gamma, \alpha : \kappa' \vdash B\ \alpha : \kappa$.
    By weakening and rule on D_2, we get $D'_2 :: \Gamma, \alpha : \kappa' \vdash B\ \alpha : \kappa$.
    By induction hypothesis, we know that $[\![D'_1]\!]\ (\gamma, (\overline{A}, R)/\alpha) = [\![D'_2]\!]\ (\gamma, (\overline{A}, R)/\alpha)$.
    However $[\![D'_1]\!]\ (\gamma, (\overline{A}, R)/\alpha) = [\![D_1]\!]\ \gamma\ (\overline{A}, R)$.
    However $[\![D'_2]\!]\ (\gamma, (\overline{A}, R)/\alpha) = [\![D_2]\!]\ \gamma\ (\overline{A}, R)$.
    So $[\![D_1]\!]\ \gamma\ (\overline{A}, R) = [\![D_2]\!]\ \gamma\ (\overline{A}, R)$.
  - Case $R \notin [\![D_4]\!]\ \gamma$:
    Then $[\![D_1]\!]\ \gamma\ (\overline{A}, R) = !_\kappa = [\![D_2]\!]\ \gamma\ (\overline{A}, R)$.

  Hence $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$.

- Case $D :: \Gamma \vdash \Pi\alpha : \kappa.\ X \equiv \Pi\alpha : \kappa'.\ X' : *$.
  By inversion, we know that $E_1 :: \Gamma \vdash \kappa \equiv \kappa' : \mathsf{kind}$
  By inversion, we know that $E_2 :: \Gamma, \alpha : \kappa \vdash X \equiv X' : *$
  Assume $D_0 :: \Gamma\ \mathsf{ok}$, $D_1' :: \Gamma \vdash \Pi\alpha : \kappa.\ X : *$ and $D_2' :: \Gamma \vdash \Pi\alpha : \kappa'.\ X' : *$.
  We want to show $[\![D_1']\!]\ \gamma = [\![D_2']\!]\ \gamma$.
  By validity, we know that $D_1 :: \Gamma \vdash \Pi\alpha : \kappa.\ X : *$ and $D_1 :: \Gamma \vdash \Pi\alpha : \kappa'.\ X' : *$.
  By coherence, it suffices to show $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$.
  By inversion on D_1, we have $D_1^K :: \Gamma \vdash \kappa : \mathsf{kind}$ and $D_1^X :: \Gamma, \alpha : \kappa \vdash X : *$.
  By inversion on D_1, we have $D_2^K :: \Gamma \vdash \kappa' : \mathsf{kind}$ and $D_2^X :: \Gamma, \alpha : \kappa' \vdash X' : *$.
  By induction, $[\![D_1^K]\!]\ \gamma = [\![D_2^K]\!]\ \gamma$.

  $\Longrightarrow$: Assume $\bar{e} \in [\![D_1]\!]\ \gamma$. We want to show $\bar{e} \in [\![D_2]\!]\ \gamma$.
  Assume $\overline{A}, R \in [\![D_2^K]\!]\ \gamma$.
  By our induction hypothesis, $R \in [\![D_1^K]\!]\ \gamma$.
  Hence $\overline{e\ A} \in [\![D_1^X]\!]\ (\gamma, (\overline{A}, R)/\alpha)$.
  By weakening and substitution, we get $D_1'^X :: \Gamma, \alpha : \kappa' \vdash X : *$,
  such that $[\![D_1'^X]\!]\ (\gamma, (\overline{A}, R)/\alpha) = [\![D_1^X]\!]\ (\gamma, (\overline{A}, R)/\alpha)$.
  By coherence, $[\![D_1'^X]\!]\ (\gamma, (\overline{A}, R)/\alpha) = [\![D_2^X]\!]\ (\gamma, (\overline{A}, R)/\alpha)$.
  Hence $\overline{e\ A} \in [\![D_2^X]\!]\ (\gamma, (\overline{A}, R)/\alpha)$.

  $\Longleftarrow$: Symmetric.

7. Case $D :: \Gamma \vdash e \equiv e' : X$.

- $D :: \Gamma \vdash e \equiv e' : X$ (identity reflection).
  By inversion, we have $D' :: \Gamma \vdash e_p : e =_X e'$ and $D'' :: \Gamma \vdash e =_X e' : *$.
  Assume we have $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash X : *$ and $\gamma \in [\![D_0]\!]$.
  By inversion on D'', $D_2 :: \Gamma \vdash e : X$ and $D_3 :: \Gamma \vdash e' : X$ and $D_4 :: \Gamma \vdash X : *$.
  Then since $\gamma \sim_{D_0} \gamma$, it follows that $\gamma(e_p) \in [\![D'']\!]\ \gamma$.
  Therefore $(\gamma_1(e), \gamma_2(e')) \in [\![D_4]\!]\ \gamma$.
  Furthermore, by induction we know that $(\gamma_1(e), \gamma_1(e)) \in [\![D_4]\!]\ \gamma$.
  Furthermore, by induction we know that $(\gamma_1(e'), \gamma_1(e')) \in [\![D_4]\!]\ \gamma$.
  By mutual induction we know that $[\![D_4]\!]$ is a candidate set, and is hence a QPER.
  Therefore $\gamma(e_1) \sim_{[\![D_4]\!]\ \gamma} \gamma(e_2)$.
  By coherence, $[\![D_4]\!]\ \gamma = [\![D_1]\!]\ \gamma$.
  Hence $\gamma(e_1) \sim_{[\![D_1]\!]\ \gamma} \gamma(e_2)$.

- $D :: \Gamma \vdash e \equiv e' : X$ (type equality).
  By inversion, we know $D' :: \Gamma \vdash e \equiv e' : Y$ and $D'' :: \Gamma \vdash X \equiv Y : *$.
  Assume we have $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash X : *$ and $\gamma \in [\![D_0]\!]$.
  By validity, we get $D_2 :: \Gamma \vdash Y : *$.
  By mutual induction, $[\![D_1]\!]\ \gamma = [\![D_2]\!]\ \gamma$.
  By induction, $\gamma(e) \sim_{[\![D_2]\!]\ \gamma} \gamma(e')$.

Hence $\gamma(e) \sim_{[\![D_1]\!]\ \gamma} \gamma(e')$.

- $D :: \Gamma \vdash [t/x]e \equiv [t'/x]e : [t/x]Y$.
  By inversion, we have $D' :: \Gamma \vdash t \equiv t' : X$ and $D'' :: \Gamma, x : X \vdash e : Y$.
  Assume we have $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash [t/x]Y : *$ and $\gamma \in [\![D_0]\!]$.
  By validity, we have $D_2 :: \Gamma \vdash t : X$ and $D_3 :: \Gamma \vdash t' : X$ and $D_4 :: \Gamma \vdash X : *$.
  Hence by rule with D_0 and D_4, we have $D_0' :: \Gamma, x : X\ \mathsf{ok}$.
  By validity, we have $D_5 :: \Gamma, x : X \vdash Y : *$.
  By induction, we know that $\gamma(t) \sim_{[\![D_4]\!]\ \gamma} \gamma(t')$.
  Hence $(\gamma, \gamma(t)/x) \sim_{D_0} (\gamma, \gamma(t')/x)$.
  Hence by mutual induction we know $(\gamma, \gamma(t)/x)(e) \sim_{[\![D_5]\!]\ (\gamma, \gamma(t)/x)} (\gamma, \gamma(t')/x)(e)$.
  By substitution, we have $D_1' :: \Gamma \vdash [t/x]Y : *$ such that $[\![D_1']\!]\ \gamma = [\![D_5]\!]\ (\gamma, \gamma(t)/x)$.
  By coherence, $[\![D_1]\!]\ \gamma = [\![D_1']\!]\ \gamma$.
  Hence $(\gamma, \gamma(t)/x)(e) \sim_{[\![D_1]\!]\ \gamma} (\gamma, \gamma(t')/x)(e)$.
  Hence $\gamma([t/x]e) \sim_{[\![D_1]\!]\ \gamma} \gamma([t'/x]e)$.

- $D :: \Gamma \vdash [A/\alpha]e \equiv [A'/\alpha]e : [A/\alpha]Y$.
  By inversion, we have $D' :: \Gamma \vdash A \equiv A' : \kappa'$ and $D'' :: \Gamma, \alpha : \kappa' \vdash e : Y$.
  Assume we have $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash [A/\alpha]Y : \mathsf{kind}$ and $\gamma \in [\![D_0]\!]$.
  By validity, we have $D_2 :: \Gamma \vdash A : \kappa'$ and $D_3 :: \Gamma \vdash A' : \kappa'$ and $D_4 :: \Gamma \vdash \kappa' : \mathsf{kind}$.
  Hence by rule with D_0 and D_4, we have $D_0' :: \Gamma, \alpha : \kappa'\ \mathsf{ok}$.
  By validity, we have $D_5 :: \Gamma, \alpha : \kappa' \vdash Y : *$.
  By induction, we know that $[\![D_2]\!]\ \gamma = [\![D_3]\!]\ \gamma$.
  Hence $(\gamma, (\gamma(A), [\![D_2]\!]\ \gamma)/\alpha) \sim_{D_0} (\gamma, (\gamma(A'), [\![D_3]\!]\ \gamma)/\alpha)$.
  By mutual induction $(\gamma, (\gamma(A), [\![D_3]\!]\ \gamma)/\alpha)(e) \sim_{[\![D_5]\!]\ (\gamma, (\gamma(A), [\![D_2]\!]\ \gamma)/\alpha)} (\gamma, (\gamma(A'), [\![D_3]\!]\ \gamma)/\alpha)(e)$.
  By substitution, we have $D_1' :: \Gamma \vdash [A/\alpha]Y : *$ such that $[\![D_1']\!]\ \gamma = [\![D_5]\!]\ (\gamma, \gamma(A)/\alpha)$.
  By coherence, $[\![D_1]\!]\ \gamma = [\![D_1']\!]\ \gamma$.
  Hence $(\gamma, (\gamma(A), [\![D_2]\!]\ \gamma)/\alpha)(e) \sim_{[\![D_1]\!]\ \gamma} (\gamma, (\gamma(A'), [\![D_3]\!]\ \gamma)/\alpha)(e)$.
  Hence $\gamma([A/\alpha]e) \sim_{[\![D_1]\!]\ \gamma} \gamma([A'/\alpha]e)$.

- $D :: \Gamma \vdash e \equiv e : X$ (reflexivity).
  By inversion, $D' :: \Gamma \vdash e : X$.
  Assume we have $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash X : *$ and $\gamma \in [\![D_0]\!]$.
  Note that $\gamma \sim_{D_0} \gamma$.
  By mutual induction $\gamma(e) \sim_{[\![D_1]\!]\ \gamma} \gamma(e)$.

- $D :: \Gamma \vdash e \equiv e' : X$ (symmetry).
  By inversion $D' :: \Gamma \vdash e' \equiv e : X$.
  Assume we have $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash X : *$ and $\gamma \in [\![D_0]\!]$.
  By induction, we know that $\gamma(e') \sim_{[\![D_1]\!]\ \gamma} \gamma(e)$.
  Hence $\gamma(e) \sim_{[\![D_1]\!]\ \gamma} \gamma(e')$.

- $D :: \Gamma \vdash e \equiv e'' : X$ (transitivity).
  By inversion, $D' :: \Gamma \vdash e \equiv e' : X$ and $D'' :: \Gamma \vdash e' : e''X$.

Assume we have $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash X : *$ and $\gamma \in [\![D_0]\!]$.
Hence $\gamma(e) \sim_{[\![D_1]\!]\ \gamma} \gamma(e')$.
Hence $\gamma(e') \sim_{[\![D_1]\!]\ \gamma} \gamma(e'')$.
Hence it follows that $\gamma(e) \sim_{[\![D_1]\!]\ \gamma} \gamma(e'')$.

- $D :: \Gamma \vdash (\lambda\alpha : \kappa.\ e)\ A \equiv [A/\alpha]e : [A/\alpha]Y$.
By inversion, we have $D' :: \Gamma \vdash \lambda\alpha : \kappa.\ e : \Pi\alpha : \kappa.\ Y$ and $D'' :: \Gamma \vdash A : \kappa$.
Assume we have $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash [A/\alpha]Y : *$ and $\gamma \in [\![D_0]\!]$.
By validity, we know $D_2 :: \Gamma \vdash \Pi\alpha : \kappa.\ Y : *$.
By inversion on D_2, we know $D_3 :: \Gamma \vdash \kappa : \mathsf{kind}$ and $D_4 :: \Gamma, \alpha : \kappa \vdash Y : *$.
From D_0 and D_3 we have $D_0' :: \Gamma, \alpha : \kappa\ \mathsf{ok}$.
By induction, we know that $\gamma(\lambda\alpha : \kappa.\ e) \sim_{[\![D_2]\!]\ \gamma} \gamma(\lambda\alpha : \kappa.\ e)$.
By induction, we know that $[\![D'']\!]\ \gamma \in [\![D_3]\!]\ \gamma$.
Hence it follows that $\gamma((\lambda\alpha : \kappa.\ e)\ A) \sim_{[\![D_4]\!]\ (\gamma, (\gamma(A), [\![D'']\!]\ \gamma)/\alpha)} \gamma((\lambda\alpha : \kappa.\ e)\ A)$.
By substitution, we have $D_1' :: \Gamma \vdash [A/\alpha]Y : *$ so $[\![D_1']\!]\ \gamma = [\![D_4]\!]\ (\gamma, (\gamma(A), [\![D'']\!]\ \gamma)/\alpha)$.
By coherence $[\![D_1]\!]\ \gamma = [\![D_1']\!]\ \gamma$.
Hence $\gamma((\lambda\alpha : \kappa.\ e)\ A) \sim_{[\![D_1]\!]\ \gamma} \gamma((\lambda\alpha : \kappa.\ e)\ A)$.
Since types are closed under reduction, $\gamma((\lambda\alpha : \kappa.\ e)\ A) \sim_{[\![D_1]\!]\ \gamma} \gamma([A/\alpha]e)$.

- $D :: \Gamma \vdash (\lambda x : X.\ e)\ e' \equiv [e'/x]e : [e'/x]Y$.
By inversion, we have $D' :: \Gamma \vdash \lambda x : X.\ e : \Pi x : X.\ Y$ and $D'' :: \Gamma \vdash e : X$.
Assume we have $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash [e'/x]Y : *$ and $\gamma \in [\![D_0]\!]$.
By validity, we know $D_2 :: \Gamma \vdash \Pi x : X.\ Y : *$.
By inversion on D_2, we know $D_3 :: \Gamma \vdash X : *$ and $D_4 :: \Gamma, x : X \vdash Y : *$.
From D_0 and D_3 we have $D_0' :: \Gamma, x : X\ \mathsf{ok}$.
By induction, we know that $\gamma(\lambda x : X.\ e) \sim_{[\![D_2]\!]\ \gamma} \gamma(\lambda x : X.\ e)$.
By induction, we know that $\gamma(e') \sim_{[\![D_3]\!]\ \gamma} \gamma(e')$.
Hence it follows that $\gamma((\lambda x : X.\ e)\ e') \sim_{[\![D_4]\!]\ (\gamma, \gamma(e')/x)} \gamma((\lambda x : X.\ e)\ e')$.
By substitution, we have $D_1' :: \Gamma \vdash [e'/x]Y : *$ so $[\![D_1']\!]\ \gamma = [\![D_4]\!]\ (\gamma, (\gamma(e'), [\![D'']\!]\ \gamma)/x)$.
By coherence $[\![D_1]\!]\ \gamma = [\![D_1']\!]\ \gamma$.
Hence $\gamma((\lambda x : X.\ e)\ e') \sim_{[\![D_1]\!]\ \gamma} \gamma((\lambda x : X.\ e)\ e')$.
Since types are closed under reduction, $\gamma((\lambda x : X.\ e)\ e') \sim_{[\![D_1]\!]\ \gamma} \gamma([e'/x]e)$.

- $D :: \Gamma \vdash e \equiv e' : \Pi x : X.\ Y$.
By inversion, we have $D' :: \Gamma, x : X \vdash e\ x : e'\ xY$.
Assume we have $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash \Pi x : X.\ Y : *$ and $\gamma \in [\![D_0]\!]$.
By inversion on D_1, we have $D_2 :: \Gamma \vdash X : *$ and $D_3 :: \Gamma, x : X \vdash Y : *$.
Hence we have $D_0' :: \Gamma, x : X\ \mathsf{ok}$.
We want to show that $\gamma(e) \sim_{[\![D_1]\!]\ \gamma} \gamma(e')$.
To show this, assume we have $\bar{t} \in [\![D_2]\!]\ \gamma$.
We want to show that $\gamma(e)\ \bar{t} \sim_{[\![D_3]\!]\ (\gamma, \bar{t}/x)} \gamma(e')\ \bar{t}$.
Now note that $(\gamma, \bar{t}/x) \in [\![D_0']\!]$.
By induction, $(\gamma, \bar{t}/x)(e\ x) \sim_{[\![D_3]\!]\ (\gamma, \bar{t}/x)} (\gamma, \bar{t}/x)(e'\ x)$.
Note that $x \notin \mathrm{FV}(e, e')$.
Hence this is equivalent to $\gamma(e)\ \bar{t} \sim_{[\![D_3]\!]\ (\gamma, \bar{t}/x)} \gamma(e')\ \bar{t}$.

Hence $\gamma(e) \sim_{[\![D_1]\!]\ \gamma} \gamma(e')$.

- $D :: \Gamma \vdash e \equiv e' : \Pi\alpha : \kappa.\, Y$.
  By inversion, we have $D' :: \Gamma, \alpha : \kappa \vdash e\ \alpha : e'\ \alpha Y$.
  Assume we have $D_0 :: \Gamma\ \mathsf{ok}$ and $D_1 :: \Gamma \vdash \Pi\alpha : \kappa.\, Y : *$ and $\gamma \in [\![D_0]\!]$.
  By inversion on D_1, we have $D_2 :: \Gamma \vdash \kappa : \mathsf{kind}$ and $D_3 :: \Gamma, \alpha : \kappa \vdash Y : *$.
  Hence we have $D_0' :: \Gamma, \alpha : \kappa\ \mathsf{ok}$.
  We want to show that $\gamma(e) \sim_{[\![D_1]\!]\ \gamma} \gamma(e')$.
  To show this, assume we have $\overline{A} \in \text{Type}^2$ and $R \in [\![D_2]\!]\ \gamma$
  We want to show that $\gamma(e)\ \overline{A} \sim_{[\![D_3]\!]\ (\gamma,(\overline{A},R)/\alpha)} \gamma(e')\ \overline{A}$.
  Now note that $(\gamma, (\overline{A}, R)/\alpha) \in [\![D_0']\!]$.
  By induction, $(\gamma, (\overline{A}, R)/\alpha)(e\ \alpha) \sim_{[\![D_3]\!]\ (\gamma,(\overline{A},R)/\alpha)} (\gamma, (\overline{A}, R)/\alpha)(e'\ \alpha)$.
  Note that $\alpha \notin \text{FV}(e, e')$.
  Hence this is equivalent to $\gamma(e)\ \overline{A} \sim_{[\![D_3]\!]\ (\gamma,(\overline{A},R)/\alpha)} \gamma(e')\ \overline{A}$.
  Hence $\gamma(e) \sim_{[\![D_1]\!]\ \gamma} \gamma(e')$.

□

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